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REPRESENTATIONS OF SUBHARMONIC HARDY FUNCTIONS IN THE COMPLEX BALL

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Abstract. For the purpose of characterizing subharmonic or $\mathcal{M}$-subharmonic Hardy classes in the unit ball of $\mathbb{C}^n$, we establish fundamental identities between integral means in terms of volume integrals and Green's functions.

1. Introduction

Let $B = B_n$ denote the open unit ball of $\mathbb{C}^n$ and S denote the boundary of B: $S = \{z \in \mathbb{C}^n : |z| = 1\}$. Let ν and σ denote respectively the Lebesgue volume measure on B and the surface measure on S normalized to be $\nu(B) = \sigma(S) = 1$. Denote $d\tau(z) = (1 - |z|^2)^{-(n+1)} d\nu(z)$.

Let $\mathcal{M}$ denote the group of all automorphism, that is, one to one biholomorphic onto map, of B. $\mathcal{M}$ consists of all maps of the form $U\varphi_a$, where U is a unitary operator of $\mathbb{C}^n$ and φ_a is defined by

$$\varphi_a(z) = \begin{cases} \frac{a - P_a z - \sqrt{1-|a|^2} Q_a z}{1 - <z,a>}, & \text{if } a \neq 0 \\ 0, & \text{if } a = 0. \end{cases}$$

Here $<\,,\,>$ is the Hermitian inner product of $\mathbb{C}^n$: $<z, w> = \sum_{j=1}^n z_j \bar{w}_j$, $z, w \in \mathbb{C}^n$, $P_a z$ is the projection of $\mathbb{C}^n$ onto the subspace generated by B:

$$P_a z = \frac{<z, a>}{<a, a>} a, \quad \text{if } a \neq 0 \quad \text{and} \quad P_0 z = 0,$$

and $Q_a(z) = z - P_a z$.

Let Δ be the complex Laplacian: $\Delta = 4 \sum_{j=1}^n D_j \bar{D}_j$, where $D_j = \frac{\partial}{\partial z_j}$ and $\bar{D}_j = \frac{\partial}{\partial \bar{z}_j}$, $j = 1, 2, \ldots, n$. In B, Δ may be decomposed into the complex tangential Laplacian and the complex radial Laplacian: $\Delta = \Delta_{tan} + \Delta_{rad}$, where Δ_{rad} is defined for $f \in C^2(B)$ and $z = r\zeta, 0 < r < 1, \zeta \in S$, to be the Laplacian of the function $\lambda \to f(z + \lambda\zeta)$ at the origin of $\mathbb{C}$ (see [3], 17.3.2).

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Let $\widetilde{\Delta}$ denote the ($\mathcal{M}$-) invariant Laplacian of B defined for $f \in C^2(B)$ by

$$\widetilde{\Delta} f(a) = \Delta\left(f \circ \varphi_a\right)(0), \quad a \in B.$$

$\widetilde{\Delta}$ is $\mathcal{M}$-invariant in the sense that

$$\left(\widetilde{\Delta} f\right) \circ \psi = \widetilde{\Delta}(f \circ \psi)$$

for all $\psi \in \mathcal{M}$, and it is known that

$$\widetilde{\Delta} f(a) = 4(1-|a|^2) \sum_{i,j=1}^{n} (\delta_{i,j} - \bar{a}_i a_j)(\bar{D}_i D_j f)(a), \quad a \in B,$$

for $f \in C^2(B)$ (see [3], 4.1.3).

A $C^2(B)$ function f is said to be harmonic (in B) if $\Delta f = 0$ in B, $\mathcal{M}$-harmonic if $\widetilde{\Delta} f = 0$ in B, pluriharmonic if $\Delta f = 0 = \widetilde{\Delta} f$ in B (see [3], 4.4.9).

An upper semicontinuous function $f : B \to [-\infty, \infty)$, $f \not\equiv -\infty$, satisfying the inequality

$$f(a) \leq \int_S f(a + r\zeta)\, d\sigma(\zeta)$$

for all $a \in B$ and for all r such that $a + r\bar{B} \subset B$ is called subharmonic (in B). An upper semicontinuous function $f : B \to [-\infty, \infty)$, $f \not\equiv -\infty$, satisfying

$$f(a) \leq \int_S f \circ \varphi_a(r\zeta)\, d\sigma(\zeta)$$

for all $a \in B$ and for all r sufficiently small is called $\mathcal{M}$-subharmonic. Also, an upper semicontinuous function $f : B \to [-\infty, \infty)$, is called plurisubharmonic if the functions

$$\lambda \to f(a + \lambda b)$$

are subharmonic in neighborhoods of the origin in $\mathbb{C}$, for all $a \in B$, $b \in \mathbb{C}^n$.

If f is subharmonic on B, then $\int_S f(r\zeta)\, d\sigma(\zeta)$ is an increasing function of r. If f is $\mathcal{M}$-subharmonic on B, then $\int_S f \circ \varphi_a(r\zeta)\, d\sigma(\zeta)$ is an increasing function of r for every a $\in B$ (see [4], 5.11).

It is known for $f \in C^2(B)$ that $\Delta f \geq 0$ if and only if f is subharmonic, and that $\widetilde{\Delta} f \geq 0$ if and only if f is $\mathcal{M}$-subharmonic. But $\Delta f \geq 0$ and $\widetilde{\Delta} f \geq 0$ does not imply that f is plurisubharmonic (see [3], 7.2.1).

For $0 < r \leq 1$, let

$$g(r, z) \;=\; \int_{|z|}^{r} \frac{1}{\rho^{2n-1}} d\rho, \quad z \in rB,$$

and

$$\widetilde{g}(r, z) \;=\; \frac{1}{2n} \int_{|z|}^{r} \frac{(1-\rho^2)^{n-1}}{\rho^{2n-1}} d\rho, \quad z \in rB.$$

Let $g(z) = g(1, z)$ and $\widetilde{g}(z) = \widetilde{g}(1, z)$. Then $g(z) = \log \frac{1}{|z|}$ if $n = 1$, and

$$g(z) \;=\; \frac{1}{2(n-1)} \left(\frac{1}{|z|^{2n-2}} - 1 \right)$$

if $n > 1$. Elementary calculation shows that $\Delta g(z) = 0$ for all $z \in B$, $z \neq 0$. So g is superharmonic (i.e - g is subharmonic) on $B\backslash\{0\}$, $g(0) = \infty$ and $\lim_{|z|\to 1} g(z) = 0$. The function

$$G(z,w) = g\left(\varphi_w(z)\right), \quad z,w \in B,$$

is called the Green's function for Δ. It satisfies $G(z,w) = G(w,z)$ and $\Delta_z G(z,w) = 0$ on $B\backslash\{0\}$.

Also, $\widetilde{\Delta}\widetilde{g}(z) = 0$ for all $z \in B$, $z \neq 0$; $\widetilde{g}$ is $\mathcal{M}$-superharmonic (i.e - $\widetilde{g}$ is $\mathcal{M}$-subharmonic) on $B\backslash\{0\}$, $\widetilde{g}(0) = \infty$ and $\lim_{|z|\to 1} \widetilde{g}(z) = 0$. The function

$$\widetilde{G}(z,w) = \widetilde{g}\left(\varphi_w(z)\right), \quad z,w \in B,$$

is called the (invariant) Green's function for $\widetilde{\Delta}$. It satisfies $\widetilde{G}(z,w) = \widetilde{G}(w,z)$ and $\widetilde{\Delta}_z\widetilde{G}(z,w) = 0$ on $B\backslash\{0\}$.

Let Rf denote the radial derivative of f: $Rf(z) = \sum_{j=1}^n z_j D_j f(z),\ z \in B$. Note that $Rf = \frac{r}{2}\phi'$ when f is radial with $f(z) = \phi(r), |z| = r$. Rf is invariant under the action of the unitary group $\mathcal{U}$.

We in this note establish fundamental identities between integral means as follows.

Theorem 1.1. *If $f \in C^2(B)$ and $0 < r < 1$, then the following (a)∼(f) are all equal.*

$$(a) \quad \int_S f(r\zeta)d\sigma(\zeta)$$

$$(b) \quad f(0) + \frac{1}{2n}\int_{rB} g(r,z)\Delta f(z)\ d\nu(z)$$

$$(c) \quad f(0) + \int_{rB} \widetilde{g}(r,z)\widetilde{\Delta} f(z)\ d\tau(z)$$

$$(d) \quad \frac{1}{r^{2n}}\int_{rB} f(z)\ d\nu(z) + \frac{1}{4nr^{2n}}\int_{rB}(r^2 - |z|^2)\Delta f(z)\ d\nu(z)$$

$$(e) \quad \frac{1}{r^{2n}(1-r^2)}\int_{rB}\left(1 - \frac{n+1}{n}|z|^2\right) f(z)\ d\nu(z) + \frac{1}{4n(n+1)r^{2n}(1-r^2)}\int_{rB}\left\{1 - \left(\frac{1-r^2}{1-|z|^2}\right)^{n+1}\right\}\widetilde{\Delta} f(z)\, d\nu(z)$$

$$(f) \quad \frac{1}{r^{2n}}\int_{rB} f(z)\, d\nu(z) + \frac{1}{nr^{2n}}\int_{rB} Rf(z)\, d\nu(z)$$

If $n \geq 2$, then each one of $(a) \sim (f)$ equals

$$(g) \quad \frac{1}{r^{2n}}\int_{rB} f(z)\, d\nu(z) + \frac{1}{4n(n-1)r^{2n}}\int_{rB} |z|^2\Delta_{tan} f(z)\, d\nu(z).$$

Theorem 1.1 can be used in characterizing various function classes, for example pluri-harmonic Hardy classes and BMO classes, in terms of volume integrals. This will be done in a forthcoming paper. Instead, we refer to [1, 2] for previous results of the same vein and present a simple illustration, which immediately follows from Theorem 1.1.

Corollary 1.2. *Let $n \geq 2$. Let $f : B \to \mathbb{C}$ with $|f|^2 \in C^2(B)$. If $\Delta_{rad}|f|^2 \geq 0$ and $\Delta_{tan}|f|^2 \geq 0$, then the following $(a) \sim (e)$ are equivalent.*

$$(a) \quad \sup_{0 \leq r < 1} \int_S |f(r\zeta)|^2 \, d\sigma(\zeta) < \infty$$

$$(b) \quad \int_B (1 - |z|)\Delta|f(z)|^2 \, d\nu(z) < \infty$$

$$(c) \quad \int_B (1 - |z|)^n \widetilde{\Delta}|f(z)|^2 \, d\tau(z) < \infty$$

$$(d) \quad \int_B R|f(z)|^2 \, d\nu(z) < \infty$$

$$(e) \quad \int_B \Delta_{tan}|f(z)|^2 \, d\nu(z) < \infty$$

2. Lemmas

Lemma 2.1. *Let $0 < r \leq 1$ be fixed.*

(a) *If $n = 1$, then $g(r, z) = \log \frac{r}{|z|} = 2\widetilde{g}(r, z)$.*
(b) *If $n \geq 2$, then*

$$\frac{\widetilde{g}(r, z)}{(1 - |z|^2)^n} \approx \frac{g(r, z)}{1 - \frac{|z|}{r}} \approx |z|^{2-2n}, \quad z \in rB.$$

Proof. (a) follows immediately. (b) follows from the following limits which can be derived by using L'Hospital's rule.

$$\lim_{t \to r} \frac{g(r, t)}{t^{2-2n}(1 - \frac{t}{r})} = \frac{1}{2n}, \quad \lim_{t \to 0} \frac{g(r, t)}{t^{2-2n}(1 - \frac{t}{r})} = \frac{1}{2(n-1)} \cdot \frac{1}{4n(n-1)};$$

$$\lim_{t \to r} \frac{\widetilde{g}(r, t)}{t^{2-2n}(1 - t^2)^n} = \frac{1}{n(n - 1 + r^2)}, \quad \lim_{t \to 0} \frac{\widetilde{g}(r, t)}{t^{2-2n}(1 - t^2)^n} = \frac{1}{4n(n-1)}.$$

□

Lemma 2.2 (See [1])**.** *Let $f \in C^2(B)$ and $a = r\zeta, 0 \leq r < 1, \zeta \in S$. Then we have the following.*

(a) $\Delta = \Delta_{tan} + \Delta_{rad}$; $\widetilde{\Delta} = (1 - r^2)\Delta_{tan} + (1 - r^2)^2\Delta_{rad}$

(b) *If f is radial, then* $\Delta_{rad}f = \dfrac{\partial f^2}{\partial^2 r} + \dfrac{1}{r}\dfrac{\partial f}{\partial r}$ *and* $\Delta_{tan}f = \dfrac{2(n-1)}{r}\dfrac{\partial f}{\partial r}$.

(c) $\Delta, \Delta_{rad}, \Delta_{tan}, \widetilde{\Delta}$ *all commutes with the action of the unitary group.*

Lemma 2.3. *If $f \in C^2(B)$ and $0 < r < 1$, then the following (a)∼ (f) are equal.*

$$(a) \quad 2nr^{2n-1} \frac{d}{dr} \int_S f(r\zeta)\, d\sigma(\zeta)$$

$$(b) \quad \int_{rB} \Delta f(z)\, d\nu(z)$$

$$(c) \quad (1-r^2)^{n-1} \int_{rB} \widetilde{\Delta} f(z)\, d\tau(z)$$

$$(d) \quad \frac{1}{2r}\frac{d}{dr} \int_{rB} (r^2 - |z|^2) \Delta f(z)\, d\nu(z)$$

$$(e) \quad \frac{1}{2(n+1)r(1-r^2)} \frac{d}{dr} \int_{rB} \left\{(1-|z|^2)^{n+1} - (1-r^2)^{n+1}\right\} \widetilde{\Delta} f(z)\, d\tau(z)$$

$$(f) \quad \frac{2}{r}\frac{d}{dr} \int_{rB} Rf(z)\, d\nu(z).$$

If $n \geq 2$, then each one of $(a) \sim (f)$ equals

$$(g) \quad \frac{1}{2(n-1)r} \frac{d}{dr} \int_{rB} |z|^2 \Delta_{tan} f(z)\, d\nu(z).$$

Proof. If we denote $f^{\#}$ the radialization of f:

$$f^{\#}(z) = \int_{\mathcal{U}} f(Uz)\, dU,$$

where $\mathcal{U}$ denote the group of unitary operators of $\mathbb{C}^n$, then by Lemma 2.2 (c)

$$\Delta_{tan}\left(f^{\#}\right) = \left(\Delta_{tan} f\right)^{\#},\ \Delta\left(f^{\#}\right) = (\Delta f)^{\#} \text{ and } \widetilde{\Delta}\left(f^{\#}\right) = \left(\widetilde{\Delta} f\right)^{\#}.$$

So it is sufficient to verify required equalities with $f^{\#}$ instead of f. Denote $f^{\#} = u$ and $u(z) = \phi(\rho)$, $\rho = |z|$ for simplicity.

Consider two representations of $r^{2n-1}\phi'(r)$:

$$r^{2n-1}\phi'(r) = \int_0^r \frac{d}{d\rho}\left\{\rho^{2n-1}\phi'(\rho)\right\} d\rho \tag{2.1}$$

and

$$r^{2n-1}\phi'(r) = (1-r^2)^{n-1} \int_0^r \frac{d}{d\rho}\left\{\frac{1}{(1-\rho^2)^{n-1}}\rho^{2n-1}\phi'(\rho)\right\} d\rho. \tag{2.2}$$

Simply from $Ru = \frac{\rho}{2}\phi'$, we have

$$2n\rho^{2n-1}\phi' = 4n\rho^{2n-2}Ru = \frac{2}{\rho}\frac{d}{d\rho}\int_0^\rho 2nr^{2n-1}Ru\, dr,$$

so that $(a) = (f)$ follows.

By Lemma 2.2

$$\Delta u(z) = \phi''(\rho) + \frac{2n-1}{\rho}\phi'(\rho),$$

$$\Delta_{rad} u(z) = \phi''(\rho) + \frac{1}{\rho}\phi'(\rho),$$

and

$$\widetilde{\Delta} u(z) = (1-\rho^2)^2 \Delta u(z) + 2(n-1)\rho(1-\rho^2)\phi'(\rho).$$

Thus, from (2.1) we obtain

$$\begin{aligned} r^{2n-1}\phi'(r) &= \int_0^r \rho^{2n-1}\left\{\phi''(\rho) + \frac{2n-1}{\rho}\phi'(\rho)\right\} d\rho \\ &= \int_0^r \rho^{2n-1}\Delta u(z)\, d\rho, \end{aligned}$$

which implies that $(a) = (b)$.

Also, from (2.2) we obtain

$$\begin{aligned} & r^{2n-1}\phi'(r) \\ =& (1-r^2)^{n-1}\int_0^r \frac{1}{(1-\rho^2)^n}\left\{(1-\rho^2)\frac{d}{d\rho}\left(\rho^{2n-1}\phi'(\rho)\right) + 2(n-1)\rho^{2n}\phi'(\rho)\right\} d\rho \\ =& (1-r^2)^{n-1}\int_0^r \frac{\rho^{2n-1}}{(1-\rho^2)^{n+1}}\widetilde{\Delta}\phi(\rho) d\rho, \end{aligned}$$

which implies that $(a) = (c)$.

Integration by parts gives that

$$\begin{aligned} & 2r\int_0^r \rho^{2n-1}\Delta\phi(\rho) d\rho \\ =& \frac{d}{dr}\left(r^2\int_0^r \rho^{2n-1}\Delta\phi(\rho) d\rho\right) - r^{2n+1}\Delta\phi(r) \\ =& \frac{d}{dr}\left(\int_0^r \rho^{2n-1} r^2 \Delta\phi(\rho) d\rho\right) - \frac{d}{dr}\int_0^r \rho^{2n+1}\Delta\phi(\rho) d\rho \\ =& \frac{d}{dr}\left(\int_0^r \rho^{2n-1}(r^2-\rho^2)\Delta\phi(\rho) d\rho\right), \end{aligned}$$

which implies that $(b) = (d)$.

By a similar way,

$$\begin{aligned} & 2(n+1)r(1-r^2)^n \int_0^r \frac{\rho^{2n-1}}{(1-\rho^2)^{n+1}}\widetilde{\Delta}\phi(\rho) d\rho \\ =& -\frac{d}{dr}\left\{(1-r^2)^{n+1}\int_0^r \frac{\rho^{2n-1}}{(1-\rho^2)^{n+1}}\widetilde{\Delta}\phi(\rho) d\rho\right\} + r^{2n-1}\widetilde{\Delta}\phi(r) \\ =& -\frac{d}{dr}\left\{(1-r^2)^{n+1}\int_0^r \frac{\rho^{2n-1}}{(1-\rho^2)^{n+1}}\widetilde{\Delta}\phi(\rho) d\rho\right\} + \frac{d}{dr}\left\{\int_0^r \rho^{2n-1}\widetilde{\Delta}\phi(\rho) d\rho\right\} \\ =& \frac{d}{dr}\int_0^r \left\{1 - \left(\frac{1-r^2}{1-\rho^2}\right)^{n+1}\right\} \rho^{2n-1}\widetilde{\Delta}\phi(\rho)\, d\rho, \end{aligned}$$

which implies that $(c) = (e)$.

Suppose $n \geq 2$. From $\Delta_{tan} u(z) = \frac{2(n-1)}{\rho}\phi'(\rho)$ and $(a) = (d)$,

$$\frac{2n}{n-1}\rho^{2n+1}\Delta_{tan} u(z) = 4n\rho^{2n}\phi'(\rho) = \frac{d}{d\rho}\int_{\rho B}(\rho^2 - |z|^2)\Delta f(z)\, d\nu(z).$$

Taking $\int_0^r d\rho$ gives $(d) = (g)$. □

3. Proof of Main Results

Proof of Theorem 1.1. That $(a) = (b)$ follows from integrating the identity

$$\frac{d}{dr}\int_S f(r\zeta) d\sigma(\zeta) \;=\; \frac{1}{2nr^{2n-1}}\int_{rB}\Delta f(z) d\nu(z)$$

(which is $(a) = (b)$ of Lemma 2.3) with respect to dr and using

$$\begin{aligned}&\frac{1}{2n}\int_0^r \frac{1}{\rho^{2n-1}} d\rho \int_{\rho B}\Delta f(z) d\nu(z)\\ &= \frac{1}{2n}\int_{rB}\Delta f(z)\left(\int_0^r \frac{1}{\rho^{2n-1}}\ \chi_{|z|<\rho} d\rho\right) d\nu(z)\\ &= \frac{1}{2n}\int_{rB} g(r,z)\Delta f(z)\ d\nu(z).\end{aligned}$$

$(a) = (c)$ follows from integrating the identity

$$\frac{d}{dr}\int_S f(r\zeta) d\sigma(\zeta) \;=\; \frac{(1-r^2)^{n-1}}{2nr^{2n-1}}\int_{rB}\widetilde{\Delta} f(z) d\tau(z)$$

(which is $(a) = (c)$ of Lemma 2.3) with respect to dr and using

$$\begin{aligned}&\frac{1}{2n}\int_0^r \frac{(1-\rho^2)^{n-1}}{\rho^{2n-1}}\ d\rho \int_{\rho B}\widetilde{\Delta} f(z)\ d\tau(z)\\ &= \frac{1}{2n}\int_{rB}\widetilde{\Delta} f(z)\left(\int_0^r \frac{(1-\rho^2)^{n-1}}{\rho^{2n-1}}\ \chi_{|z|<\rho}\ d\rho\right) d\tau(z)\\ &= \int_{rB}\widetilde{g}(r,z)\widetilde{\Delta} f(z)\ d\tau(z).\end{aligned}$$

$(a) = (d)$ follows from integrating the identity

$$r^{2n}\frac{d}{dr}\int_S f(r\zeta) d\sigma(\zeta) \;=\; \frac{1}{4n}\frac{d}{dr}\int_{rB}(r^2 - |z|^2)\Delta f(z)\ d\nu(z)$$

(which is $(a) = (d)$ of Lemma 2.3) with respect to dr and using

$$\begin{aligned}&\int_0^r \rho^{2n}\left(\frac{d}{d\rho}\int_S f(\rho\zeta) d\sigma(\zeta)\right) d\rho\\ &= r^{2n}\int_S f(r\zeta) d\sigma(\zeta) - 2n\int_0^r \rho^{2n-1} d\rho \int_S f(\rho\zeta)\ d\sigma(\zeta) \qquad (3.1)\\ &= r^{2n}\int_S f(r\zeta) d\sigma(\zeta) - \int_{rB} f(z)\ d\nu(z).\end{aligned}$$

Also, $(a) = (e)$ follows from integrating the identity

$$r^{2n}(1-r^2)\,\frac{d}{dr}\int_S f(r\zeta)\,d\sigma(\zeta)$$
$$= \frac{1}{4n(n+1)}\,\frac{d}{dr}\int_{rB}\left\{(1-|z|^2)^{n+1}-(1-r^2)^{n+1}\right\}\widetilde{\Delta}f(z)\,d\tau(z)$$

(which is $(a) = (e)$ of Lemma 2.3) with respect to dr and using

$$\int_0^r \rho^{2n}(1-\rho^2)\left(\frac{d}{d\rho}\int_S f(\rho\zeta)\ d\sigma(\zeta)\right)\,d\rho$$
$$= r^{2n}(1-r^2)\int_S f(r\zeta)\ d\sigma(\zeta) - 2n\int_0^r \rho^{2n-1}\left(1-\frac{n+1}{n}\rho^2\right)\,d\rho\int_S f(\rho\zeta)d\sigma(\zeta)$$
$$= r^{2n}(1-r^2)\int_S f(r\zeta)\ d\sigma(\zeta) - \int_{rB}\left(1-\frac{n+1}{n}|z|^2\right)f(z)\ d\nu(z).$$

$(a) = (f)$ follows from integrating the identity

$$r^{2n}\frac{d}{dr}\int_S f(r\zeta)d\sigma(\zeta) \ = \ \frac{1}{n}\frac{d}{dr}\int_{rB} Rf(z)\ d\nu(z)$$

(which is $(a) = (f)$ of Lemma 2.3) with respect to dr and using (3.1).

If $n \geq 2$, then $(a) = (g)$ follows from integrating the identity

$$r^{2n}\frac{d}{dr}\int_S f(r\zeta)d\sigma(\zeta) \ = \ \frac{1}{4n(n-1)}\frac{d}{dr}\int_{rB}|z|^2\Delta_{tan}f(z)\ d\nu(z)$$

(which is $(a) = (g)$ of Lemma 2.3) with respect to dr and using

$$\int_0^r \rho^{2n}\left(\frac{d}{d\rho}\int_S f(\rho\zeta)d\sigma(\zeta)\right)d\rho \ = \ r^{2n}\int_S f(r\zeta)d\sigma(\zeta) - \int_{rB} f(z)\ d\nu(z).$$

□

Proof of Corollary 1.2. That $\Delta_{rad}|f|^2 \geq 0$ and $\Delta_{tan}|f|^2 \geq 0$ imply $\Delta|f|^2 \geq 0$ and $\widetilde{\Delta}|f|^2 \geq 0$. These subharmonicity imply

$$\sup_{0\leq r<1}\int_S |f(r\zeta)|^2\ d\sigma(\zeta) = \lim_{r\to 1}\int_S |f(r\zeta)|^2\ d\sigma(\zeta).$$

Whence by Lemma 2.1 and Theorem 1.1 the result follows. □

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