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THE COHEN TYPE THEOREM FOR S-$*_w$-PRINCIPAL IDEAL DOMAINS

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Abstract. Let D be an integral domain, $*$ a star-operation on D, and S a (not necessarily saturated) multiplicative subset of D. In this article, we prove the Cohen type theorem for S-$*_w$-principal ideal domains, which states that D is an S-$*_w$-principal ideal domain if and only if every nonzero prime ideal of D (disjoint from S) is S-$*_w$-principal.

1. Introduction

For the sake of clarity, we first review some terminologies for star-operations. Let D be an integral domain with quotient field K and $\mathbf{F}(D)$ the set of nonzero fractional ideals of D. A *star-operation* on D is a mapping $I \mapsto I_*$ from $\mathbf{F}(D)$ into itself which satisfies the following three conditions for all $0 \neq a \in K$ and all $I, J \in \mathbf{F}(D)$:

(1) $(a)_* = (a)$ and $(aI)_* = aI_*$;
(2) $I \subseteq I_*$, and if $I \subseteq J$, then $I_* \subseteq J_*$; and
(3) $(I_*)_* = I_*$.

The most important examples of star-operations are the d-operation, v-operation, and w-operation. The *d-operation* is the identity mapping, *i.e.*, $I \mapsto I_d := I$. For an $I \in \mathbf{F}(D)$, set $I^{-1} = \{a \in K \,|\, aI \subseteq D\}$. The *$v$-operation* is the mapping defined by $I \mapsto I_v := (I^{-1})^{-1}$. The *$w$-operation* is the mapping defined by $I \mapsto I_w := \{a \in K \,|\, Ja \subseteq I$ for some finitely generated ideal J of D with $J_v = D\}$. Let $*$ be a star-operation on D. Then $*$ induces a new star-operation $*_w$ on D. The *$*_w$-operation* is the mapping defined by $I \mapsto I_{*_w} := \{a \in K \,|\, Ja \subseteq I$ for some $J \in \mathrm{GV}^*(D)\}$, where $\mathrm{GV}^*(D)$ is the set of nonzero finitely generated ideals J of D with $J_* = D$. (We call an element J of $\mathrm{GV}^*(D)$ a *$*$-Glaz-Vasconcelos ideal* ($*$-GV-ideal) of D.) When $* = d$

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(resp., $* = v$), the $*_w$-operation is precisely the same as the d-operation (resp., w-operation).

Let D be an integral domain, $*$ a star-operation on D, and S a (not necessarily saturated) multiplicative subset of D. In [1, Definition 1], Anderson and Dumitrescu introduced the notion of S-principal ideal domains. They defined an ideal I of D to be *S-principal* if there exist an element $s \in S$ and a principal ideal (c) of D such that $sI \subseteq (c) \subseteq I$; and the domain D to be an *S-principal ideal domain* (S-PID) if each ideal of D is S-principal. In [4, Section 1], the authors studied the w-operation analogue of S-PIDs. They defined a nonzero ideal I of D to be *S-w-principal* if there exist an element $s \in S$ and a principal ideal (c) of D such that $sI \subseteq (c) \subseteq I_w$; and the domain D to be an *S-unique factorization domain* (S-UFD) (or *S-factorial domain*) if each nonzero ideal of D is S-w-principal. Recently, in [5, Definition 1], the authors generalized these notions by using star-operations and introduced the concept of S-$*_w$-principal ideal domains. They defined a nonzero ideal I of D to be *S-$*_w$-principal* if there exist an element $s \in S$ and a principal ideal (c) of D such that $sI \subseteq (c) \subseteq I_{*_w}$; and the domain D to be an *S-$*_w$-principal ideal domain* (S-$*_w$-PID) if each nonzero ideal of D is S-$*_w$-principal. If $* = d$ (resp., $* = v$), then the notion of S-$*_w$-PIDs is precisely the same as that of S-PIDs (resp., S-factorial domains).

The purpose of this article is to give the Cohen type theorem for S-$*_w$-PIDs. As corollaries, we recover the characterizations of PIDs and UFDs. More precisely, we show that D is an S-$*_w$-PID if and only if every nonzero prime ideal of D (disjoint from S) is S-$*_w$-principal (Theorem 3). We also regain that D is a PID if and only if every prime ideal of D is principal; and D is a UFD if and only if for any nonzero prime ideal P of D, P_w is principal (Corollaries 6 and 7).

2. Main results

In this section, we give the Cohen type theorem for S-$*_w$-PIDs. To do this, we need the following two lemmas.

Lemma 1. *Let D be an integral domain and $*$ a star-operation on D.*

(1) *If I is a nonzero ideal of D and c is an element of D, then $(I_{*_w} : c) = (I : c)_{*_w}$.*

(2) *If $\{I_\alpha\}_{\alpha\in\Lambda}$ is a chain of nonzero ideals of D, then $\left(\bigcup_{\alpha\in\Lambda} I_\alpha\right)_{*_w} = \bigcup_{\alpha\in\Lambda}(I_\alpha)_{*_w}$.*

Proof. (1) Let $a \in (I_{*_w} : c)$. Then $ac \in I_{*_w}$; so there exists an element $J_1 \in \mathrm{GV}^*(D)$ such that $acJ_1 \subseteq I$. Hence $aJ_1 \subseteq (I : c)$, and thus $a \in (I : c)_{*_w}$. For the reverse containment, let $b \in (I : c)_{*_w}$. Then we can find a $*$-GV-ideal J_2 of D such that $bJ_2 \subseteq (I : c)$; so $bcJ_2 \subseteq I$. Hence $bc \in I_{*_w}$, and thus $b \in (I_{*_w} : c)$.

(2) Let $a \in \left(\bigcup_{\alpha\in\Lambda} I_\alpha\right)_{*_w}$. Then there exists an element $J \in \mathrm{GV}^*(D)$ such that $Ja \subseteq \bigcup_{\alpha\in\Lambda} I_\alpha$. Since J is finitely generated, $Ja \subseteq I_\beta$ for some $\beta \in \Lambda$. Hence $a \in (I_\beta)_{*_w}$. Thus $\left(\bigcup_{\alpha\in\Lambda} I_\alpha\right)_{*_w} \subseteq \bigcup_{\alpha\in\Lambda}(I_\alpha)_{*_w}$. For the

reverse containment, note that $(I_\gamma)_{*_w} \subseteq \left(\bigcup_{\alpha\in\Lambda} I_\alpha\right)_{*_w}$ for all $\gamma \in \Lambda$. Thus $\bigcup_{\alpha\in\Lambda}(I_\alpha)_{*_w} \subseteq \left(\bigcup_{\alpha\in\Lambda} I_\alpha\right)_{*_w}$. □

Lemma 2. *Let D be an integral domain, $*$ a star-operation on D, and S a multiplicative subset of D. Then an ideal of D maximal among non-S-$*_w$-principal ideals is a prime ideal of D which is disjoint from S.*

Proof. Let P be an ideal of D maximal among non-S-$*_w$-principal ideals of D, and suppose to the contrary that P is not a prime ideal of D. Then we can find $a, b \in D \setminus P$ such that $ab \in P$. By the maximality of P, $P + (a)$ is an S-$*_w$-principal ideal of D; so we can choose an element $s \in S$ and a principal ideal (c) of D such that

$$s(P + (a)) \subseteq (c) \subseteq (P + (a))_{*_w}.$$

Note that $(P_{*_w} : c)$ is an ideal of D containing P and b; so $(P_{*_w} : c)$ is an S-$*_w$-principal ideal of D by the maximality of P. Therefore there exist an element $t \in S$ and a principal ideal (d) of D such that

$$t(P : c) \subseteq t(P_{*_w} : c) \subseteq (d) \subseteq (P_{*_w} : c)_{*_w} = (P_{*_w} : c),$$

where the equality follows from Lemma 1(1). Let $x \in P$. Then $sx = cy$ for some $y \in (P : c)$; so $sP \subseteq (P : c)c \subseteq P$. Hence we obtain

$$stP \subseteq t(P : c)c \subseteq (cd) \subseteq (P_{*_w} : c)c \subseteq P_{*_w},$$

which shows that P is S-$*_w$-principal. However, this is a contradiction to the fact that P is not S-$*_w$-principal. Thus P is a prime ideal of D.

If P intersects S, then we can find an element $s \in P \cap S$; so $sP \subseteq (s) \subseteq P_{*_w}$. Hence P is S-$*_w$-principal. This is absurd, because P is not S-$*_w$-principal. Thus $P \cap S = \emptyset$. □

We are now ready to prove the main result in this article.

Theorem 3. *Let D be an integral domain, $*$ a star-operation on D, and S a multiplicative subset of D. Then the following statements are equivalent.*

(1) *D is an S-$*_w$-PID.*

(2) *Every nonzero prime ideal of D (disjoint from S) is S-$*_w$-principal.*

Proof. (1) ⇒ (2) This implication follows directly from the definition of S-$*_w$-PIDs.

(2) ⇒ (1) Suppose that every nonzero prime ideal of D (disjoint from S) is S-$*_w$-principal, and let $\mathcal{A}$ be the set of nonzero non-S-$*_w$-principal ideals of D. If D is not an S-$*_w$-PID, then $\mathcal{A}$ is a nonempty set. Also, note that $\mathcal{A}$ is partially ordered under inclusion. Let $\{I_\alpha\}_{\alpha\in\Lambda}$ be a chain in $\mathcal{A}$, and set $I = \bigcup_{\alpha\in\Lambda} I_\alpha$. Then I is a nonzero ideal of D. If I is S-$*_w$-principal, then there exist an element $s \in S$ and a principal ideal (c) of D such that $sI \subseteq (c) \subseteq I_{*_w}$; so by Lemma 1(2), $(c) \subseteq (I_\beta)_{*_w}$ for some $\beta \in \Lambda$. Therefore $sI_\beta \subseteq (c) \subseteq (I_\beta)_{*_w}$. However, this is impossible, because I_β is not S-$*_w$-principal. Hence I is not

S-$*_w$-principal. Note that I is an upper bound of the chain $\{I_\alpha\}_{\alpha\in\Lambda}$; so Zorn's lemma guarantees the existence of a maximal element. Let P be a maximal element in $\mathcal{A}$. By Lemma 2, P is a (nonzero) prime ideal of D (disjoint from S), which is absurd. Thus D is an S-$*_w$-PID. □

Corollary 4. ([1, Proposition 16]) *Let D be an integral domain and S a multiplicative subset of D. Then D is an S-PID if and only if every* (*nonzero*) *prime ideal of D is S-principal.*

Proof. This equivalence is an immediate consequence of Theorem 3 by taking $* = d$. □

Corollary 5. (cf. [4, Theorem 3.2]) *Let D be an integral domain and S a multiplicative subset of D. Then D is an S-factorial domain if and only if every nonzero prime ideal of D is S-w-principal.*

Proof. This equivalence follows directly from Theorem 3 by applying $* = v$. □

Corollary 6. ([3, Section 1.1, Exercise 10]) *Let D be an integral domain. Then D is a PID if and only if every* (*nonzero*) *prime ideal of D is principal.*

Proof. Let S be the set of units in D. By applying $* = d$, the equivalence is an immediate consequence of Theorem 3. □

Let D be an integral domain. It was shown that D is a UFD if and only if every w-ideal of D is principal (cf. [2, pages 284-285]).

Corollary 7. *Let D be an integral domain. Then D is a UFD if and only if for any nonzero prime ideal P of D, P_w is principal.*

Proof. Let S be the set of units in D. By applying $* = v$ to Theorem 3, we obtain the desired equivalence. □

Let D be an integral domain. It is known that D is a PID if and only if every countably generated ideal of D is principal. We end this article with the following question.

Question 8. *Let D be an integral domain, $*$ a star-operation on D, and S a multiplicative subset of D. Is it true that D is an S-$*_w$-PID if and only if every nonzero countably generated ideal of D is S-$*_w$-principal?*

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