East Asian Math. J.
Vol. 34 (2018), No. 5, pp. 549–570
http://dx.doi.org/10.7858/eamj.2018.035

PARAMETRIZED PERTURBATION RESULTS ON GLOBAL POSITIVE SOLUTIONS FOR ELLIPTIC EQUATIONS INVOLVING CRITICAL SOBOLEV-HARDY EXPONENTS AND HARDY TEREMS

Wan Se Kim

Abstract. We establish existence and bifurcation of global positive solutions for parametrized nonhomogeneous elliptic equations involving critical Sobolev-Hardy exponents and Hardy terms. The main approach to the problem is the variational method.

1. Introduction

In this paper, we are concerned with the multiple existence and bifurcation of global positive solutions of the following nonhomogeneous problem:

$$(P_\nu) \qquad \begin{cases} -\Delta u - \mu \dfrac{u}{|x|^2} = |u|^{2^*-2}u + \nu f \quad \text{in } \mathbb{R}^N, \\ u \in H \quad \text{in } \mathbb{R}^N, \end{cases}$$

where $\nu \in \mathbb{R}^+$, $f \in H^{-1}$, $f \geq 0$ and $f \not\equiv 0$ in $\mathbb{R}^N$.

Let $N \geq 3$, $0 \leq s < 2$, $2^*(S) := 2(N-s)/(N-2)$, and $2^* = 2^*(0)$. We put $||u||^p = \int_{\mathbb{R}^N} |u|^p dx$, $||u||_\infty = \text{ess sup}_{x\in\Omega}|u(x)|$. The space $D^{1,2}(\mathbb{R}^N) := \{u \in L^{2^*}(\mathbb{R}^N); \nabla u \in L^2(\mathbb{R}^N)\}$ with inner product $(u,v) = \int_{\mathbb{R}^N}(\nabla u \cdot \nabla v)dx$ and the corresponding norm $\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{1/2}$ is a Hilbert space. The space $H := H_0^1(\mathbb{R}^N)$ is the closure of $C_0^\infty(\mathbb{R}^N)$ by $(\cdot,\cdot)$.

By the Sobolev-Hardy inequality(see. [8]):

$$\frac{(N-2)^2}{4}\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2}dx \leq \int_{\mathbb{R}^N} |\nabla u|^2 dx \quad \text{for all} \quad u \in D^{1,2}(\mathbb{R}^N).$$

Received December 11, 2017; Accepted January 30, 2018.
2010 *Mathematics Subject Classification.* 34B16, 34B18, 34C23, 35J20.
Key words and phrases. elliptic equations; critical Sobolev exponents, Hardy term, bifurcation, multiplicity, parametrized nonhomogeneous elliptic problem.
This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(2018R1D1A1B07047804).

(pISSN 1226-6973, eISSN 2287-2833)

We note that H is a Hilbert space with the equvalent norm(cf. [9], [10]):

$$||u|| := \left[\int_{\mathbb{R}^N} \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2}\right) dx\right]^{1/2},$$

where $0 \le \mu < \bar{\mu} := (N-2)^2/4$; $\bar{\mu}$ is the best Sobolev-Hardy constant. By H^{-1}, we denote its dual with norn $||\cdot||_*$ and by $<,>$ the pairing of H.

It is known that the following Sobolev-Hardy inequality in [8] and [10]:Assume that $0 \le s \le 2,\ 2 \le r \le 2^*(s)$, then there exist a constant $C > 0$ such that

$$C\left(\int_{\mathbb{R}^N} \frac{|u|^r}{|x|^s}\right)^{2/r} \le ||u||^2,\ \forall u \in H. \tag{1.1}$$

Let $A_{s,r}$ to denote the best Sobolev-Hardy constant, i.e., the largest constant C satisfying the above inequality, that is,

$$A_{s,r} := \inf_{0 \ne u \in H} \frac{\int_{\mathbb{R}^N} \left(|\nabla u|^2 - \mu |u|^2/|x|^2\right) dx}{\left[\int_{\mathbb{R}^N} |u|^r/|x|^s dx\right]^{2/r}}$$

In the important Sobolev-Hardy critical case where $r = 2^*(s)$, we shall simply denote $A_{s,2^*(s)}$ as A_s.

Remark 1. We note the case: $s = 0$ i.e., $A_0 = A_{0,2^*}$. Usually, we denote

$$S := \inf_{0 \ne u \in D^{1,2}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{[\int_{\mathbb{R}^N} |u|^{2^*}]^{2/2^*}}$$

and since the above norm $||\cdot||$ and the usual morm are equivalent in $D^{1,2}(\mathbb{R}^N)$, we may assume that A_0 by some contant works as S, so we may assume $A_0 = S$.

In [10], we see that for $\epsilon > 0,\ 0 \le s < 2$ and $\beta = \sqrt{\bar{\mu} - \mu}$, the function

$$\omega_{\epsilon,s}(x) := \frac{\left[\frac{2\epsilon\beta^2(N-s)}{\sqrt{\bar{\mu}}}\right]^{\sqrt{\bar{\mu}}/(2-s)}}{\left[|x|^{\sqrt{\bar{\mu}}-\beta}\left(\epsilon + |x|^{(2-s)\beta/\sqrt{\bar{\mu}}}\right)^{(N-2)/(2-s)}\right]},\ 0 \le \mu < \bar{\mu}.$$

solve the equation

$$-\Delta u - \mu \frac{u}{|x|^2} = \frac{|u|^{2^*(s)-2}}{|x|^s} u \ \text{ in } \ \mathbb{R}^N \setminus \{0\} \tag{1.2}$$

and satisfy

$$||\omega_{\epsilon,s}||^2 = \int_{\mathbb{R}^N} \frac{|\omega_{\epsilon,s}|^{2^*(s)}}{|x|^s} = A_s^{(N-s)/(2-s)}. \tag{1.3}$$

Moreover, A_s is attained by $\omega_{\epsilon,s}$ only on $\mathbb{R}^N$.
where $\nu \in R^+$, $f \in H^{-1}(\mathbb{R}^N)$, $f \ge 0$ and $f \not\equiv 0$ in $\mathbb{R}^N$.

Our attempt to show multiplicity of positive solutions for problem (P_μ) relies on the Ekeland's variational principle in [6] and the Mountain Pass Theorem in [1].

Since our problem (P_ν) posesses the critical nonlinearity and the embedding $H(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ is not compact, in taking the opportunity of variational structure of problem, the (PS) condition is no longer valid and so the Mountain Pass Theorem in [1] could not be applied directly. However, we can use the Mountain Pass Theorem *without* the (PS) condition in [4] to get some $(PS)_c$ sequence of the variational functional for the second solution with $c > 0$.

For convenience, we omit "$\mathbb{R}^N$" and "dx" in integration and, throughtout this paper, we will use the letter C to denote the natural various constants independent of u. From now on, we put $p = 2^*$.

2. Existence of minimal positive solutions

As a consequence of Hardy inequality, it is ease to see:

Lemma 2.1. *The operator* $-\Delta - \mu\frac{u}{|x|^2}$ *is positive, has discrete spectrum and has the maximum principle in* H.

Proof. See [10] and [12]. ∎

In order to get the existence of positive solutions of (P_ν), we consider the energy functional I_ν of the problem (P_ν) defined by

$$I_\nu(u) := \frac{1}{2}\int\left(|\nabla u|^2 - \mu\frac{|u|^2}{|x|^2}\right) - \frac{1}{p}\int\left(u^+\right)^p - \nu\int fu, \quad \text{for} \quad u \in H.$$

First, we study the existence of the first solution for the problem (P_ν) by finding a local mininum for energy functional I_ν . We denote

$$C_N^* := \frac{1}{2}\left(\frac{N}{N+2}\right)^{(N-2)/4}\left(\frac{4}{N+2}\right)A_0^{(N-2)/4}. \tag{2.1}$$

Lemma 2.2. *Assume* $f \in H^{-1}$, $f(x) \geq 0$, $f(x) \not\equiv 0$ *and* $||\nu f||_* \leq C_N^*$, *then there exits a positive constant* $R_0 > 0$ *such that* $I_\nu(u) \geq 0$ *for any* $u \in \partial\bar{B}_{R_0} = \{u \in H : ||u|| = R_0\}$.

Proof. We consider the function $h(t) : [0, +\infty) \to R$ defined by

$$h(t) = \frac{1}{2}t - \frac{1}{p}A_0^{-p/2}t^{p-1}.$$

Note that $h(0) = 0$, $p > 2$ and $h(t) \to -\infty$ as t $\to \infty$. We can show easly there a unique $t_0 > 0$ achieving the maxinum of $h(t)$ at t_0. Since

$$h'(t_0) = \frac{1}{2} - \frac{p-1}{p}A_0^{-p/2}t_0^{p-2} = 0,$$

we have

$$t_0 = \left(\frac{p}{2(p-1)}\right)^{1/(p-2)} A_0^{p/2(p-2)}.$$

Hence, we have

$$h(t_0) = \frac{1}{2}\left(\frac{N}{N+2}\right)^{(N-2)/4}\left(\frac{4}{N+2}\right) A_0^{(N-2)/4}. \tag{2.2}$$

Taking $R_0 = t_0$, for $u \in \partial \bar{B}_{R_0}$,

$$\begin{aligned} I_\nu(u) &= \frac{1}{2}\int \left(|\nabla u|^2 - \mu\frac{|u|^2}{|x|^2}\right) - \frac{1}{p}\int (u^+)^p - \nu \int fu \\ &\geq \frac{1}{2}||u||^2 - \frac{1}{p}A_0^{-p/2}||u||^p - ||\nu f||_*||u|| \\ &= t_0\left[h(t_0) - ||\nu f||_*\right] \end{aligned} \tag{2.3}$$

From (2.2) and (2.3), we have $I_\nu(u)|_{\partial \bar{B}_{R_0}} \geq 0$. This completes the proof. ∎

Proposition 2.3. *Assume $f \in H^{-1}$, $f(x) \geq 0$, $f(x) \not\equiv 0$ in $\mathbb{R}^N$ and $||\nu f||_* \leq C_N^*$, then problem (P_ν) has at least one positive solution u_ν such that*

$$I_\nu(u_\nu) := c_1 = \inf\{I_\nu : u \in \bar{B}_{R_0}\}, \tag{2.4}$$

where $\bar{B}_{R_0} = \{u \in H : ||u|| \leq R_0\}$.

Proof. By Sobolev inequality, the generalized Hölder and Young's inequality with $\epsilon > 0$, there exists $C_\epsilon > 0$, we have

$$\begin{aligned} I_\nu(u) &= \frac{1}{2}\int \left(|\nabla u|^2 - \nu\frac{|u|^2}{|x|^2}\right) - \frac{1}{p}\int (u^+)^p - \nu \int fu \\ &\geq \frac{1}{2}||u||^2 - \frac{1}{p}S^{-p/2}||u||^p - ||\nu f||_*||u|| \\ &\geq \left(\frac{1}{2} - \epsilon\right)||u||^2 - \frac{1}{p}S^{-p/2}||u||^p - C_\epsilon||\nu f||_*^2. \end{aligned}$$

Taking $\epsilon < \frac{1}{2}$, then, for $R_0 = t_0$ as in Lemma 2,2, we can find a $C_{R_0} > 0$ small enough such that

$$I_\nu(u)|_{\partial B_{R_0}} \geq C_{R_0} \text{ for } ||\nu f||_* \leq C_N^*. \tag{2.5}$$

Since there exists a $\tilde{C}_{R_0} > 0$ such that $|I_\nu(u)| \leq \tilde{C}_{R_0}$ for all $u \in \bar{B}_{R_0}$ and $\bar{B}_{R_0}$ is a complete metric space with respect to the metric $d(u,v) = ||u-v||, u, v \in \bar{B}_{R_0}$, by using the Ekeland's variational principle, from (2.5), we can prove that there exists a sequence $\{u_n\} \subset \bar{B}_{R_0}$ and $u_\nu \in \bar{B}_{R_0}$ such that

$$I_\nu(u_n) \to c_1, \tag{2.6}$$

$$I'_\nu(u_n) \to 0, \tag{2.7}$$

$$u_n \to u_\nu \text{ weakly in } H, \tag{2.8}$$

$$u_n \to u_\nu \quad \text{a.e.} \quad \text{in} \quad \mathbb{R}^N,$$
$$\nabla u_n \to \nabla u_\nu \quad \text{a.e.} \quad \text{in} \quad \mathbb{R}^N$$

and

$$u_n{}^{p-1} \to u_\nu{}^{p-1} \quad \text{weakly} \quad \text{in} \quad \left(L^p(\mathbb{R}^N)\right)^* \quad \text{as } n \to \infty.$$

Therefore, u_ν is a weak solution of (P_ν). Hence,

$$\langle I'_\nu(u_\nu), \varphi\rangle = 0 \quad \forall \varphi \in H. \tag{2.9}$$

Moreover, by Lemma 2.1, u_ν is positive on $\mathbb{R}^N$, where I'_ν is the Frėchlet derivative of I_ν.

Next, we are going to prove (2.4). In fact, by the definition of c_1, we know that $I_\nu(u_\nu) \geq c_1$ since $u_\nu \in \bar{B}_{R_0}$, that is,

$$I_\nu(u_\nu) = \frac{1}{2}\int \left(|\nabla u_\nu|^2 - \mu\frac{|u_\nu|^2}{|x|^2}\right) - \frac{1}{p}\int |u_\nu|^p - \nu\int f u_\nu \geq c_1 \tag{2.10}$$

By (2.9) and (2.10), we have

$$\left(\frac{1}{2} - \frac{1}{p}\right)\int \left(|\nabla u_\mu|^2 - \mu\frac{|u_\nu|^2}{|x|^2}\right) - \left(1 - \frac{1}{p}\right)\nu\int f u_\nu \geq c_1 \tag{2.11}$$

On the other hand, by (2.6) - (2.8) and Fatou's lemma, we get

$$\begin{aligned} c_1 &= \liminf_n \left(\frac{1}{2} - \frac{1}{p}\right)\int \left(|\nabla u_n|^2 - \mu\frac{|u_n|^2}{|x|^2}\right) - \limsup_n \left(1 - \frac{1}{p}\right)\nu\int f u_n \\ &\geq \left(\frac{1}{2} - \frac{1}{p}\right)\int \left(|\nabla u_\nu|^2 - \mu\frac{|u_\nu|^2}{|x|^2}\right) - \left(1 - \frac{1}{p}\right)\nu\int f u_\nu. \end{aligned} \tag{2.12}$$

Thus, (2.10) and (2.12) imply (2.4) holds. This completes the proof. ■

Remark 2. (i) $c_1 < 0$, (ii) c_1 is bounded below, (iii) $||u_\nu|| = o(1)$ as $\nu \to 0^+$.

Indeed: (i) For $t > 0$ and $\varphi > 0$, we have

$$I_\nu(t\varphi) = \frac{t^2}{2}\int \left(|\nabla \varphi|^2 - \mu\frac{|\varphi|^2}{|x|^2}\right) - \frac{t^p}{p}\int |\varphi|^p - t\nu\int f\varphi \leq \frac{t^2}{2}||\varphi||^2 - t\nu\int f\varphi.$$

By taking $t > 0$ sufficiently small, we can see $c_1 < 0$.

(ii) By (2.9) with $\varphi = u_\nu$, and $c_1 = I_\nu(u_\nu)$, we have

$$\begin{aligned} c_1 &= \left(\frac{1}{2} - \frac{1}{p}\right)\int \left(|\nabla u_\nu|^2 - \mu\frac{|u_\nu|^2}{|x|^2}\right) - \left(1 - \frac{1}{p}\right)\nu\int f u_\nu \\ &\geq \left(\frac{1}{2} - \frac{1}{p}\right)||u_\nu||^2 - \left(1 - \frac{1}{p}\right)||\nu f||_*||u_\nu|| \\ &\geq -\frac{1}{2p}\left[\frac{(p-1)^2}{p-2}\right]||\nu f||_*^2 \end{aligned} \tag{2.13}$$

by Young's inequality.

(iii) Since $c_1 < 0$, from (2.13), we see that $||u_\nu|| \to 0$ as $\nu \to 0^+$. Hence, $||u_\nu|| = o(1)$ as $\nu \to 0^+$. We also have that $\{u_\nu\}$ is uniformly bounded with respect to ν. We will restate results relating to this remark in Proposition 3.4 more precisely.

Proposition 2.4. *Problem* (P_ν) *possesses at least one minimal positive solution of* (P_ν).

Proof. Let $\mathscr{N}$ be the Nehari manifold (cf. [15]):

$$\mathscr{N} := \left\{ u \in H : \int \left(|\nabla u|^2 - \mu \frac{|u|^2}{|x|^2} \right) = \int |u|^p + \int \nu f u \right\} \setminus \{0\} .$$

Note that $||\nu f||_* \ll 1$ for ν small enough and for each $u \in H \setminus \{0\}$, there exists a unique $t_u > 0$ such that

$$t_u^2 \int \left(|\nabla u|^2 - \mu \frac{|u|^2}{|x|^2} \right) - t_u^p \int |u|^p - t_u \int \nu f u = 0$$

and $I_\nu(t_u u) > 0$. Then

$$\mathscr{N} = \{ t_u u \ : \ u \in H \setminus \{0\} \}$$

and

$$\mathscr{N} \cong S^{N-1} = \{ u \in H \ : \ ||u|| = 1 \} .$$

Hence,

$$H = H_1 \cup H_2 \cup \mathscr{N}, \quad H_1 \cap H_2 = \phi \text{ and } 0 \in H_1,$$

where

$$\begin{aligned} H_1 &= \{ tu \ : \ u \in H \setminus \{0\} , \ t \in [0, t_u[\} \\ H_2 &= \{ tu \ : \ u \in H \setminus \{0\} , \ t > t_u \} . \end{aligned}$$

This implies that for $t > 0$ with $t < t_u, \ tu \in H_1$.

Here, we need to switch our view point, by associating with v a mapping

$$v : [0, \infty[\to H$$

defined by

$$(v(t))x = v(x,t), \ x \in \mathbb{R}^N, \ t \in [0, \infty[.$$

In other words, we consider v not as a function of x and t together, but rather as a mapping v of t into the space H of a function of x.

We have, for any $v_0 \in H_1$, the solution v of the initial value problem:

$$\begin{cases} \dfrac{dv}{dt} - \Delta v - \mu \dfrac{v}{|x|^2} = & v^{p-1} + \nu f(x) \text{ in } \mathbb{R}^N \\ v(0) = v_0, \end{cases}$$

converges to u_ν as $t \to \infty$,

Indeed, in the proof of Proposition 2.3, we know that $I_\nu(v(t))$ is decreasing and $\lim_{t \to \infty} I_\nu(v(t)) = I_\nu(u_\nu)$, where $I_\nu(u_\nu)$ is the local minimum.

Since

$$\begin{aligned} I_\nu(v(t)) - I_\nu(v(s)) &= \int_s^t \frac{d}{dt} I_\nu(v(t))dt \\ &= \int_s^t \left\langle \frac{d}{dt}v, \nabla I_\nu(v(t)) \right\rangle dt \\ &= -\int_t^s \left\| \frac{d}{dt}v \right\|^2 dt, \end{aligned}$$

we have, $\lim_{s,t\to\infty} \left\| \frac{d}{dt}v \right\|^2 = 0$. Thus, $v' \to 0$ a.e. in $\mathbb{R}^N$ as $t \to \infty$ and hence, $\langle I'_\nu(v), \varphi\rangle \to 0$, $\forall \varphi \in C^\infty(\mathbb{R}^N)$. Therefore, we have $v \to u_\nu$ as $t \to \infty$, since $I_\nu(v(t))$ is decreasing and converges to the local minimum $I_\nu(u_\nu)$.
Now, let $v_0 = tu$, where $t \in]0,1[$ and u is a positive solution. Then $u \in \mathscr{N}$ and $v_0 \in H_1$. Since $v_0 \le u$ and the solution v converges u_ν as $t \to \infty$, by the order preserving principle, $u_\nu \le u$. This completes the proof. ∎

Proposition 2.5. *Suppose that $f \in H^{-1}$, $f \ge 0$, $f \not\equiv 0$ in $\mathbb{R}^N$ and $||\nu f||_* \le C_N^*$. Then there exist $\tilde{\nu} \ge \bar{\nu} > 0$ such that (P_ν) possesses a positive solution for $0 < \nu \le \bar{\nu}$ and no positive solution for $\nu > \bar{\nu}$.*

Proof. By Proposition 2.3, (P_ν) has a positive solution if $\nu \le C_N^*/||f||_*$. Suppose (P_ν) has a positive solution for some $\nu = \bar{\nu}$. We will show that (P_ν) has a positive solution for any $0 < \nu \le \bar{\nu}$. For fixed $0 < \nu < \bar{\nu}$, using the Lax-Milgram Theorem, we construct a positive sequence $\{u_n\}$ as following;

Let

$$-\Delta u_1 - \mu \frac{u_1}{|x|^2} = \nu f \quad \text{in} \quad \mathbb{R}^N,$$

and

$$-\Delta u_n - \mu \frac{u_n}{|x|^2} = u_{n-1}^{p-1} + \nu f \quad \text{for} \quad n \ge 2. \tag{2.14}$$

Then, by the maximum principle, we have $0 < u_n < u_{n+1} < \cdots < \bar{u}$ for $n \ge 1$. And $||u_1|| \le \nu ||f||_*$. Multiplying (2.14) by u_n, we have $||u_n|| \le A^{-p/2}||\bar{u}||^{p-1} + \nu||f||_*$.

Therefore, there exists u in H such that

$$u_n \to u \text{ weakly in } H \text{ as } n \to \infty,$$

$$u_n \to u \text{ a.e. in } \mathbb{R}^N \text{ as } n \to \infty,$$

$$\nabla u_n \to \nabla u \text{ a.e. in } \mathrm{\mathbb{R}^N},$$

$$u_n^{p-1} \to u^{p-1} \text{ weakly in } \left(L^p(\mathbb{R}^N)\right)^* \text{ as } n \to \infty.$$

Thus, u is a positive solution of (P_ν).

Next, let u be a positive solution of (P_ν). Then, for any $\epsilon > 0$, multiplying (P_ν) by $\omega_{\epsilon,s}$, we have

$$-\int \Delta u \cdot \omega_{\epsilon,s} - \mu \frac{u}{|x|^2}\omega_{\epsilon,s} = \int u^{p-1}\omega_{\epsilon,s} + \nu \int f(x)\omega_{\epsilon,s}. \tag{2.15}$$

By Green's formular, we have, for any $R > 1$, we have

$$\begin{aligned}\int_{\partial B_R} \Delta u \cdot \omega_{\epsilon,s} - \int_{\partial B_R} u \cdot \Delta\omega_{\epsilon,s} &= \int \left(\frac{\partial u}{\partial n} - \frac{\partial \omega_{\epsilon,s}}{\partial n}\right) dS \\ &\leq \omega_{\epsilon,s}(R) \int_{\partial B_R} |\nabla u| dS + |\nabla \omega_{\epsilon,s}|(R) \int_{\partial B_R} |u| dS \\ &\leq O\left(R^{-N+2}\right) \left(\int_{\partial B_R} |\nabla u| dS + \int_{\partial B_R} |u| dS\right).\end{aligned}$$

Hence, the right-hand side approaches 0. Therefore, we have

$$\int \Delta u \cdot \omega_{\epsilon,s} = \int u \cdot \Delta\omega_{\epsilon,s}. \tag{2.16}$$

Since $u \in H$ is a positive solution to (P_ν),

$$\int \left(-\Delta u - \mu \frac{u}{|x|^2}\right) \omega_{\epsilon,s} = \int |u|^{p-1} \omega_{\epsilon,s} + \int \nu f(x) \omega_{\epsilon,s}.$$

From (2.16), we have

$$\int \left(-\Delta \omega_{\epsilon,s} - \mu \frac{\omega_{\epsilon,s}}{|x|^2}\right) u = \int |u|^{p-1} \omega_{\epsilon,s} + \nu \int f(x) \omega_{\epsilon,s}.$$

Since $p > 2$, for any $M > 0$, there exists a constant $C > 0$ such that

$$u^{p-1} \geq Mu - C\omega_{\epsilon,s}^{p-1}, \quad \forall u > 0.$$

Hence, we have, from (2.15),

$$\int \left(-\Delta \omega_{\epsilon,s} - \mu \frac{\omega_{\epsilon,s}}{|x|^2}\right) u \geq \int \left[\left(Mu - C\omega_{\epsilon,s}^{p-1}\right) \omega_{\epsilon,s} + \nu f(x) \omega_{\epsilon,s}\right].$$

Therefore, by (1.2), we have

$$\begin{aligned}\nu \int f(x)\omega_{\epsilon,s} &\leq \int \left(-\Delta \omega_{\epsilon,s} - \mu \frac{\omega_{\epsilon,s}}{|x|^2}\right) u - M \int \omega_{\epsilon,s} u + C \int \omega_{\epsilon,s}^p \\ &\leq \int \omega_{\epsilon,s}^{p-1} u - M \int \omega_{\epsilon,s} u + C \int \omega_{\epsilon,s}^p \\ &\leq ||\omega_{\epsilon,s}||_\infty^{p-2} \int \omega_{\epsilon,s} u - M \int \omega_{\epsilon,s} u + C \int \omega_{\epsilon,s}^p.\end{aligned}$$

Taking $M = ||\omega_{\epsilon,s}||_\infty^{p-2}$, then, by (1.1), we have

$$\nu \leq \frac{C \int \omega_{\epsilon,s}^p}{\int f(x)\omega_{\epsilon,s}} < \infty.$$

Hence, there exists $\bar{\nu} > 0$ such that, by (1.3),

$$\bar{\nu} \leq \tilde{\nu} := \inf_{\epsilon > 0} \frac{C \int w_{\epsilon,s}^p}{\int f(x)\omega_{\epsilon,s}} = \inf_{\epsilon > 0} \frac{CS^{N/2}}{\int f(x)\omega_{\epsilon,s}} < \infty. \tag{2.17}$$

Therefore, if $\nu > \bar{\nu}$, then (P_ν) has no solution and this completes the proof. ∎

3. Multiplicity of positive solutions

From now on, we assume that $f \in H^{-1}$, $f \geq 0$, $f \not\equiv 0$ in $\mathbb{R}^N$ and f satisfies $||\nu f||_* \leq C_N^*$.

We set

$$\nu^* := \sup\{\nu \in \mathbb{R}^+ : (P_\nu) \text{ has at least one positive solution in } H\}.$$

Then, by Proposition 2.5, we have $0 < \bar{\nu} \leq \nu^* < \infty$.

Remark. The minimal solution u_ν of (P_ν) is increasing with respect to ν. Indeed, suppose $\nu^* > \nu > \eta$. Since

$$-\Delta u_\nu - \mu \frac{u_\nu}{|x|^2} - u_\nu^{p-1} - \eta f(x) = (\nu - \eta) f \geq 0,$$

$u_\nu > 0$ is a supersolution of (P_η). Since $f(x) \geq 0$ and $f(x) \not\equiv 0$, $u \equiv 0$ is a subsolution of (P_η) for $\eta > 0$. By the standard barrier method, we can obtain a solution u_η of (P_η) such that $0 \leq u_\eta \leq u_\nu$ on $\mathbb{R}^N$. We note that 0 is not a solution of (P_η), $\nu > \eta$ and u_η is a minimal solution of (P_η). Therefore, because u_η also can be derived by an iteration scheme with initial value $u_{(0)} = 0$, by the maximal principle, $0 < u_\eta < u_\nu$ in $\mathbb{R}^N$ which completes the proof. ∎

Now, consider the corresponding eigenvalue problem:

$$(3.1)_\nu \qquad \begin{cases} -\Delta \varphi - \mu \dfrac{\varphi}{|x|^2} = \lambda(\nu)(p-1) u_\nu^{p-2} \varphi \text{ in } \mathbb{R}^N, \\ \varphi \text{ in } H. \end{cases}$$

Let λ_1 be the first eigenvalue of $(3.1)_\nu$; i.e.,

$$\lambda_1 = \lambda_1(\nu) := \inf\{\int \left(|\nabla \varphi|^2 - \mu \frac{|\varphi|^2}{|x|^2} \right) : \varphi \in H, (p-1) \int u_\nu^{p-2} \varphi^2 dx = 1\}.$$

Then, $0 < \lambda_1 < \infty$ and we can achieve the minimum by some function $\varphi_1 = \varphi_1(\nu) \in H$ and $\varphi_1 > 0$ in Ω if $\nu \in]0, \nu^*[$ (*cf.* [17]).

We summarize basic properties for $\lambda_1(\nu)$:

Lemma 3.1. *(i) For* $\nu \in]0, \nu^*[$, $\lambda_1(\nu) > 1$,
(ii) If $0 < \eta < \nu \leq \nu^*$, *then* $\lambda_1(\nu) < \lambda_1(\eta)$,
(iii) $\lambda_1(\nu) \to +\infty$ *as* $\nu \to 0^+$.

Proof. (i) For given $0 < \eta < \nu \leq \nu^*$, every solution u_ν of (P_ν) with $\nu \in]0, \nu^*[$ is a supersolution of (P_ν). By Taylor expansion, we have

$$\begin{aligned} -\Delta(u_\nu - u_\eta) - \mu \frac{1}{|x|^2}(u_\nu - u_\eta) &= \left(u_\nu^{p-1} - u_\eta^{p-1}\right) + (\nu - \eta) f \\ &> (p-1) u_\eta^{p-2}(u_\nu - u_\eta) \end{aligned}$$

and moreover, we get

$$\int \nabla(u_\nu - u_\mu)\nabla\varphi_1 - \mu\int \frac{(u_\nu - u_\eta)}{|x|^2}\varphi_1 = \int \left(u_\nu^{p-1} - u_\eta^{p-1}\right)\varphi_1 + \int(\nu-\eta)f\varphi_1$$
$$> (p-1)\int u_\eta^{p-2}(u_\nu - u_\eta)\varphi_1.$$

Therefore, from $(3.1)_\nu$, we have

$$\int \nabla(u_\nu - u_\eta)\nabla\varphi_1 - \mu\int \frac{(u_\nu - u_\eta)}{|x|^2}\varphi_1 = \lambda_1(\nu)(p-1)\int u_\eta^{p-2}(u_\nu - u_\eta)\varphi_1,$$

which implies $\lambda_1(\nu) > 1$.

(ii) Since, for $0 < \eta < \nu \leq \nu^*$, $u_\eta < u_\nu$ and

$$\lambda_1(\eta)(p-1)\int u_\eta^{p-2}\varphi_1(\eta)\varphi_1(\nu) = \int \left(\nabla\varphi_1(\eta) - \mu\frac{\varphi_1(\eta)}{|x|^2}\right)\varphi_1(\nu)$$
$$= \lambda_1(\nu)(p-1)\int u_\nu^{p-2}\varphi_1(\nu)\varphi_1(\eta),$$

we have $\lambda_1(\eta) > \lambda_1(\nu)$.

(iii) First, we show that $||u_\nu|| \to 0$ as $\nu \to 0^+$. Let $\varphi = u_\nu$, Multiplying (P_ν) by u_ν, we have,

$$\int \left(|\nabla u_\nu|^2 - \mu\frac{|u_\nu|^2}{|x|^2}\right) = \int u_\nu^p + \nu\int f u_\nu$$

and hence, for $\epsilon > 0$, we have, by Young's inequality with ϵ,

$$\left(1 - \frac{1}{\lambda_1(p-1)} - \frac{\epsilon}{2}\right)||u_\nu||^2 \leq C_\epsilon \nu^2 ||f||_*^2 \text{ for } \epsilon > 0.$$

Thus, for $\epsilon > 0$ small, we have $||u_\nu||^2 \leq C_\epsilon \nu^2$ for some constant $C_\epsilon > 0$, and hence, $||u_\nu|| = o(1)$ as $\nu \to 0^+$.

Next, Multiplying $(3.1)_\nu$ by φ_1,

we have,

$$||\varphi_1||^2 = \lambda_1(\nu)(p-1)\int u_\nu^{p-2}\varphi_1^2$$
$$\leq \lambda_1(\nu)(p-1)\left(\int |u_\nu|^p\right)^{(p-2)/p}\left(\int \varphi_1^p\right)^{2/p}$$
$$\leq \lambda_1(p-1)A_0^{-p/2}||u_\nu||^{p-2}\left(\int |\nabla\varphi_1|^2 - \mu\frac{|\varphi_1|^2}{|x^2|}\right) \text{ for some } C > 0$$

and thus, $0 < A_0^{p/2} \leq \lambda_1(\nu)(p-1)||u_\nu||^{p-2}$. Therefore, from (iii), we have the desired result. This completes the proof. ∎

Lemma 3.2. *Let u_ν be a positive solution of* $(1.3)_\nu$ *for which* $\lambda_1(\nu) > 1$. *Then, for any* $g \in H$, *the problem:*

$$-\Delta w - \mu \frac{w}{|x|^2} = (p-1)u_\nu^{p-2} w + g(x), \quad w \in H \tag{3.2}$$

has a solution.

Proof. Consider the functional defined by

$$J(w) = \frac{1}{2}\int \left(|\nabla w|^2 - \nu \frac{|w|^2}{|x|^2}\right) - \frac{1}{2}(p-1)\int u_\nu^{p-2}w^2 - \int gw, \quad w \in H.$$

From Hölder's inequality and Young's inequality, we have, for any $\epsilon > 0$,

$$\begin{aligned} J(w) &\geq \left(\frac{1}{2} - \frac{1}{2\lambda_1(\nu)}\right)||w||^2 - \frac{\epsilon}{2}||w||^2 - C_\epsilon ||g||_*^2 \\ &= \left(\frac{1}{2} - \frac{1}{2\lambda_1(\nu)} - \frac{\epsilon}{2}\right)||w||^2 - C_\epsilon ||g||_*^2 \end{aligned}$$

and hence, for small $\epsilon > 0$, there exist $C_{1,\epsilon} > 0$ and $C_{2,\epsilon} > 0$ such that

$$J(w) \geq C_{1,\epsilon}||w||^2 - C_{2,\epsilon}||g||_*^2. \tag{3.3}$$

Let $\{w_n\} \subset H$ be the minimizing sequence of $J(\cdot)$. From (3.3), we have $\{w_n\}$ is bounded in H. Hence, passing subsequence, we may have that there exists $w \in H$ such that

$$w_n \to w \text{ weakly in } H \text{ as } n \to \infty,$$
$$w_n \to w \text{ a.e. in } \text{ as } n \to \infty$$

Here, we also note that

$$\nabla w_n \to \nabla w \text{ a.e. in } \mathbb{R}^N \text{ as } n \to \infty.$$

And

$$u_n^{p-1} \to \tilde{u}^{p-1} \text{ weakly in } (L^p(\mathbb{R}^N))^* \text{ as } n \to \infty.$$

By Fatou's Lemma

$$||w||^2 \leq \liminf_{n\to\infty} ||w_n||^2.$$

Since $\{w_n\}$ is bounded in H, from (1.1), $\int u_\nu^{p-2}w_n^2 < \infty$ for $n \geq 1$ imply

$$\lim_{n\to\infty}\int gw_n = \int gw, \ \lim_{n\to\infty}\int u_\nu^{p-2}w_n^2 = \int u_\nu^{p-2}w^2$$

and hence,

$$J(w) \leq \lim_{n\to\infty} J(w_n) = d.$$

Then, $J(w) = d$ and w is a minimizer of J. Therefore, w is a critical point of J and w is a solution of (3.2). This completes the proof. ∎

Proposition 3.3. *For* $\nu = \nu^*$, *the problem* (P_ν) *has a positive solution* u_{ν^*} *and* $\lambda_1(\nu^*) = 1$. *Moreover, the solution* u_{ν^*} *is unique in* H.

Proof. For $\nu \in]0,\nu^*[$, multiplying (P_ν) by u_ν, we have, by $(3.1)_\nu$,

$$\begin{aligned}\int\left(|\nabla u_\nu|^2-\mu\frac{|u_\nu|^2}{|x|^2}\right)&=\int u_\nu^{2^*}+\nu\int fu_\nu\\&\le\frac{1}{\lambda_1(\nu)(p-1)}\int\left(|\nabla u_\nu|^2-\mu\frac{|u_\nu|^2}{|x|^2}\right)+\nu^*||f||_*||u_\nu||\\&=\left(\frac{1}{\lambda_1(\nu)(p-1)}+\frac{\epsilon\nu^*}{2}\right)||u_\nu||^2+\frac{\nu^*}{2\epsilon}||f||_*^2.\end{aligned}$$

By taking $\epsilon > 0$ small enough, there exists an constant $C_\epsilon > 0$ such that $||u_\nu|| \le C_\epsilon$ for all $\nu \in]0,\nu^*[$. Then, there exists u_{ν^*} in H such that u_ν monotonically increasing to u_{ν^*} as $\nu \to \nu^*$ and $u_\nu \to u_{\nu^*}$ weakly in H as $\nu \to \nu^*$. Hence, u_{ν^*} is a positive solution of (P_ν) with $\nu = \nu^*$. We note that $\lambda_1(\nu)$ is a continuous function of $\nu \in]0,\nu^*]$.

Define $F : \mathbb{R}^1 \times H \to H^{-1}$ by

$$F(\nu,u) := \Delta u + \mu\frac{u}{|x|^2} + (u^+)^{p-1} + \nu f(x) \ \text{ in } \ \mathbb{R}^N \setminus \{0\}.$$

Since $u_\nu \to u_{\nu*}$ weakly as $\nu \to \nu^*$, from Lemma 3.1, $\lambda(\nu^*) \ge 1$. If $\lambda_1(\nu^*) > 1$, then $F_u(\nu^*,u_{\nu^*})\varphi = \Delta\varphi + \mu\frac{\varphi}{|x|^2} + (p-1)u_{\nu^*}^{p-2}\varphi = 0$ has no nontrivial solution. From Lemma 3.2, $F(\nu^*,u_{\nu^*})$ is an isomorphism of $\mathbb{R}^1 \times H$ onto H^{-1}, and by the implicitly function theorem to F, we find a neighborhood $]\nu^*-\delta,\ \nu^*+\delta[$ of ν^* such that (P_ν) possesses a positive solution if $\nu \in]\nu^*-\delta,\ \nu^*+\delta[$, which contradicts the definition of ν^*. Therefore, $\lambda_1(\mu^*) = 1$.

Suppose v_{ν^*} is a positive solution of (P_{ν^*}). Then $v_{\nu*} \ge u_{\nu^*}$ since u_{ν^*} is minimal. Let $w = v_{\nu^*} - u_{\nu^*}$. Then, since $\lambda_1(\nu^*) = 1$, we have

$$-\Delta w - \mu\frac{w}{|x|^2} \ge (p-1)u_{\nu^*}^{p-2}w.$$

Since $\varphi_1 = \varphi_1(\nu^*)$ is the eigenfunction of the problem $(3,1)_{\nu^*}$, we have,

$$(p-1)\int u_{\nu^*}^{p-2}\varphi_1 w = \int \nabla w\nabla\varphi_1 - \mu\int w\frac{\varphi_1}{|x|^2} \ge (p-1)\int u_{\nu^*}^{p-1}w\varphi_1$$

and hence, $w \equiv 0$. This completes the proof. ∎

Proposition 3.4. *The minimal solution u_ν of (P_ν) increasing continuously to u_{ν^*} as $\nu \to \nu^*$ and uniformly bounded in H for all $\mu \in]0,\nu^*]$. Moreover, $||u_\nu|| \le O(\nu^2)$ as $\nu \to 0^+$.*

Proof. It suffices to find the uniform bound of u_ν. Multiplying (P_ν) by u_ν, we have

$$\int\left(|\nabla u_\nu|^2-\mu\frac{|u_\nu|^2}{|x|^2}\right)=\int u_\nu^p+\int \nu fu_\nu$$

and hence, for $\epsilon > 0$, we have

$$\left(1-\frac{1}{\lambda_1(\nu)(p-1)}-\frac{\epsilon}{2}\right)||u_\nu||^2 \le \frac{\nu^2}{2\epsilon}||f||_*^2 \ \text{ for } \epsilon > 0.$$

Therefore, for $\epsilon > 0$ small, we have $||u_\nu|| \leq C_\epsilon \nu$ for some constant $C_\epsilon > 0$. Next, fix $\tau \in]0, \nu^*]$. If ν increases to τ, then u_ν is increasing up to u_τ and $u_\nu \to u_\tau$ in H. If it is not the case, then, by multiplying u_τ on (P_ν) again, we have, Lemma 4.3 in [8],

$$||u_\nu||^2 \leq \int u_\tau^{p-1} u_\tau + \nu^* \langle f, u_\tau \rangle$$

and so

$$||u_\nu||^2 \leq S^{-p/2} ||u_\tau||^p + \nu^* ||f||_* ||u_\tau||.$$

Hence, there exists a sequence $\{u_{\nu_j}\}$ in H conversing weakly to a solution $\tilde{u}$ of (P_τ) but $\tilde{u} \neq u_\tau$. Since $\{u_{\nu_j}\}$ coverge to $\tilde{u}$ strongly in local L^1 sense, by the maximum principle, we have $u_{\nu_j} \leq \tilde{u} < u_\tau$ which leads a contradiction to the minimality of u_τ. This completes the proof. ■

Remark 3. From Proposition 3.4 , we have that $\lambda(\nu)$ is a continuously decreasing function from $[0, \nu^*]$ onto $[1, \infty[$ and $||u_\nu|| = o(1)$ as $\nu \to 0^+$.

Next, we are going to find the second solutions bigger than minimal solutions. In order to get another positive solution of (P_ν), we consider the following problem:

$$(3.4)_\nu \qquad \begin{cases} -\Delta v - \mu \dfrac{v}{|x|^2} = (v^+ + u_\nu)^{p-1} - u_\nu^{p-1} \quad \text{in} \quad \Omega, \\ v \in H, \;\; v > 0 \text{ in } \Omega \end{cases}$$

and the corresponding variational functional:

$$J_\nu(v) := \frac{1}{2} \int \left(|\nabla v|^2 - \mu \frac{|v|^2}{|x|^2} \right) - \frac{1}{p} \int \left((v^+ + u_\nu)^p - u_\nu^p - p u_\nu^{p-1} v^+ \right)$$

for $v \in H$.

Clearly, we can have another positive solution $U_\nu = u_\nu + v_\nu$ if we show the problem $(3.4)_\nu$ possesses a positive solution for $\nu \in]0, \nu^*[$. We look for a critical point of J_ν which is a weak solution of $(3.4)_\nu$ by employing standard argument of the Mountain Pass method without the (PS) condition.

In the proof of the existance second solution, we make use of some arguments in [7].

Theorem 3.5. *The problem (P_μ) possesses at least two positive solutions for all $\nu \in]0, \nu^*[$.*

Proof. (i) Let $v \in H \setminus \{0\}$, Then, for $\epsilon > 0$, by Young's inequality,

$$\begin{aligned} J_\nu(v) &= \frac{1}{2}\int\left(|\nabla v|^2 - \mu\frac{|v|^2}{|x|^2}\right)dx - \int\int_0^{v^+}\left((u_\nu+t)^{p-1} - u_\nu^{p-1}\right)dtdx \\ &\geq \frac{1}{2}\left(1-\frac{1}{\lambda_1}\right)\int\left(|\nabla v|^2 - \mu\frac{|v|^2}{|x|^2}\right)dx- \\ &\quad -\int\int_0^{v^+}\left[(u_\nu+t)^{p-1} - u_\nu^{p-1} - (p-1)u_\nu^{p-2}t\right]dtdx \\ &\geq \frac{1}{2}\left(1-\frac{1}{\lambda_1}\right)\int\left(|\nabla v|^2 - \mu\frac{|v|^2}{|x|^2}\right)dx - \int\int_0^{v^+}\left(\epsilon u_\nu^{p-2}t + C_\epsilon t^{p-1}\right)dtdx \\ &\geq \frac{1}{2}\left(1-\frac{1}{\lambda_1}\right)||v||^2 - \frac{\epsilon}{2}\int u_\nu^{p-2}\left(v^+\right)^2dx - \frac{C_\epsilon}{p}\int\left(v^+\right)^pdx \\ &\geq \frac{1}{2}\left(1-\frac{1}{\lambda_1}-\frac{\epsilon}{2(p-1)\lambda_1}\right)||v||^2 - \frac{C_\epsilon}{p}S^{-1/2}||v||^p \end{aligned}$$

for some constant $C_\epsilon > 0$. Hence, for sufficiently small $\epsilon > 0$, there exist $\rho > 0, \delta > 0$ such that

$$J_\nu(v)|_{\partial\tilde{B}_\rho} \geq \delta > 0,$$

where $\tilde{B}_\rho = \{u \in H : \ ||u|| \leq \rho\}$.

(ii) Let $v \in H, v \geq 0$ and $v \not\equiv 0$, then, for $t > 0$, we have

(3.5)
$$\begin{aligned} J_\nu(tv) &= \frac{t^2}{2}\int\left(|\nabla v|^2 - \mu\frac{|v|^2}{|x|^2}\right)dx - \frac{1}{p}\int\left[(u_\nu+tv)^p - u_\nu^p - pu_\nu^{p-1}tv\right]dx \\ &\leq \frac{t^2}{2}\int\left(|\nabla v|^2 - \mu\frac{|v|^2}{|x|^2}\right)dx - \frac{t^p}{p}\int|v|^pdx \\ &\leq \frac{t^2}{2}||v||^2 - \frac{t^p}{p}||v||_p^p \end{aligned}$$

Hence, we deduce

$$J_\mu(tv) \to -\infty$$

as $t \to \infty$. Therefore, for any $0 \not\equiv v \in H$ with $v \geq 0$, there exists a constant $t_0 > 0$ such that $J_\nu(t_0 v) \leq 0$ for $t \geq t_0$.

Observe that

Next, we are going to show that

$$\sup_{t\geq 0} J_\nu(tu_0) < \frac{1}{N}S^{N/2}$$

for some u_0.

Indeed, for small $t_1 > 0$, by Proposition 2.3 and its remark, any $0 < t < t_1$, $J_\nu(tu_0) < \frac{1}{N}S^{N/2}$ for some $u_0 \in H$. Choose $t_2 > t_1$ such that $J_\nu(tu_0) \leq 0$ for all $t \geq t_2$, For $t_1 \leq t \leq t_2$, from (3.5), we have

$$\begin{aligned} J_\nu(tu_0) &< \frac{t^2}{2}\int \left(|\nabla u_0|^2 - \mu\frac{|u_0|^2}{|x|^2}\right)dx - \frac{t^p}{p}\int |u_0|^p dx \\ &= \left(\frac{t^2}{2} - \frac{t^p}{p}\right)S^{N/2} \leq \frac{1}{N}S^{N/2}. \end{aligned}$$

Let

$$\Gamma := \{\gamma \in \mathscr{C}([0,1], H); \gamma(0) = 0,\ \gamma(1) = t_2 u_0\}$$

and

$$c_\nu = \inf_{\gamma\in\Gamma}\max_{s\in[0,1]} J_\nu(\gamma(s)).$$

Then, we have

$$0 < \alpha \leq c_\nu \leq \sup_{t\geq 0} J_\nu(tu_0) < \frac{1}{N}S^{N/2}. \tag{3.6}$$

We now applying the Mountain Pass Theorem without Palais-Smale condition in [4] to get a sequence $\{v_n\} \subset H$ such that

$$J_\nu(v_n) \to c_\nu, \quad J'_\nu(v_n) \to 0 \quad \text{in} \quad H. \tag{3.7}$$

Then, we see that $\{v_n\}$ is bounded in H. Hence, there exists a subsequence, say again, $\{v_n\}$ such that

$$v_n \to v_\nu \text{ weakly in } H,$$

$$v_n \to v_\nu \text{ a.e. in } \Omega,$$

$$\nabla v_n \to \nabla v_\nu \text{ a.e. in } \Omega,$$

and

$$(v_n + u_\nu)^{p-1} - u_\nu^{p-1} \to \left(v^+ + u_\nu\right)^{p-1} - u_\nu^{p-1} \text{ weakly in } \left(L^p(\Omega)\right)^*.$$

Hence, v_ν is a weak solution of $-\Delta v - \mu\frac{v}{|x|^2} = (v^+ + u_\nu)^{p-1} - u_\nu^{p-1}$.

Using the maximal principle, we get $v_\nu \geq 0$ in Ω. Furthermore, $||v_n^-|| = o(1)$ since $< J'_\nu(v_n), v_n^- > \to 0$ as $n \to \infty$. Set $u_n := v_n + u_\nu$ and $u := v + u_\nu$. We claim that $u \not\equiv u_\nu$. Suppose $u \equiv u_\nu$. Then $v_n = u_n - u$ converges weakly but not strongy to 0 in H because $c_\nu > 0$. Now, we observe that, by Hölder's inequality,

$$\begin{aligned} &\int \left[\left(v_n^+ + u_\nu\right)^{p-1} - \left(v_n^+\right)^{p-1}\right] v_n^+ \\ &= (p-1)\int \left(v_n^+ + \theta u_\nu\right)^{p-2} u_\nu v_n^+ \\ &\leq (p-1)\left[\int \left(v_n^+ + \theta u_\nu\right)^{p-1} v_n^+\right]^{(p-2)/(p-1)} \left[\int u_\nu^{p-1} v_n^+\right]^{1/(p-1)} \\ &= o(1) \end{aligned}$$

for some $0 < \theta < u_\nu$ and thus

$$
\begin{aligned}
||v_n^+||^2 &= \int \left[\left(v_n^+ + u_\nu\right)^{p-1} - \left(v_n^+\right)^{p-1} \right] v_n^+ + o(1) \\
&= \int \left(v_n^+ + u_\nu\right)^{p-1} v_n^+ + o(1) \\
&= ||v_n^+||_p^p + o(1).
\end{aligned}
$$

Then, by the Sobolev-Hardy inequality:(1.1),

$$
S||v_n^+||_p^2 \leq ||v_n^+||^2 = ||v_n^+||_p^p + o(1),
$$

which gives us that $||v_n^+|| \geq S^{N/2}$. On the other hand,

$$
\begin{aligned}
K_\nu(u_n) &:= \frac{1}{2}||u_n||^2 + \frac{1}{p}||v_n^+ + u_\nu||_p^p - \nu < f, u_n > \\
&= \frac{1}{2}||u_\nu||^2 - \frac{1}{p}||u_\nu||_p^p - \nu < f, u_\nu > + J_\nu(v_n) \\
&= H_\nu(u_\nu) + J_\nu(v_n) \\
&= K_\nu(u_\nu) + c_\nu + o(1).
\end{aligned}
$$

Moreover, from Brezis-Leb Lemma[$cf.$[3]] that,

$$
\begin{aligned}
K_\nu(u_n) &:= \frac{1}{2}\left(||u_\nu||^2 + ||v_n||^2\right) - \frac{1}{p}\left(||u_\nu||_p^p + ||v_n^+||_p^p\right) - \nu < f, u_n > + o(1) \\
&= K_\nu(u_\nu) + \frac{1}{2}||v_n^+||^2 - \frac{1}{p}||v_n^+||_p^p + o(1) \\
&= K_\nu(u_\nu) + \frac{1}{N}||v_n^+||_p^p + o(1).
\end{aligned}
$$

Then, we have

$$
c_\nu < \frac{1}{N}S^{N/2} \leq ||v_n^+||_p^p = c_\nu + o(1),
$$

a contraction. Therefore, $v_\nu := v > 0$ and $U_\nu := v_\nu + u_\nu$ is a second solution to (P_ν). This completes the proof. ∎

Consequently, we have:

Theorem 3.6. *Assume $f \in H$, $f \geq 0$, $f \not\equiv 0$ in Ω and $||\nu f||_* \leq C_N^*$. Then there exists a positive constant $\nu^* > 0$ such that (P_ν) possesses at least two positive solutions for $0 < \nu < \nu^*$, a unique solution for $\nu = \nu^*$ and no positive solution if $\nu > \nu^*$.*

4. Bifurcation

In order to study the uniqueness of second the solutions U_ν and bifurcation phenomenon, we consider following eigenvalue problem:

$$(4.1)_\nu \qquad \begin{cases} -\Delta\phi - \mu\dfrac{\phi}{|x|^2} = \eta(\nu)(p-1)U_\nu^{p-2}\phi, \\ \qquad \phi \text{ in } H. \end{cases}$$

Let η_1 be the first eigenvalue of $(4.1)_\nu$;i.e.,

$$\eta_1 = \eta_1(\nu) = \inf_{0\neq\phi\in H}\left\{\int|\nabla\phi|^2 - \mu\frac{|\phi|^2}{|x|^2} : \int(p-1)U_\nu^{p-2}\phi^2 = 1\right\}.$$

The infinum is achieveed by some function ϕ and $\phi > 0$ in Ω.

In the proof of the following lemma, we make use of arguments in [2].

Lemma 4.1. *Let U_ν be a second positive solution of (P_ν) obtained in Theorem 3.5. Then $\eta_1(\nu) < 1$ for $0 < \nu < \nu^*$.*

Proof. Suppose contrary that $\eta_1(\mu) \geq 1$. Let $\phi_1 > 0$ be the eigenfunction for the eigenvalue η_1 and $\psi := U_\nu - u_\nu > 0$. Then ϕ_1 and ψ satisfies

$$(4.2) \quad \Delta\phi_1 + \mu\frac{\phi_1}{|x|^2} + (p-1)U_\nu^{p-2}\phi_1 \leq 0 \text{ and } \Delta\psi + \mu\frac{\psi}{|x|^2} + (p-1)U_\nu^{p-2}\psi \geq 0,$$

respectively. Set $\sigma = \psi/\phi_1$;i.e., $\psi = \sigma\phi_1$. Then, by (4.2),

$$(4.3) \qquad \sigma\nabla(\phi_1^2\nabla\sigma) = \psi\Delta\psi - \Delta\phi_1\frac{\psi^2}{\phi_1} \geq 0.$$

Let ζ be a C^∞ function on $\mathbb{R}^+$ such that $0 \leq \zeta(t) \leq 1$,

$$\zeta(t) := \begin{cases} 1 \text{ for } 0 \leq t \leq 1, \\ 0 \text{ for } t \geq 2. \end{cases}$$

For $R > 0$, set $\zeta_R(t) := \zeta\left(\frac{|x|}{R}\right)$ in $\mathbb{R}^N$. Multiplying (4.3) by ζ_R^2 and intergrating over $\mathbb{R}^N$, we have by Green' theorem,

$$(4.4) \qquad \begin{aligned} \int \zeta_R^2\phi_1^2|\nabla\sigma|^2 &\leq 2\left|\int \phi_1^2\zeta_R\sigma\nabla\sigma\cdot\nabla\zeta_R\right| \\ &\leq 2\left[\int_{R<|x|<2R}\zeta_R^2\phi_1^2|\nabla\sigma|^2\right]^{1/2}\left[\int\phi_1^2\sigma^2|\nabla\zeta_R|^2\right]^{1/2} \\ &\leq C_1\left[\int_{R<|x|<2R}\zeta_R^2\phi_1^2|\nabla\sigma|^2\right]^{1/2}\left[\int_{R<|x|<2R}\psi^p\right]^{1/2} \\ &\leq C_2\left[\int_{R<|x|<2R}\zeta_R^2\phi_1^2|\nabla\sigma|^2\right]^{1/2} \end{aligned}$$

for some constants C_1 and C_2 independent of R. Then,

$$\int \zeta_R^2 \phi_1^2 |\nabla\sigma|^2 \le C_3$$

for some constant $C_3 > 0$ independent of R.
Letting $R \to \infty$, we see that

$$\int_{\mathbb{R}^N} \phi_1^2 |\nabla\sigma|^2 \le C_3.$$

But then it follows that the last term in (4.4) tends to 0 as $R \to \infty$, so that

$$\int_{\mathbb{R}^n} \phi_1^2 |\nabla\sigma|^2 = 0.$$

Therefore, σ is a positive constant and by (4.2), $\phi \equiv \psi = U_\nu - u_\nu$, and thus $U_\nu \equiv u_\nu$, which leads a contradiction. This completes the proof. ■

Lemma 4.2. *For $\nu \in]0, \nu^*[$, U_ν decreases contonusely to u_{ν^*} as $\nu \to \nu^*$ in H. Moreover,*
(i) $U_\nu \to u_{\nu^*}$ as $\nu \to \nu^*$ by the uniqueness of u_{ν^*},
(ii) $\lim_{\nu \to 0^+} ||U_\nu|| = S^{N/4}$.

Proof. First, we note that

$$\begin{aligned}\left(\frac{1}{2} - \frac{1}{p}\right) ||U_\nu||^2 &= \frac{1}{2}||U_\nu||^2 - \frac{1}{p}\int \left(U_\nu^p + \nu \int f U_\nu\right) \\ &= \nu\left(1 - \frac{1}{p}\right)\int fU_\nu - \nu\int fu_\nu - \nu \int fv_\nu \\ &+ \frac{1}{2}||u_\nu||^2 + \frac{1}{2}||v_\nu||^2 + \int \nabla u_\nu \nabla v_\nu + \int u_\nu v_\nu - \frac{1}{p}\int U_\nu^p \\ &\ge \nu\left(1 - \frac{1}{p}\right)\int fU_\nu + J_\nu(v_\nu) + H(u_\nu),\end{aligned}$$

where $H(u) := \frac{1}{2}||u||^2 - \frac{1}{p}\int u^p - \nu \int fu.$

From Hölder's and Young's inequality, for $\epsilon > 0$, we have

$$\left(\frac{p-2}{2p} - \frac{\epsilon(p-1)}{2p}\right)||U_\nu||^2 \le \frac{p-1}{\epsilon 2p}\nu^2 ||f||_*^2 + \frac{1}{N}S^{N/2} + H(u_\nu).$$

Since

$$\begin{aligned}H(u_\nu) &= \left(\frac{1}{2} - \frac{1}{p}\right)||u_\nu||^2 - \nu\left(1 - \frac{1}{p}\right)\int fu_\nu \\ &\le \left(\frac{1}{2} - \frac{1}{p}\right)||u_{\nu^*}||^2,\end{aligned}$$

$H(u_\nu)$ is uniformly bounded for $\nu \in (0, \nu^*]$. Moreover, by the remark of Proposition 3.4, $H(u_\nu) = o(1)$ as $\nu \to 0^+$. Taking $\epsilon > 0$ small enought, we have $||U_\nu|| \le$ C for some C > 0. Since $0 < u_\nu \le U_\mu$, (i) follows from Proposition 3.3 and Proposition 3.4.

For (ii). By (ii) of Lemma 3.1, and (i) and (iii) in the proof of Theorem 3.5, there exists $d > 0$ such that

$$0 < d < J_\nu(v_\nu) = H(U_\nu) - H(u_\nu) < \frac{1}{N} S^{N/2}$$

and thus, since $J'_\nu(U_\nu)U_\nu = 0$,

$$d + H(u_\nu) \leq \frac{1}{N}||U_\nu||^2 - \frac{p-1}{p}\nu \int fU_\nu \leq H(u_\nu) + \frac{1}{N}S^{N/2}.$$

Since U_ν is uniformly bounded,

$$d + o(1) \leq \frac{1}{N}||U_\nu||^2 \leq \frac{1}{N}S^{N/2} + o(1). \tag{4.5}$$

By Sobolev's inequality, $S||U_\nu||_p^2 \leq ||U_\nu||^2 = ||U_\nu||_p^p + o(1)$. Then $||U_\nu||_p^p \geq S^{N/2} + o(1)$ and so $||U_\nu||^2 \geq S^{N/2} + o(1)$. Therefore by (4.5), we have

$$\lim_{\nu \to 0^+} ||U_\nu|| = S^{N/2}.$$

Now, fix $\rho \in]0, \nu^*]$. Suppose μ increase to ρ, then U_ν is decreasing to U_ρ in H and we have

$$||U_\nu|| \leq S^{-p/2}||U_\rho||^{p-1} + \rho||f||_*$$

and so, there exists a sequence U_{ν_j} conving weakly to a solution $\tilde{U}$ of (P_ν) in H with $\rho = \nu$ but $\tilde{U} \neq U_\rho$. By the maximum principle, we have $U_\rho < \tilde{U} \leq U_{\nu^*}$ which contradicts the uniqueness of solutions bigger than u_ν. Therefore, U_ν is decreasing continuously to U_ρ and $U_\nu \to U_\rho$ in H. This completes the proof. ∎

Lemma 4.3. *et V be a positive supersolution of (P_ν) bigger than u_ν then $V \leq U_\nu$.*

Proof. Suppose $V > U_\nu$ in Ω, then $W = V - U_\nu$ satisfies

$$(p-1)\int U_\nu^{p-2} W\phi_1 \leq \int \nabla W \cdot \nabla \phi_1 = \eta_1(p-1)\int U_\nu^{p-2} W\phi_1$$

and thus, $\eta_1(\nu) \geq 1$, which leads a conrradiction. This completes the proof. ∎

Remark 4. From Lemma 4.1 and Lemma 4.3, we can see the uniqueness of second solutions which are bigger than the minimal solutions u_ν.

Now, we state basic properties of the eigenvalue problem $(4.1)_\nu$:

Lemma 4.4. *(i)* $1/(p-1) < \eta_1(\nu) < 1$ *for* $0 < \nu < \nu^*$,
(ii) $\eta_1(\nu) \to 1/(p-1) \to 1/(p-1)$ *as* $\nu \to 0^+$,
(iii) $\eta_1(\nu) \to 1$ *as* $\nu \to \nu^*$.

Proof. (i) Since $\phi_1 > 0$ is an eigenvector corresponding to the the first eigenvalue $\eta_1(\mu)$, we know

$$\eta_1(\nu)(p-1)\int U_\nu^{p-1}\phi_1 = \int \nabla U_\nu \cdot \nabla \phi_1 = \int U_\nu^{p-1}\phi_1 + \nu \int f\phi_1.$$

and so,

$$\eta_1(\nu)(p-2)\int U_\nu^{p-1}\phi_1 = \nu\int f\phi_1.$$

Therefore, by Lemma 4.1, $1 > \eta_1(\mu) > \frac{1}{p-1}$.

(ii) As $\mu \to 0^+$,

$$\frac{1}{p-1} < \eta_1(\nu) \le \frac{||U_\nu||^2}{(p-1)\,||U_\nu||_p^p} \le \frac{S^{N/2}+o(1)}{(p-1)\left(S^{N/2}+o(1)\right)} \to \frac{1}{p-1}.$$

Thus, $\eta_1(\nu) \to 1/(p-1)$ as $\nu \to 0^+$.

(iii) follows from (i) of Lemma 3.1, Proposition 3.3, Lemma 4.1 and (i) of Lemma 4.2. This completes the proof. ∎

In order to show the existence of a bifurcation point, we make use of Theorem 3.2 is in [5].

Now, we have:

Theorem 4.5. *(i) The set $\{U_\nu\}$ is bounded uniformly in H,*
(ii) (ν^*, u_{ν^*}) is a bifurcation point.

Proof. (i) It follows immediately from the proof of Lemma 4.2.
(ii) For this, define $F : R \times H \to H^{-1}$ by

$$F(\nu, u) := \Delta u - u + (u^+)^{2^*-1} + \nu f(x).$$

It is easy to see that $F(\nu, u)$ is differentiable at solution point (ν, u) for $]0, \nu^*[$ and

$$F_u(\nu, u_\nu)w = \Delta w - w + (2^*-1)u_\nu^{2^*-2}w$$

is an ismorphism of $R \times H$ onto H^{-1}. Then, by the Implicit Function Theorem, the solution of $F(\nu, u)$ near (ν, u_ν) are given by a single continuous cuver and $u_m n \to 0$ in H^-1 as $\nu \to 0$.

We now are going to prove that (ν^*, u_{ν^*}) is a bifurcation point of F. Since $F_u(\mu^*, u_{\mu^*})\phi = 0, \phi \in H^1(\mathbb{R}^N)$ has a solution $\phi_1 > 0$ in $\mathbb{R}^N$, $\mathscr{N}\left(F_u\left(\mu^*, u_{\mu^*}\right)\right) = \operatorname{span}\{\phi_1\}$ is one dimensional and $\operatorname{codim}\mathscr{R}\left(F_u\left(\mu^*, u_{\mu^*}\right)\right) = 1$ by the Fredholm alternative. Suppose there exists $v \in H^1(\mathbb{R}^N)$ satisfying

$$\Delta v - v + (2^*-1)\,u_{\mu^*}^{2^*-2}v = -f(x).$$

Then

$$0 = \int\left(\nabla v\cdot\nabla\phi_1 + v\phi_1 - (2^*-1)\,u_{\mu^*}^{2^*-2}v\phi_1\right) = \int f\phi_1,$$

which is impossible because $0 \not\equiv f \ge 0$. Hence, $F_u\left(\mu^*, u_{\mu^*}\right) \notin \mathscr{R}\left(F_u\left(\mu^*, u_{\mu^*}\right)\right)$. Thus, by Theorem 3.2 in [5], (μ^*, u_{μ^*}) is the bifurcation point near which, the solution of (p_μ) form a curve $(\mu^* + \tau(s), u_{\mu^*} + s\phi_1 + z(s))$ with s near $s = 0$ and $\tau(0) = \tau'(0) = 0, z(0) = z'(0) = 0$. Finally, we will show that $\tau''(0) < 0$ which implies that the bifurcation curve only turns to the left in the $\mu u-$plane. For this, differentiate (P_μ) in s, we have

$$\Delta u_s - u_s + (2^*-1)\,u^{2^*-2}u_s + \tau'(s)f(x) = 0, \tag{4.6}$$

where $u_s = \phi_1 + z'(s)$. Multiplying $F_u(\mu^*, u_{\mu^*})\phi_1 = 0$ by u_s and (4,6) by ϕ_1, integrating and substracting, we have

$$\tau'(s)\int f\phi_1 = (2^*-1)\int \left(u_{\mu^*}^{2^*-2} - (u_{\mu^*}+s\phi_1+z(s))^{2^*-2}\right)(\phi_1+z'(s))\phi_1$$
$$= -s(2^*-1)(2^*-2)\int (u_{\mu^*}+\theta(s\phi_1+z(s)))^{2^*-3}\left(\phi_1+\frac{z(s)}{s}\right)(\phi_1+z'(s))\,\phi_1$$

for some $\theta(s)\in(0,1)$. Therefore,

$$\tau''(0)\int f\phi_1 = \left(lim_{s\to 0}\frac{\tau'(s)}{s}\right)\int f\phi_1 = -(2^*-1)(2^*-2)\int (u_{\mu^*})^{2^*-3}\phi_1^3$$

and $\tau''(0) < 0$. This completes proof. ∎

References

[1] A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funt. Anal. **14**, (1973), 349-381.

[2] H. Berestycki, L. Capffarelli and L. Nirenberg, *Futher qualitative properties for elliptic equations in unbounded domains*, Ann. Scuola Norm. Sup. Pisa CI. Sci. **4**25(1997),69-94.

[3] H. Brezis and E. Lieb, *A relation between pointwise convergence of functionals and convergence of functions*, Proc. Amer. Soc., **88** (1983), 486-490.

[4] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponent*, Comm. Pure Appl. Math., **36**(1983), 437-477 .

[5] M. Crandall and P. Rabiowitz, *Bifurcation, Perturbation of simple eigenvalues, and linealized stability*, Arch. Rational Mech. Mech. Anal., **52**(1973), 161-180.

[6] L. Ekeland, *Convex minimization problem*, Bull. Amer. Math. Soc., (NS)**1** (1976), 443-474.

[7] Y. Deng and Y. Li, *Existence and bifurcation of positive solutions for a semilinear elliptic equation with critical exponent*, J. Diff. Equa., **130** (1996), 179-200.

[8] N. Ghoussoub and C. Yuan, *Multiple solutions for qusi-linear PDEs involving critical Sobolev and Hardy exponentials*, Trans. Amer. Math. Soc., **352** (2000)5703-5743.

[9] D. Kang, *Solutions of quasilinear elliptic problem with a critical Sobolev-Hardy exponent and a Hardy-type term*, J. Math. Anal. Appl., **341** (2008)464-781.

[10] D. Kang and Y. Deng, *Multiple solutions for inhomogeneous elliptic problems involving critical Sobolev-Hardy exponentis*, Nonlinear Analysis T.M.A., **60** (2005)297-753.

[11] D. Kang and S. Peng, *Positive solutions for singular critical elliptic problems*, Appl. Math. Lett., **17** (2004)411-416.

[12] W.S. Kim, *Multiple existence of positive global solutions parameterized nonhomogeneous elliptic equations involving critical exponents*, East Asian Math. J., **30(3)**(2014)335-353.

[13] D. Smets, *Nonlinear Schrödinger equations with Hardy potentials and critical nonlearitie*, Trans. Amer. Math. Soc., **357(7)** (2004) 2909-2938.

[14] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pure Appl., **110** (1976). 353-372.

[15] M. Willem, *Minimax theorems*, Birkhauser. Boston, Basel, Berlin. (1996).

[16] X.-P. Zhu, *A perturbation result on positive entire solutions of a semilinear elliptic equation*, J. Diff. Equa, **92** (1991) 163-178.

[17] X.-P. Zhu and H.-S. Zhu, *Existence of multiple positive solutions of inhomogeneous semilinear elliptic problems in unbounded domain*, Proc. Roy. Soc. Edinburgh , **115 A**(1990)301-318.

Wan Se Kim
Department of Mathematics Research Institute for Natural Sciences Hanyang University Seoul 133-791, Korea
E-mail address: wanskim@hanyang.ac.kr