Bull. Korean Math. Soc. **55** (2018), No. 5, pp. 1529–1561
https://doi.org/10.4134/BKMS.b170892
pISSN: 1015-8634 / eISSN: 2234-3016

CERTAIN DIFFERENCE POLYNOMIALS AND SHARED VALUES

Xiao-Min Li and Hui Yu

Abstract. Let f and g be nonconstant meromorphic (entire, respectively) functions in the complex plane such that f and g are of finite order, let a and b be nonzero complex numbers and let n be a positive integer satisfying $n \geq 21$ ($n \geq 12$, respectively). We show that if the difference polynomials $f^n(z) + af(z+\eta)$ and $g^n(z) + ag(z+\eta)$ share b CM, and if f and g share 0 and ∞ CM, where $\eta \neq 0$ is a complex number, then f and g are either equal or at least closely related. The results in this paper are difference analogues of the corresponding results from [4].

1. Introduction and main results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in (cf. [1, 8, 13, 16]). It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying $S(r,h) = o(T(r,h))$ as r runs to infinity outside of a set of finite logarithmic measure. We say that a is a small function of f, if a is a meromorphic function satisfying $T(r,a) = S(r,f)$ (cf. [16]). Throughout this paper, we denote by $\rho(f)$ the order of f. The definition of the order of a meromorphic function can be found in [1, 13].

Let f and g be two nonconstant meromorphic functions, and let a be a value in the extended plane. We say that f and g share the value a CM, provided that f and g have the same a-points in the complex plane, and each common a-point of f and g has the same multiplicities related to f and g. We say that f and g share the value a IM, provided that f and g have the same a-points in the complex plane (cf. [16]). In this paper, we also need the following definitions:

Received October 13, 2017; Revised March 19, 2018; Accepted April 3, 2018.
2010 *Mathematics Subject Classification.* Primary 30D35; Secondary 39A05.
Key words and phrases. Nevanlinna theory, difference polynomials, uniqueness theorems.
This work was financially supported in part by the NSFC (No.11171184), the NSFC (No.11461042) and the NSF of Shandong Province, China (No.ZR2014AM011).

Definition 1.1 ([12])**.** Let f be a nonconstant meromorphic function, let p be a positive integer, and let $a \in \mathbb{C}\bigcup\{\infty\}$. Then, by $N_{p)}(r, \frac{1}{f-a})$, we denote the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not greater than p, by $\overline{N}_{p)}(r, \frac{1}{f-a})$, we denote the corresponding reduced counting function (ignoring multiplicities). By $N_{(p}(r, \frac{1}{f-a})$, we denote the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not less than p, by $\overline{N}_{(p}(r, \frac{1}{f-a})$ we denote the corresponding reduced counting function (ignoring multiplicities), where $N_{p)}(r, \frac{1}{f-a})$,$\overline{N}_{p)}(r, \frac{1}{f-a})$, $N_{(p}(r, \frac{1}{f-a})$ and $\overline{N}_{(p}(r, \frac{1}{f-a})$ mean $N_{p)}(r, f)$, $\overline{N}_{p)}(r, f)$, $N_{(p}(r, f)$ and $\overline{N}_{(p}(r, f)$, respectively, if $a = \infty$.

Definition 1.2. Let f be a nonconstant meromorphic function, let a be any value in the extended complex plane, and let k be a positive integer. We define the k-th order truncated counting functions of a-points of f in $|z| < r$ as

$$N_k\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-a}\right) + \cdots + \overline{N}_{(k}\left(r, \frac{1}{f-a}\right).$$

Accordingly we define the deficiency related to $N_k\left(r, \frac{1}{f-a}\right)$ as

$$\delta_k(a, f) = 1 - \limsup_{r\to\infty} \frac{N_k\left(r, \frac{1}{f-a}\right)}{T(r, f)}.$$

Remark 1.3. By Definition 1.2 we have

$$0 \le \delta_k(a, f) \le \delta_{k-1}(a, f) \le \delta_1(a, f) = \Theta(a, f) \le 1,$$

which appears that the ramification index $\Theta(a, f)$ and the deficiency $\delta_1(a, f)$ are the same.

Recently, the difference variant of the Nevanlinna theory has been established in [2,5] and, in particular, in [6], by Halburd-Korhonen and by Chiang-Feng, independently. Using these theories, some mathematicians from Finland and China began to consider the uniqueness questions of meromorphic functions sharing values with their shifts, and produced many fine works, for example, see [9,10,17]. In this paper, we will consider a uniqueness question of meromorphic functions whose certain difference polynomials share three values. Here the meromorphic functions are of finite order.

In 2011, Grahl and Nevo proved the following results:

Theorem 1.4 ([4, Theorem 1])**.** *Let f and g be nonconstant meromorphic functions the complex plane, let a and b be two nonzero complex numbers, and let n and k be two positive integers satisfying $n \ge 5k + 17$. Assume that the functions $\psi_f = f^n + af^{(k)}$ and $\psi_g = g^n + ag^{(k)}$ share the value b CM. Then*

$$\frac{\psi_f - b}{\psi_g - b} = \frac{f^n}{g^n} = \frac{af^{(k)} - b}{ag^{(k)} - b} \quad or \quad \frac{\psi_f - b}{\psi_g - b} = \frac{f^n}{ag^{(k)} - b} = \frac{af^{(k)} - b}{g^n}$$

or $f = g$, $f^{(k)} = g^{(k)} = \frac{b}{a}$.

Theorem 1.5 ([4, Theorem 2])**.** *Let f and g be nonconstant entire functions in the complex plane, let a and b be two nonzero complex numbers, and let n and k be two positive integers satisfying $n \geq 11$ and $n \geq k+2$. Assume that the functions ψ_f and ψ_g defined as in Theorem 1.4 share the value b CM. Then*

$$\frac{\psi_f - b}{\psi_g - b} = \frac{f^n}{g^n} = \frac{af^{(k)} - b}{ag^{(k)} - b} \quad \text{or} \quad f = g, \quad f^{(k)} = g^{(k)} = \frac{b}{a}.$$

Regarding Theorem 1.4 and Theorem 1.5, one may ask, what can be said about the relationship between f and g, if certain difference polynomial of f and the same difference polynomial of g share a finite nonzero value or share at least two distinct finite values? In this direction, we will prove the following result:

Theorem 1.6. *Let f and g be two transcendental meromorphic functions in the complex plane such that $\rho(f) < \infty$ and $\rho(g) < \infty$, let a and b be two nonzero complex numbers, and let n be a positive integer satisfying $n \geq 21$. Assume that the functions*

$$\psi_f(z) = f^n(z) + af(z+\eta) \quad \text{and} \quad \psi_g(z) = g^n(z) + ag(z+\eta) \tag{1}$$

share the value b CM, and that f and g share 0 and ∞ CM. Then

$$\frac{\psi_f(z) - b}{\psi_g(z) - b} = \frac{f^n(z)}{g^n(z)} = \frac{af(z+\eta) - b}{ag(z+\eta) - b} \quad \text{or} \quad f = g.$$

Proceeding as in the proof of Theorem 1.6 in Section 3 of this paper, we get the following result:

Theorem 1.7. *Let f and g be two transcendental entire functions such that $\rho(f) < \infty$, let a and b be two nonzero complex numbers, and let n be a positive integer satisfying $n \geq 12$. Assume that the functions ψ_f and ψ_g defined as in* (1) *share the value b CM, and that f and g share 0 and ∞ CM. Then*

$$\frac{\psi_f(z) - b}{\psi_g(z) - b} = \frac{f^n(z)}{g^n(z)} = \frac{af(z+\eta) - b}{ag(z+\eta) - b} \quad \text{or} \quad f = g.$$

2. Preliminaries

In this section, we introduce the following lemmas to prove the main results in this paper. First of all, we introduce the following result by Halburd, Korhonen and Tohge:

Lemma 2.1 ([7, Lemma 8.3])**.** *Let $T(r) : [0, +\infty) \to [0, +\infty)$ be a nondecreasing continuous function and let $s \in \mathbb{R}^+$. If the hyper-order of T is strictly less than one, i.e., $\limsup\limits_{r\to\infty} \frac{\log\log T(r)}{\log r} = \zeta < 1$, and let $\delta \in (0, 1-\zeta)$, then $T(r+s) = T(r) + o\left(\frac{T(r)}{r^\delta}\right)$, where r runs to infinity outside of a set of finite logarithmic measure.*

The following result is proved by Chiang and Feng:

Lemma 2.2 ([2, Corollary 2.5])**.** *Let f be a meromorphic function of finite order ρ, and let η be a fixed nonzero complex number. Then, for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+\eta)}\right) = O(r^{\rho-1+\varepsilon}).$$

Lemma 2.3 ([2, Theorem 2.1])**.** *Let f be a meromorphic function of order $\rho(f) = \rho < \infty$, and let η be a fixed nonzero complex number. Then for each $\varepsilon > 0$, we have*

$$T(r, f(z+\eta)) = T(r, f(z)) + O(r^{\rho-1+\varepsilon}) + O(\log r).$$

The following result plays an important role in proving the main results of the present paper:

Lemma 2.4. *Let f be a transcendental meromorphic function of finite order, let a, b and η be three nonzero complex numbers. Then, for a positive integer $n > 1$ and a fixed positive number $\delta_0 \in (0,1)$, we have*

$$\begin{aligned}(n-1)T(r, f(z)) \le \overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}\left(r, \frac{1}{f^n(z) + af(z+\eta) - b}\right) + 4\overline{N}(r, f(z)) \\ + o\left(\frac{T(r, f(z))}{r^{\delta_0}}\right) + O\left(r^{\rho-1+\varepsilon}\right) + O(\log r)\end{aligned}$$

as r runs to infinity outside of a set of finite logarithmic measure.

Proof. Now we set

$$g_0(z) = -f^n(z), \quad g_1(z) = f^n(z) + af(z+\eta) - b. \tag{2}$$

Then, g_0 and g_1 are linearly independent. Indeed, if there exist two constant c_1 and c_2 satisfying $|c_0| + |c_1| > 0$, such that $c_0 g_0 + c_1 g_1 = 0$, i.e.,

$$(c_0 - c_1) f^n(z) = c_1(af(z+\eta) - b). \tag{3}$$

Obviously, $c_1 \neq 0$. Then, by (3) we have $c_0 - c_1 \neq 0$. This together with Lemma 2.1 and the assumption $\rho(f) < \infty$, and so $\rho_2(f) = 0$ gives

$$nT(r, f(z)) = T(r, f(z+\eta)) + O(1) = T(r, f(z)) + o\left(\frac{T(r, f(z))}{r^{\delta_0}}\right),$$

and so

$$T(r, f) = o\left(\frac{T(r, f)}{r^{\delta_0}}\right)$$

as r runs to infinity outside of a set of finite logarithmic measure. This is impossible. Here and in what follows, $\delta_0 \in (0,1)$ is some fixed positive number. Hence, g_0 and g_1 are linearly independent, and so

$$\frac{g_1'}{g_1} - \frac{g_0'}{g_0} \not\equiv 0. \tag{4}$$

By (2) we have $g_0(z)+g_1(z)=af(z+\eta)-b$, and so

$$g_0'(z)+g_1'(z)=\frac{d(af(z+\eta)-b)}{dz}:=af'(z+\eta).$$

Therefore,

$$g_0(z)=(af(z+\eta)-b)\left(\frac{g_1'(z)}{g_1(z)}-\frac{af'(z+\eta)}{af(z+\eta)-b}\right)\left(\frac{g_1'(z)}{g_1(z)}-\frac{g_0'(z)}{g_0(z)}\right)^{-1}. \tag{5}$$

By (5), Lemma 2.1 and Lemma 2.2, the left equality of (2) and the first fundamental theorem we have

$$\begin{aligned}
nm(r,f(z))=m(r,g_0(z))&\le m(r,af(z+\eta))+m\left(r,\frac{g_1'(z)}{g_1(z)}-\frac{af'(z+\eta)}{af(z+\eta)-b}\right)\\
&\quad+m\left(r,\left(\frac{g_1'(z)}{g_1(z)}-\frac{g_0'(z)}{g_0(z)}\right)^{-1}\right)+O(1)\\
&\le m(r,f(z))+m\left(r,\frac{g_1'(z)}{g_1(z)}-\frac{af'(z+\eta)}{af(z+\eta)-b}\right)\\
&\quad+N\left(r,\frac{g_1'(z)}{g_1(z)}-\frac{g_0'(z)}{g_0(z)}\right)-N\left(r,\left(\frac{g_1'(z)}{g_1(z)}-\frac{g_0'(z)}{g_0(z)}\right)^{-1}\right)\\
&\quad+m\left(r,\frac{g_1'(z)}{g_1(z)}-\frac{g_0'(z)}{g_0(z)}\right)+O(r^{\rho-1+\varepsilon})
\end{aligned}\tag{6}$$

as r runs to infinity outside of a set of finite logarithmic measure. By (1), (2), (6), Lemma 2.3, the assumption $\rho(f)=\rho<\infty$ and the definition of the order of a meromorphic function we deduce that g_0, g_1 and $af(z+\eta)-b$ are of order $\le\rho$. Combining this with the lemma on the logarithmic derivatives (cf. [13, Theorem 2.3.3]) we have

$$\begin{aligned}
(n-1)m(r,f)&\le N\left(r,\frac{g_1'}{g_1}-\frac{g_0'}{g_0}\right)-N\left(r,\left(\frac{g_1'}{g_1}-\frac{g_0'}{g_0}\right)^{-1}\right)\\
&\quad+o\left(\frac{T(r,f(z))}{r^{\delta_0}}\right)+O(r^{\rho-1+\varepsilon})+O(\log r).
\end{aligned}\tag{7}$$

Now we suppose that $z_0\in\mathbb{C}$ is a pole of $f(z)$ with multiplicity $p\ge1$. Next we let p_1 be the order of z_0 as the pole of $f(z+\eta)$, which means that p_1 is a positive integer, when $f(z_0+\eta)=\infty$ with p_1 being the multiplicity of z_0 as a pole of $f(z+\eta)$, and such that $p_1=0$, when $f(z_0+\eta)\ne\infty$. For convenience we introduce the notation $p_1^+(f,z_0)$. Let p_1 be an integer, next we denote by $p_1^+(f,z_0)$ a nonnegative integer, which is defined in the following manner:

$$p_1^+(f,z_0)=\begin{cases}p_1, & \text{if } z_0 \text{ is a pole of } f(z+\eta) \text{ of multiplicity } p_1\ge1,\\ 0, & \text{if } z_0 \text{ is not a pole of } f(z+\eta).\end{cases}\tag{8}$$

By (1) and (2) and the above supposition we can see that z_0 is a pole of $f^n(z)$ with multiplicity np, and deduce that, possibly z_0 is a pole of $\frac{g_1'(z)}{g_1(z)}-\frac{af'(z+\eta)}{af(z+\eta)-b}$

and $\frac{g_1'(z)}{g_1(z)} - \frac{g_0'(z)}{g_0(z)}$ with multiplicity equal to 1 at most. This gives

$$\frac{g_1'(z)}{g_1(z)} - \frac{g_0'(z)}{g_0(z)} = c(z-z_0)^{\mu} + \text{higher powers of } z - z_0 \tag{9}$$

for each $z \in \Delta'(z_0, \delta)$, where c is a nonzero constant, $\mu \geq -1$ is an integer. Here $\Delta'(z_0, \delta) = \{z : 0 < |z - z_0| < \delta\}$, δ is some sufficiently small positive number.

By comparing the multiplicities of z_0 as the poles of both sides of (5), we deduce by (1), (2), (8), (9) and the above analysis that

$$np \leq 1 + \mu + p_1^+(f, z_0). \tag{10}$$

By (10) we have

$$\mu \geq np - 1 - p_1^+(f, z_0). \tag{11}$$

By (9) and (11) we deduce

$$N\left(r, \left(\frac{g_1'(z)}{g_1(z)} - \frac{g_0'(z)}{g_0(z)}\right)^{-1}\right) \geq nN(r, f(z)) - \overline{N}(r, f(z)) - N(r, f(z+\eta)). \tag{12}$$

By noting that

$$\limsup_{r\to\infty} \frac{\log N(r, f)}{\log r} \leq \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r} = \rho < \infty,$$

and so

$$\limsup_{r\to\infty} \frac{\log\log N(r, f)}{\log r} = 0,$$

we have by Lemma 2.1 that for a fixed positive number $\delta_0 \in (0, 1)$ we have

$$N(r, f(z+\eta)) = N(r, f(z)) + o\left(\frac{T(r, f(z))}{r^{\delta_0}}\right) \tag{13}$$

and

$$\overline{N}(r, f(z+\eta)) = \overline{N}(r, f(z)) + o\left(\frac{T(r, f(z))}{r^{\delta_0}}\right) \tag{14}$$

as r runs to infinity outside of a set of finite logarithmic measure. By substituting (13) into (12) we have

$$N\left(r, \left(\frac{g_1'(z)}{g_1(z)} - \frac{g_0'(z)}{g_0(z)}\right)^{-1}\right) \geq (n-1)N(r, f(z)) - \overline{N}(r, f(z)) + o\left(\frac{T(r, f(z))}{r^{\delta_0}}\right) \tag{15}$$

as r runs to infinity outside of a set of finite logarithmic measure. By (2) and (14) we have

$$\begin{aligned} &N\left(r, \frac{g_1'(z)}{g_1(z)} - \frac{g_0'(z)}{g_0(z)}\right) \\ &\leq \overline{N}\left(r, \frac{1}{g_0(z)}\right) + \overline{N}\left(r, \frac{1}{g_1(z)}\right) + \overline{N}(r, g_0(z)) + \overline{N}(r, g_1(z)) \end{aligned} \tag{16}$$

$$\leq \overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}\left(r, \frac{1}{f^n(z) + af(z+\eta) - b}\right)$$
$$+ 3\overline{N}(r, f(z)) + o\left(\frac{T(r, f(z))}{r^{\delta_0}}\right)$$

as r runs to infinity outside of a set of finite logarithmic measure. By (7), (15) and (16) we have the conclusion of Lemma 2.4. □

The following result is given by Li and Yang:

Lemma 2.5 ([14, Lemma 2]). *Let $f_1, f_2, \ldots, f_n$ be nonconstant meromorphic functions such that $\sum_{j=1}^{n} f_j = 1$. If $f_1, f_2, \ldots, f_n$ are linearly independent, then*

$$T(r, f_1) \leq \sum_{j=1}^{n} N_{n-1}\left(r, \frac{1}{f_j}\right) + (n-1)\sum_{j=2}^{n} \overline{N}(r, f_j) + o(T(r))$$

as r runs to infinity outside of a subset $E \subset (0, +\infty)$ of finite linear measure. Here $T(r) = \max\limits_{1\leq j\leq n}\{T(r, f_j)\}$.

We also need the following three results from [16]:

Lemma 2.6 ([16, Lemma 1.10]). *Let f_1 and f_2 be nonconstant meromorphic functions in the complex plane and c_1, c_2, c_3 be nonzero constants. If $c_1 f_1 + c_2 f_2 = c_3$, then*

$$T(r, f_1) \leq \overline{N}\left(r, \frac{1}{f_1}\right) + \overline{N}\left(r, \frac{1}{f_2}\right) + \overline{N}(r, f_1) + S(r, f_1).$$

Lemma 2.7 ([16, Theorem 1.62]). *Let $f_1, f_2, \ldots, f_n$ be nonconstant meromorphic functions, and let $f_{n+1} \not\equiv 0$ be a meromorphic function such that $\sum_{j=1}^{n+1} f_j = 1$. Suppose that there exists a subset $I \subseteq \mathbb{R}^+$ with linear measure $mesI = \infty$ such that*

$$\sum_{i=1}^{n+1} N\left(r, \frac{1}{f_i}\right) + n\sum_{\substack{i=1\\ i\neq j}}^{n+1} \overline{N}(r, f_i) < (\lambda + o(1))T(r, f_j), \quad j = 1, 2, \ldots, n,$$

as $r \in I$ and $r \to \infty$, where λ is a real number satisfying $0 \leq \lambda < 1$. Then $f_{n+1} = 1$.

Lemma 2.8 ([16, Lemma 9.1]). *Let f and g be two nonconstant meromorphic functions that share 0, 1 and ∞ CM. Suppose that*

$$\delta_{1)}(0, f) + \delta_{1)}(1, f) > \frac{3}{2},$$

where

$$\delta_{1)}(0, f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}_{1)}\left(r, \frac{1}{f}\right)}{T(r, f)} \text{ and } \delta_{1)}(1, f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}_{1)}\left(r, \frac{1}{f-1}\right)}{T(r, f)}.$$

Then $f + g = 1$.

The following result is given by Doeringer:

Lemma 2.9 ([3, Lemma 2]). *Let f be a meromorphic function, and let $Q^*[f]$ and $Q[f]$ denote differential polynomials in f with arbitrary meromorphic coefficients $q_1^*, \dots, q_n^*$ and $q_1, \dots, q_k$ respectively. Suppose that $P[f]$ is given as*

$$P[f] = a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0,$$

where $a_n \not\equiv 0$ and that $a_n, a_{n-1}, \dots, a_1, a_0$ are small functions of f. If $P[f]Q^*[f] = Q[f]$ and $\gamma_Q \le n$, then

$$m(r, Q^*[f]) \le \sum_{j=1}^{n} m(r, q_j^*) + \sum_{j=1}^{k} m(r, q_j) + S(r, f).$$

Lemma 2.10 ([11, Lemma 2.2]). *Let $\varphi(r)$ be a nondecreasing, continuous function on $\mathbb{R}^+$. Suppose that*

$$0 < \rho < \limsup_{r\to\infty} \frac{\log \varphi(r)}{\log r},$$

and set

$$I = \{t : t \in \mathbb{R}^+,\ \varphi(r) \ge r^\rho\}.$$

Then we have

$$\overline{\log dens} I = \limsup_{r\to\infty} \frac{\int_{I\cap[1,r]} \frac{dr}{r}}{\log r} > 0.$$

3. Proof of theorems

Proof of Theorem 1.6. Suppose that $f \not\equiv g$. Without loss of generality, we assume that $a = 1$. Otherwise, we replace f, g and b with cf, cg and bc^n respectively, where c is a constant satisfying $c^{n-1} = a$. Next we define

$$\varphi_f = \frac{f^n}{\psi_f - b} \quad \text{and} \quad \varphi_g = \frac{g^n}{\psi_g - b}, \tag{17}$$

where ψ_f and ψ_g are the functions defined as in (1). Assume that φ_f is a constant, say $\varphi_f = c$, where $c \ne 0$ is a constant. If $c = 1$, by (1) and the left equality of (17) we have $af(z+\eta) = b$ for all $z \in \mathbb{C}$, this is impossible. Therefore $c \ne 1$, this together with the left equality of (17) gives

$$f^n(z) = \frac{c(f(z+\eta) - b)}{1-c}$$

for all $z \in \mathbb{C}$, this together with Lemma 2.1 and Valiron-Mokhonoko lemma (cf. [15]) implies that for each $\varepsilon > 0$ we have

$$\begin{aligned} nT(r, f(z)) &= T(r, f^n(z)) = T(r, f(z+\eta)) + O(1) \\ &= T(r, f(z)) + o\left(\frac{T(r, f(z))}{r^{\delta_0}}\right) + O(1), \end{aligned} \tag{18}$$

and so

$$T(r,f) = o\left(\frac{T(r,f)}{r^{\delta_0}}\right) + O(1)$$

as r runs to infinity outside of a set of finite logarithmic measure. This is impossible. Here and in what follows, $\delta_0 \in (0,1)$ is some fixed positive number. Similarly, φ_g is a nonconstant meromorphic function.

On the other hand, by the left equality of (1) and the left equality of (17) we deduce

$$\overline{N}(r,\varphi_f) \le \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{\psi_f - b}\right). \tag{19}$$

Next we denote by $\overline{N}_L\left(r, \frac{1}{f^n(z)}\right)$ the reduced counting function of those common zeros of $f^n(z)$ and $f(z+\eta)-b$ in $|z|<r$, where the multiplicity of each such common zero as the zero of $f^n(z)$ is greater than its multiplicity as the zero of $f(z+\eta)-b$, denote by $\overline{N}\left(r, \frac{1}{f(z)} | f(z+\eta)-b \neq 0\right)$ the reduced counting function of those zeros of $f(z)$ in $\overline{N}\left(r, \frac{1}{f(z)}\right)$ that are not zeros of $f(z+\eta)-b$. Then, by noting that

$$\frac{1}{\varphi_f(z)} = 1 + \frac{f(z+\eta)-b}{f^n(z)}, \ \limsup_{r\to\infty} \frac{\log \overline{N}(r,f)}{\log r} \le \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} = \rho < \infty,$$

and so

$$\limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r} = 0,$$

we deduce by Lemma 2.1 that for some fixed positive number $\delta_0 \in (0,1)$, we have

$$\begin{aligned} &\overline{N}\left(r, \frac{1}{\varphi_f(z)}\right) \\ &\le \overline{N}\left(r, \frac{1}{f(z)} \mid f(z+\eta)-b \neq 0\right) + \overline{N}_L\left(r, \frac{1}{f^n(z)}\right) + \overline{N}(r, f(z+\eta)) \\ &\le \frac{1}{n} N\left(r, \frac{1}{\varphi_f(z)}\right) + \overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}(r, f(z)) + o\left(\frac{T(r,f(z))}{r^{\delta_0}}\right) \\ &\le \frac{1}{n} T(r, \varphi_f(z)) + \overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}(r, f(z)) + o\left(\frac{T(r,f(z))}{r^{\delta_0}}\right) \end{aligned} \tag{20}$$

as r runs to infinity outside of a set of finite logarithmic measure.

By noting that

$$\frac{1}{\varphi_f(z)-1} = -1 - \frac{f^n(z)}{f(z+\eta)-b},$$

we have

$$\begin{aligned}\overline{N}\left(r, \frac{1}{\varphi_f(z)-1}\right) &= \overline{N}\left(r, \frac{f^n(z)}{f(z+\eta)-b}\right)\\ &\le \overline{N}(r, f(z)) + \overline{N}\left(r, \frac{1}{f(z+\eta)-b}\right).\end{aligned} \tag{21}$$

By (1), (17), (19)-(21) and the second fundamental theorem we have

$$\begin{aligned}T(r, \varphi_f(z)) &\le \overline{N}(r, \varphi_f(z)) + \overline{N}\left(r, \frac{1}{\varphi_f(z)}\right) + \overline{N}\left(r, \frac{1}{\varphi_f(z)-1}\right)\\ &\quad + O(\log(rT(r, \varphi_f(z))))\\ &\le 3\overline{N}(r, f(z)) + \overline{N}\left(r, \frac{1}{\psi_f(z)-b}\right) + \frac{1}{n}T(r, \varphi_f(z)) + \overline{N}\left(r, \frac{1}{f(z)}\right)\\ &\quad + \overline{N}\left(r, \frac{1}{f(z+\eta)-b}\right) + o\left(\frac{T(r, f(z))}{r^{\delta_0}}\right) + O(\log(rT(r, f(z)))),\end{aligned} \tag{22}$$

and so

$$\begin{aligned}\left(1-\frac{1}{n}\right)T(r, \varphi_f(z)) &\le 3\overline{N}(r, f(z)) + \overline{N}\left(r, \frac{1}{\psi_f(z)-b}\right) + \overline{N}\left(r, \frac{1}{f(z)}\right)\\ &\quad + \overline{N}\left(r, \frac{1}{f(z+\eta)-b}\right) + o\left(\frac{T(r, f(z))}{r^{\delta_0}}\right)\\ &\quad + O(\log(rT(r, f(z))))\end{aligned} \tag{23}$$

as r runs to infinity outside of a set of finite logarithmic measure. Now we prove

$$\rho(f) = \rho(g) =: \rho < \infty. \tag{24}$$

Indeed, by Lemmas 2.1 and 2.4, and the assumptions of shared values of Theorem 1.6 we have

$$\begin{aligned}(n-1)T(r, f(z)) &\le \overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}\left(r, \frac{1}{f^n(z)+af(z+\eta)-b}\right) + 4\overline{N}(r, f(z))\\ &\quad + o\left(\frac{T(r, f(z))}{r^{\delta_0}}\right) + O(r^{\rho-1+\varepsilon}) + O(\log r)\\ &\le \overline{N}\left(r, \frac{1}{g(z)}\right) + \overline{N}\left(r, \frac{1}{g^n(z)+ag(z+\eta)-b}\right) + 4\overline{N}(r, g(z))\\ &\quad + o\left(\frac{T(r, f(z))}{r^{\delta_0}}\right) + O(r^{\rho-1+\varepsilon}) + O(\log r)\\ &\le (n+6)T(r, g(z)) + o\left(\frac{T(r, f(z))}{r^{\delta_0}}\right) + o\left(\frac{T(r, g(z))}{r^{\delta_0}}\right)\\ &\quad + O(r^{\rho-1+\varepsilon}) + O(\log r)\end{aligned} \tag{25}$$

and

$$(n-1)T(r,g(z)) \le (n+6)T(r,f(z)) + o\left(\frac{T(r,f(z))}{r^{\delta_0}}\right) + o\left(\frac{T(r,g(z))}{r^{\delta_0}}\right) + O(r^{\rho-1+\varepsilon}) + O(\log r) \tag{26}$$

as r runs to infinity outside of a set of finite logarithmic measure.

By (25), (26) and the standard reasoning of removing an exceptional set (cf. [13, Lemma 1.1.2]) we deduce $\rho(f) \le \rho(g)$ and $\rho(g) \le \rho(f)$, this gives (24). For a nonconstant meromorphic function w, next we define

$$2w^2(z)w'(z+\eta) + 2nw(z)w'(z)(b - w(z+\eta)) =: P(z, w(z)). \tag{27}$$

Assume that $P(z,f) = 0$. Then, by (27) we have

$$\frac{f'(z+\eta)}{f(z+\eta)-b} = \frac{nf'(z)}{f(z)} \tag{28}$$

for all $z \in \mathbb{C}$. By integrating two sides of (28) we have $f^n(z) = c(f(z+\eta)-b)$, where $c \neq 0$ is a constant. Similar to (18), we have a contradiction. Therefore, $P(z,f) \not\equiv 0$. Similarly, we have $P(z,g) \not\equiv 0$. For a nonconstant meromorphic function w, we also set

$$\frac{A(z,w)}{P(z,w)} =: Q(z,w), \tag{29}$$

where

$$\begin{aligned} &A(z,w(z)) \\ &= n(n-1){w'}^2(z)(b-w(z+\eta)) + nw(z)w''(z)(b-w(z+\eta)) + w^2(z)w''(z+\eta). \end{aligned} \tag{30}$$

Furthermore, we define

$$\frac{{\psi_f}'}{\psi_f - b} - \frac{{\psi_g}'}{\psi_g - b} =: D \quad \text{and} \quad D - Q(z,f) + Q(z,g) =: H. \tag{31}$$

We consider the following two cases:

Case 1. Suppose that $H \not\equiv 0$. First of all, we deduce an estimate for the counting function of the simple zeros of $\psi_f - b$. For this purpose, now we let z_0 be a common simple zero of $\psi_f - b$ and $\psi_g - b$. Then $\frac{\psi_f'}{\psi_f - b}$ has the Laurent expansion

$$\begin{aligned} \frac{\psi_f'(z)}{\psi_f(z)-b} &= \frac{\psi_f'(z_0) + \psi_f''(z_0)(z-z_0) + \cdots}{\psi_f'(z_0)(z-z_0) + \frac{1}{2}\psi_f''(z_0)(z-z_0)^2 + \cdots} \\ &= \frac{1}{z-z_0} + \frac{1}{2}\cdot\frac{\psi_f''}{\psi_f'}(z_0) + \cdots \end{aligned} \tag{32}$$

for each $z \in \Delta'(z_0,\delta)$, where $\Delta'(z_0,\delta) = \{z : 0 < |z-z_0| < \delta\}$, δ is some sufficiently small positive number. Since an analogous expansion holds for

$\frac{\psi_g'}{\psi_g - b}$, we obtain by the left equality of (31) that

$$D(z_0) = \frac{1}{2} \cdot \frac{\psi_f''}{\psi_f'}(z_0) - \frac{1}{2} \cdot \frac{\psi_g''}{\psi_g'}(z_0). \tag{33}$$

By noting (29) and $\psi_f(z_0) = f^n(z_0) + f(z_0 + \eta) = b$, we deduce

$$\frac{\psi_f''(z_0)}{\psi_f'(z_0)} = \frac{2A(z, f(z_0))}{P(z, f(z_0))} = 2Q(z, f(z_0)). \tag{34}$$

Similarly, we have

$$\frac{\psi_g''(z_0)}{\psi_g'(z_0)} = 2Q(z, g(z_0)). \tag{35}$$

By inserting (27), (33)-(35) into the right equality of (31), we have $H(z_0) = 0$. This together with the assumption $H \not\equiv 0$ gives

$$\begin{aligned} \overline{N}_{1)}\left(r, \frac{1}{\psi_f - b}\right) &= \overline{N}_{1)}\left(r, \frac{1}{\psi_g - b}\right) \le \overline{N}(r, \frac{1}{H}) \\ &\le T(r, H) + O(1) = m(r, H) + N(r, H) + O(1). \end{aligned} \tag{36}$$

Next we estimate $m(r, H)$ and $N(r, H)$ separately. Indeed, by (1), left equality of (31), (24) and the lemma on the logarithmic derivatives (cf. [13, Theorem 2.3.3]) we deduce

$$m(r, D) = O(\log r). \tag{37}$$

For a nonconstant meromorphic function w, we also set

$$\begin{aligned} &\frac{w'(z+\eta)}{w(z+\eta) - b} =: L(w(z)), \frac{w''(z+\eta)}{w(z+\eta) - b} =: \tilde{L}(w(z)) \quad \text{and} \\ &\frac{nw'}{w} - L(w) =: V(w). \end{aligned} \tag{38}$$

Then, we have $(L(w))' = \tilde{L}(w) - (L(w))^2$, and so it follows by (27), (29), (30) and (38) that

$$\begin{aligned} 2Q(z, w) &= \frac{n(n-1){w'}^2 + nww'' - w^2\tilde{L}(w)}{nww' - w^2 L(w)} \\ &= n \cdot \frac{w'}{w} + \frac{nww'' - w^2(L(w))' - w^2(L(w))^2 - n{w'}^2 + nw'L(w)}{nww' - w^2 L(w)} \\ &= n \cdot \frac{w'}{w} + L(w) + \frac{n\frac{w''}{w} - n\frac{{w'}^2}{w^2} - (L(w))'}{n\frac{w'}{w} - L(w)} \\ &= n \cdot \frac{w'}{w} + L(w) + \frac{V'(w)}{V(w)}. \end{aligned} \tag{39}$$

By the obtained result $P(z,w) \not\equiv 0$ we deduce by (27) and (38) that $V(w) \not\equiv 0$. In particular, $V(f) \not\equiv 0$ and $V(g) \not\equiv 0$. Moreover, by (24), (38), Lemma 2.3 and the lemma on logarithmic derivatives we deduce

$$T(r, V(f)) \leq AT(r,f) + O(r^{\rho-1+\varepsilon}) + O(\log r), \tag{40}$$

where $A > 0$ is a constant. By (38)-(40) we deduce

$$m(r, Q(z,f)) = O(\log r). \tag{41}$$

Similarly, we have

$$m(r, Q(z,g)) = O(\log r). \tag{42}$$

By (31), (37), (41) and (42) we have

$$m(r, H) = O(\log r). \tag{43}$$

Since ψ_f and ψ_g share b CM, f and g share 0 and ∞ CM, we have by (31) and (39) that

$$\begin{aligned} N(r, H(z)) &= N(r, D(z) - Q(z, f(z)) + Q(z, g(z))) \\ &\leq \overline{N}\left(r, \frac{1}{V(f(z))}\right) + \overline{N}\left(r, \frac{1}{V(g(z))}\right) + \overline{N}\left(r, \frac{1}{f(z+\eta)-b}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{g(z+\eta)-b}\right), \end{aligned} \tag{44}$$

where

$$\begin{aligned} &D - Q(z,f) + Q(z,g) \\ &= \frac{\psi_f'}{\psi_f - b} - \frac{1}{2}\left(\frac{nf'}{f} + L(f) + \frac{V'(f)}{V(f)}\right) - \frac{\psi_g'}{\psi_g - b} + \frac{1}{2}\left(\frac{ng'}{g} + L(g) + \frac{V'(g)}{V(g)}\right). \end{aligned} \tag{45}$$

By (24), (28), Lemma 2.1, the first fundamental theorem and the lemma on the logarithmic derivatives, we have

$$\begin{aligned} \overline{N}(r, \frac{1}{V(f(z))}) &\leq T\left(r, \frac{nf'(z)}{f(z)} - \frac{f'(z+\eta)}{f(z+\eta)-b}\right) + O(1) \\ &\leq \overline{N}(r, f(z)) + \overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}(r, f(z+\eta)) \\ &\quad + \overline{N}\left(r, \frac{1}{f(z+\eta)-b}\right) + O(\log r) \\ &= 2\overline{N}(r, f(z)) + \overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}\left(r, \frac{1}{f(z+\eta)-b}\right) \\ &\quad + o\left(\frac{T(r, f(z))}{r^{\delta_0}}\right) + O(\log r) \end{aligned} \tag{46}$$

and

$$\overline{N}(r, \frac{1}{V(g(z))}) \leq 2\overline{N}(r, g(z)) + \overline{N}\left(r, \frac{1}{g(z)}\right) + \overline{N}\left(r, \frac{1}{g(z+\eta)-b}\right)$$

$$+ o\left(\frac{T(r,g(z))}{r^{\delta_0}}\right) + O(\log r) \tag{47}$$

as r runs to infinity outside of a set of finite logarithmic measure. By inserting (46) and (47) into (44) we have

$$\begin{aligned} N(r,H(z)) \le{}& 2\overline{N}(r,f(z)) + 2\overline{N}(r,g(z)) + \overline{N}\left(r,\frac{1}{f(z)}\right) + \overline{N}\left(r,\frac{1}{g(z)}\right) \\ &+ 2\overline{N}\left(r,\frac{1}{f(z+\eta)-b}\right)) + 2\overline{N}\left(r,\frac{1}{g(z+\eta)-b}\right) \\ &+ o\left(\frac{T(r,f(z))}{r^{\delta_0}}\right) + o\left(\frac{T(r,g(z))}{r^{\delta_0}}\right) + O(\log r) \end{aligned} \tag{48}$$

as r runs to infinity outside of a set of finite logarithmic measure. By inserting (43) and (48) into (36) we have

$$\begin{aligned} \overline{N}_{1)}\left(r,\frac{1}{\psi_f(z)-b}\right) &= \overline{N}_{1)}\left(r,\frac{1}{\psi_g(z)-b}\right) \\ &\le 2\overline{N}(r,f(z)) + 2\overline{N}(r,g(z)) + \overline{N}\left(r,\frac{1}{f(z)}\right) + \overline{N}\left(r,\frac{1}{g(z)}\right) \\ &\quad + 2\overline{N}\left(r,\frac{1}{f(z+\eta)-b}\right) + 2\overline{N}\left(r,\frac{1}{g(z+\eta)-b}\right) \\ &\quad + o\left(\frac{T(r,f(z))}{r^{\delta_0}}\right) + o\left(\frac{T(r,g(z))}{r^{\delta_0}}\right) + O(\log r) \end{aligned} \tag{49}$$

as r runs to infinity outside of a set of finite logarithmic measure.

Next we let z_0 be a common multiple zero of $\psi_f - b$ and $\psi_g - b$. Then, we have

$$\psi_f(z_0) = f^n(z_0) + f(z_0+\eta) = b, \quad \psi_g(z_0) = g^n(z_0) + g(z_0+\eta) = b$$

and

$$nf^{n-1}(z_0)f'(z_0) + f'(z_0+\eta) = 0, \quad ng^{n-1}(z_0)g'(z_0) + g'(z_0+\eta) = 0.$$

Then we conclude that either

$$f(z_0) = g(z_0) = f'(z_0+\eta) = g'(z_0+\eta) = f(z_0+\eta)-b = g(z_0+\eta)-b = 0, \tag{50}$$

or

$$f(z_0) \ne 0, \quad f(z_0+\eta) \ne b, \quad g(z_0) \ne 0, \quad g(z_0+\eta) \ne b \tag{51}$$

such that

$$\frac{nf^{n-1}(z_0)f'(z_0)}{f^n(z_0)} = \frac{f'(z_0+\eta)}{f(z_0+\eta)-b} \quad \text{and} \quad \frac{ng^{n-1}(z_0)g'(z_0)}{g^n(z_0)} = \frac{g'(z_0+\eta)}{g(z_0+\eta)-b},$$

and so

$$V(f(z_0)) = n\cdot\frac{f'(z_0)}{f(z_0)} - \frac{f'(z_0+\eta)}{f(z_0+\eta)-b} = 0 \tag{52}$$

and

$$V(g(z_0)) = n \cdot \frac{g'(z_0)}{g(z_0)} - \frac{g'(z_0+\eta)}{g(z_0+\eta)-b} = 0. \tag{53}$$

By (46), (50)-(53) and the obtained result $V(f) \not\equiv 0$ we have

$$\begin{aligned}\overline{N}_{(2}\left(r, \frac{1}{\psi_f(z)-b}\right) &\le \overline{N}\left(\frac{1}{V(f(z))}\right) + \overline{N}_{(2}\left(r, \frac{1}{f(z+\eta)-b}\right)\\ &\le 2\overline{N}(r, f(z)) + \overline{N}\left(r, \frac{1}{f(z)}\right) + N_2\left(r, \frac{1}{f(z+\eta)-b}\right)\\ &\quad + o\left(\frac{T(r, f(z))}{r^{\delta_0}}\right) + O(\log r)\end{aligned} \tag{54}$$

as r runs to infinity outside of a set of finite logarithmic measure. Finally, by (49) and (54), we have

$$\begin{aligned}\overline{N}\left(r, \frac{1}{\psi_f(z)-b}\right) &= \overline{N}_{1)}\left(r, \frac{1}{\psi_f(z)-b}\right) + \overline{N}_{(2}\left(r, \frac{1}{\psi_f(z)-b}\right)\\ &\le 4\overline{N}(r, f(z)) + 2\overline{N}(r, g(z)) + 2\overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}\left(r, \frac{1}{g(z)}\right)\\ &\quad + 2\overline{N}\left(r, \frac{1}{f(z+\eta)-b}\right) + N_2\left(r, \frac{1}{f(z+\eta)-b}\right)\\ &\quad + 2\overline{N}\left(r, \frac{1}{g(z+\eta)-b}\right) + o\left(\frac{T(r, f(z))}{r^{\delta_0}}\right) + o\left(\frac{T(r, g(z))}{r^{\delta_0}}\right)\\ &\quad + O(\log r).\end{aligned} \tag{55}$$

By substituting (55) into (23) we have

$$\begin{aligned}&\left(1-\frac{1}{n}\right) T(r, \varphi_f(z))\\ &\le 7\overline{N}(r, f(z)) + 2\overline{N}(r, g(z)) + 3\overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}\left(r, \frac{1}{g(z)}\right)\\ &\quad + 3\overline{N}\left(r, \frac{1}{f(z+\eta)-b}\right) + N_2\left(r, \frac{1}{f(z+\eta)-b}\right) + 2\overline{N}\left(r, \frac{1}{g(z+\eta)-b}\right)\\ &\quad + o\left(\frac{T(r, f(z))}{r^{\delta_0}}\right) + o\left(\frac{T(r, g(z))}{r^{\delta_0}}\right) + O(\log r)\\ &\le 14T(r, f(z)) + 5T(r, g(z)) + o\left(\frac{T(r, f(z))}{r^{\delta_0}}\right) + o\left(\frac{T(r, g(z))}{r^{\delta_0}}\right) + O(\log r)\end{aligned} \tag{56}$$

as r runs to infinity outside of a set of finite logarithmic measure. Similarly,

$$\begin{aligned}&\left(1-\frac{1}{n}\right) T(r, \varphi_g)\\ &\le 14T(r, g) + 5T(r, f) + o\left(\frac{T(r, f)}{r^{\delta_0}}\right) + o\left(\frac{T(r, g)}{r^{\delta_0}}\right) + O(\log r)\end{aligned} \tag{57}$$

as r runs to infinity outside of a set of finite logarithmic measure. Now we set $T(r) = \max\{T(r,f), T(r,g)\}$ and $S(r) = o(T(r))$ as r runs to infinity outside of a set of finite logarithmic measure. Then, by (56) and (57) we have

$$\left(1-\frac{1}{n}\right)T(r,\varphi_f) \le 19T(r) + o\left(\frac{T(r)}{r^{\delta_0}}\right) + O(\log r) \tag{58}$$

and

$$\left(1-\frac{1}{n}\right)T(r,\varphi_g) \le 19T(r) + o\left(\frac{T(r)}{r^{\delta_0}}\right) + O(\log r) \tag{59}$$

as r runs to infinity outside of a set of finite logarithmic measure. On the other hand, by the left equality of (1) and the left equality of (17) we have

$$\frac{1}{\varphi_f(z)} = 1 + \frac{f(z+\eta)-b}{f^n(z)}. \tag{60}$$

By (24), (60) and Lemma 2.3 we have

$$\begin{aligned} nT(r,f(z)) = T(r,f^n(z)) &= T\left(r, \frac{f(z+\eta)-b}{f^n(z)} \cdot \frac{1}{f(z+\eta)-b}\right) + O(1) \\ &\le T\left(r, \frac{f(z+\eta)-b}{f^n(z)}\right) + T\left(r, \frac{1}{f(z+\eta)-b}\right) + O(1) \\ &= T\left(r, 1 + \frac{f(z+\eta)-b}{g^n(z)}\right) + T\left(r, \frac{1}{f(z+\eta)-b}\right) + O(1) \\ &= T(r,\varphi_f(z)) + T(r,f(z)) + O(r^{\rho-1+\varepsilon}) + O(1), \end{aligned}$$

i.e.,

$$(n-1)T(r,f) \le T(r,\varphi_f) + O(r^{\rho-1+\varepsilon}) + O(1). \tag{61}$$

Similarly,

$$(n-1)T(r,g) \le T(r,\varphi_g) + O(r^{\rho-1+\varepsilon}) + O(1). \tag{62}$$

By (58) and (61) we have

$$\begin{aligned} \left(1-\frac{1}{n}\right)(n-1)T(r,f) &\le (1-\frac{1}{n})T(r,\varphi_f) + O(r^{\rho-1+\varepsilon}) + O(1) \\ &\le 19T(r) + o\left(\frac{T(r)}{r^{\delta_0}}\right) + O(r^{\rho-1+\varepsilon}) + O(\log r), \end{aligned}$$

i.e.,

$$(n-1)^2T(r,f) \le 19nT(r) + o\left(\frac{T(r)}{r^{\delta_0}}\right) + O(r^{\rho-1+\varepsilon}) + O(\log r) \tag{63}$$

as r runs to infinity outside of a set of finite logarithmic measure. Similarly, by (59) and (62) we have

$$(n-1)^2T(r,g) \le 19nT(r) + o\left(\frac{T(r)}{r^{\delta_0}}\right) + O(r^{\rho-1+\varepsilon}) + O(\log r) \tag{64}$$

as r runs to infinity outside of a set of finite logarithmic measure. By (63) and (64) we have

$$(n-1)^2T(r) \leq 19nT(r) + o\left(\frac{T(r)}{r^{\delta_0}}\right) + O(r^{\rho-1+\varepsilon}) + O(\log r) \tag{65}$$

as r runs to infinity outside of a set of finite logarithmic measure. By (24), the assumption $n \geq 21$, the standard reasoning of removing an exceptional set (cf. [13, Lemma 1.1.2]) and the definition of the order of a meromorphic function, we deduce

$$\rho \leq \limsup_{r\to\infty} \frac{\log T(r)}{\log r} \leq \rho - 1,$$

which is impossible.

Case 2. Suppose that $H = 0$. Then, by (31) we have

$$2H = \frac{2\psi_f'}{\psi_f - b} - \frac{2\psi_g'}{\psi_g - b} - 2Q(z,f) + 2Q(z,g) = 0. \tag{66}$$

By substituting (38) and (39) into (66) and then taking the integration on two sides of (66) we deduce

$$\left(\frac{\psi_f(z)-b}{\psi_g(z)-b}\right)^2 = \frac{cf^n(z)}{g^n(z)} \cdot \frac{f(z+\eta)-b}{g(z+\eta)-b} \cdot \frac{V(f(z))}{V(g(z))}, \tag{67}$$

where $c \neq 0$ is some constant. We consider the following two subcases:

Subcase 2.1. Assume that $V(f) \not\equiv cV(g)$. Now we let $z_0 \in \mathbb{C}$ be a common zero of $\psi_f - b$ and $\psi_g - b$ such that z_0 is neither a zero nor a pole of f and g.

Suppose that z_0 is a common simple zero $\psi_f - b$ and $\psi_g - b$. Then, we have $f^n(z_0)+f(z_0+\eta) = g^n(z_0)+g(z_0+\eta) = b$. Combining this with the assumption that f and g share $0, \infty$ CM, we deduce that z_0 is neither a zero or a pole of $f(z+\eta)-b$ and $g(z+\eta)-b$. Therefore,

$$\begin{aligned}\frac{\psi_f(z_0)-b}{\psi_g(z_0)-b} = \frac{\psi_f'(z_0)}{\psi_g'(z_0)} &= \frac{nf^{n-1}(z_0)f'(z_0)+f'(z_0+\eta)}{ng^{n-1}(z_0)g'(z_0)+g'(z_0+\eta)} \\ &= \frac{f^n(z_0)}{g^n(z_0)} \cdot \frac{n\cdot\frac{f'(z_0)}{f(z_0)} - \frac{f'(z_0+\eta)}{f(z_0+\eta)-b}}{n\cdot\frac{g'(z_0)}{g(z_0)} - \frac{g'(z_0+\eta)}{g(z_0+\eta)-b}} = \frac{f^n(z_0)}{g^n(z_0)} \cdot \frac{V(f(z_0))}{V(g(z_0))},\end{aligned}$$

which together with (67) gives

$$\begin{aligned}\frac{f^{2n}(z_0)}{g^{2n}(z_0)} \cdot \left(\frac{V(f(z_0))}{V(g(z_0))}\right)^2 &= \frac{cf^n(z_0)}{g^n(z_0)} \cdot \frac{f(z_0+\eta)-b}{g(z_0+\eta)-b} \cdot \frac{V(f(z_0))}{V(g(z_0))} \\ &= \frac{cf^{2n}(z_0)}{g^{2n}(z_0)} \cdot \frac{V(f(z_0))}{V(g(z_0))}.\end{aligned} \tag{68}$$

By the above analysis we know that z_0 is neither a zero or a pole of $\frac{f^n}{g^n}$ and $\frac{\psi_f-b}{\psi_g-b}$, we deduce by (67) that z_0 is neither a zero nor a pole of $V(f)/V(g)$. Furthermore, by (38) and the above analysis we can see that z_0 is not a pole

of $V(f)$ and $V(g)$. Therefore, by (68) we deduce $V(f(z_0))/V(g(z_0)) = c$, and so $V(f(z_0)) - cV(g(z_0)) = 0$.

Suppose that z_0 is a common multiple zero of $\psi_f - b$ and $\psi_g - b$. Then, by (50)-(53) and the assumption that f and g share $0, \infty$ CM, we deduce that z_0 is either a common zero of f and g or a common zero of $V(f)$ and $V(g)$, and so a zero of $V(f) - cV(g)$.

Therefore, by Lemma 2.1, the assumption of shared values and the above analysis we deduce

$$\begin{aligned}
&\overline{N}\left(\frac{1}{\psi_f(z)-b}\right)\\
&\le \overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}\left(r, \frac{1}{V(f(z)) - cV(g(z))}\right) + \overline{N}(r, f(z))\\
&\le \overline{N}\left(r, \frac{1}{f(z)}\right) + T(r, V(f(z)) - cV(g(z))) + \overline{N}(r, f(z)) + O(1)\\
&\le 5\overline{N}(r, f(z)) + 3\overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}\left(r, \frac{1}{f(z+\eta)-b}\right)\\
&\quad + \overline{N}\left(r, \frac{1}{g(z+\eta)-b}\right) + O(r^{\rho-1+\varepsilon}).
\end{aligned}$$

Next, in the same manner as the proceeding from (55) to the end of Case 1, we have a contradiction by the assumption that $n \ge 21$.

Subcase 2.2. Assume that $V(f) = cV(g)$. Then, (67) can be rewritten as

$$\left(\frac{\psi_f(z)-b}{\psi_g(z)-b}\right)^2 = \frac{c^2 f^n(z)}{g^n(z)} \cdot \frac{f(z+\eta)-b}{g(z+\eta)-b}. \tag{69}$$

By the assumption that $f(z)$ and $g(z)$ share $0, \infty$ CM, we know that $f(z+\eta)$ and $g(z+\eta)$ share $0, \infty$ CM. Combining this with (1), (69) and the assumption that ψ_f and ψ_g share b CM, we deduce $f(z+\eta)$ and $g(z+\eta)$ share b CM. Therefore,

$$\frac{f}{g} = e^{\alpha}, \quad \frac{f(z+\eta)-b}{g(z+\eta)-b} = e^{\beta(z)}, \quad \frac{f^n(z)+f(z+\eta)-b}{g^n(z)+g(z+\eta)-b} = e^{\gamma(z)}, \tag{70}$$

where α, β, and γ are entire functions. By (24), (70) and Lemma 2.3 we deduce that α, β, and γ are polynomials of order not more than ρ.

Next we first prove that e^{α} is a nonconstant entire function. Indeed, assume that e^{α} is a constant, say $e^{\alpha} = c_1$. Then, by (70) we have $f = c_1 g$. Combining this with the supposition $f \not\equiv g$ and the assumption that f and g share b CM, we deduce that $c_1 \notin \{0, 1\}$ such that b and bc_1 are two distinct Picard exceptional values of f. On the other hand, by inserting $f = c_1 g$ into the third equality of (70) we have

$$(c_1^n - e^{\gamma(z)})g^n(z) = (e^{\gamma(z)} - c_1)g(z+\eta) + b(1 - e^{\gamma(z)}) \tag{71}$$

for all $z \in \mathbb{C}$.

Suppose that $c_1^n - e^\gamma \not\equiv 0$, and that γ is not a constant. Then, by (71) we have

$$g^n(z) = \frac{(e^{\gamma(z)} - c_1)g(z+\eta)}{c_1^n - e^{\gamma(z)}} + \frac{b(1-e^{\gamma(z)})}{c_1^n - e^{\gamma(z)}} \tag{72}$$

for all $z \in \mathbb{C}$. By (72), Lemma 2.1 and the assumption $n \geq 21$ we deduce

$$\begin{aligned} nN(r, g(z)) &\leq N(r, g(z+\eta)) + N_{(n}\left(r, \frac{1}{c_1^n - e^{\gamma(z)}}\right) \\ &\leq N(r, g(z)) + 2N\left(r, \frac{1}{(c_1^n - e^{\gamma(z)})'}\right) + o\left(\frac{T(r, g(z))}{r^{\delta_0}}\right) \\ &= N(r, g(z)) + 2N\left(r, \frac{1}{\gamma'(z)}\right) + o\left(\frac{T(r, g(z))}{r^{\delta_0}}\right) \\ &\leq N(r, g(z)) + o\left(\frac{T(r, g(z))}{r^{\delta_0}}\right) + O(\log r), \end{aligned}$$

i.e.,

$$N(r, g) = o\left(\frac{T(r, g)}{r^{\delta_0}}\right) + O(\log r) \tag{73}$$

as r runs to infinity outside of a set of finite logarithmic measure. By (24), (73) and the assumption that f and g share ∞ CM, we have

$$N(r, f) = N(r, g) = o\left(\frac{T(r, g)}{r^{\delta_0}}\right) + O(\log r) \leq O\left(r^{\rho - \delta_0 + \varepsilon}\right) + O(\log r) \tag{74}$$

as r runs to infinity outside of a set of finite logarithmic measure. By (74) and the obtained result that b and bc_1 are two distinct Picard exceptional values of f, and the second fundamental theorem we have

$$\begin{aligned} T(r, f) &\leq \overline{N}\left(r, \frac{1}{f-b}\right) + \overline{N}\left(r, \frac{1}{f - bc_1}\right) + \overline{N}(r, f) + O(\log r) \\ &\leq O\left(r^{\rho - \delta_0 + \varepsilon}\right) + O(\log r) \end{aligned} \tag{75}$$

as r runs to infinity outside of a set of finite logarithmic measure. By (24), (75) and the standard reasoning of removing an exceptional set (cf. [13, Lemma 1.1.2]) and the definition of the order of a meromorphic function, we deduce

$$\rho = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r} \leq \rho - \delta_0 + \varepsilon, \tag{76}$$

which is impossible.

Suppose that γ is a constant, and that $c_1^n - e^\gamma \neq 0$. Then, by (72) we deduce (73)-(76), and so we have a contradiction.

Suppose that $c_1^n - e^\gamma = 0$. Then, by (71) we have

$$(c_1 - c_1^n)g(z+\eta) = b(1 - c_1^n) \tag{77}$$

for all $z \in \mathbb{C}$. If $c_1 = c_1^n$, by (77) and the assumption $b \neq 0$ we have $c_1^n = c_1 = 1$, which contradicts the obtained result $c_1 \neq 1$. Hence, $c_1 \neq c_1^n$, and (77) can be rewritten as

$$g(z+\eta) = \frac{b(1-c_1^n)}{c_1 - c_1^n}, \tag{78}$$

which contradicts the fact that $g(z+\eta)$ is a nonconstant meromorphic function.

Secondly we prove that e^β is a nonconstant entire function. Indeed, we assume that e^β is a constant, say $e^\beta = c_2$. Then, by the second equation of (70) we have $f - b = c_2(g-b)$. Combining this with the supposition $f \not\equiv g$ and the obtained result that f and g share 0, b CM, we deduce that $c_2 \notin \{0,1\}$ such that 0 and $b(1-c_2)$ are two distinct Picard exceptional values of f. Therefore, by (70) we deduce

$$\frac{e^{n\alpha(z)}g^n(z) + c_2(g(z+\eta)-b)}{g^n(z) + g(z+\eta) - b} = e^{\gamma(z)},$$

and so

$$(e^{n\alpha(z)} - e^{\gamma(z)})g^n(z) = (e^{\gamma(z)} - c_2)g(z+\eta) + b(c_2 - e^{\gamma(z)}) \tag{79}$$

for all $z \in \mathbb{C}$.

Suppose that $e^{n\alpha} - e^\gamma \not\equiv 0$. Then

$$g^n(z) = \frac{(e^{\gamma(z)} - c_2)g(z+\eta)}{e^{n\alpha(z)} - e^{\gamma(z)}} + \frac{b(c_2 - e^{\gamma(z)})}{e^{n\alpha(z)} - e^{\gamma(z)}}$$

for all $z \in \mathbb{C}$. Next, in the same manner as from (72) to (78) we have a contradiction.

Suppose that $e^{n\alpha} - e^\gamma = 0$. Then, by (70) we have

$$e^{n\alpha(z)} = e^{\gamma(z)} = \frac{f^n(z)}{g^n(z)} = \frac{f^n(z) + f(z+\eta) - b}{g^n(z) + g(z+\eta) - b} = \frac{f(z+\eta) - b}{g(z+\eta) - b} = c_2,$$

which contradicts the obtained result that e^α is a nonconstant entire function.

By rewriting the third equation of (70), we have

$$f^n(z) + f(z+\eta) - e^{\gamma(z)}g^n(z) - e^{\gamma(z)}(g(z+\eta) - b) = b,$$

and so

$$f_1 + f_2 + f_3 + f_4 = b, \tag{80}$$

where and in what follows,

$$f_1 = f^n,\ f_2(z) = f(z+\eta),\ f_3 = -e^\gamma g^n,\ f_4(z) = -e^{\gamma(z)}(g(z+\eta) - b). \tag{81}$$

By (24), (41), (44) and Lemma 2.3 and the obtained result that $f(z+\eta)$ and $g(z+\eta)$ share 0, b, ∞ CM, we deduce for $1 \le j \le 4$ that

$$T(r, f_j) = O(T(r,f)) + O(r^{\rho-1+\varepsilon}), \quad T(r) = O(T(r,f)) \tag{82}$$

as $r \to \infty$. Here and in what follows, $T(r) = \max\limits_{1\le j\le 4}\{T(r,f_j)\}$.

We consider the following two subcases.

(2.2.1) Suppose that f_1, f_2, f_3, f_4 are linearly independent. By (80), (81), Lemmas 2.3 and 2.5, the assumption that f and g share 0, ∞ CM, and the obtained result that $f(z+\eta)$ and $g(z+\eta)$ share b CM, we deduce

$$\begin{aligned}
nT(r, f(z)) &= T(r, f_1(z)) \\
&\le \sum_{j=1}^{4} N_3\left(r, \frac{1}{f_j(z)}\right) + 3\sum_{j=2}^{4} \overline{N}(r, f_j(z)) + o(T(r, f(z))) \\
&\le 3\overline{N}\left(r, \frac{1}{f(z)}\right) + 3\overline{N}\left(r, \frac{1}{g(z)}\right) + N\left(r, \frac{1}{f(z+\eta)}\right) \\
&\quad + N\left(r, \frac{1}{g(z+\eta)-b}\right) + 3\overline{N}(r, f(z+\eta)) + 3\overline{N}(r, g(z)) \\
&\quad + 3\overline{N}(r, g(z+\eta))) + o(T(r, f(z))) \\
&= 6\overline{N}\left(r, \frac{1}{f(z)}\right) + N\left(r, \frac{1}{f(z+\eta)}\right) + N\left(r, \frac{1}{g(z+\eta)-b}\right) \\
&\quad + 6\overline{N}(r, f(z+\eta)) + 3\overline{N}(r, f(z)) + o(T(r, f(z))) \\
&\le 17T(r, f(z)) + O(r^{\rho-1+\varepsilon}) + o(T(r, f(z))) \qquad (83)
\end{aligned}$$

as r runs to infinity outside of a set of finite linear measure. By (83) we deduce $n \le 17$, which contradicts the assumption $n \ge 21$.

(2.2.2) Suppose that f_1, f_2, f_3, f_4 are linearly dependent. Then we have

$$c_1 f_1 + c_2 f_2 + c_3 f_3 + c_4 f_4 = 0, \qquad (84)$$

where and in what follows, c_1, c_2, c_3, c_4 are constants satisfying $\sum_{j=1}^{4} |c_j| > 0$. By substituting (81) into (84) we have

$$c_1 f^n(z) + c_2 f(z+\eta) - e^{\gamma(z)}(c_3 g^n(z) + c_4(g(z+\eta) - b)) = 0 \qquad (85)$$

for all $z \in \mathbb{C}$. Obviously, c_1 and c_2 are constants satisfying $|c_1| + |c_2| > 0$. Indeed, if $c_1 = c_2 = 0$, by (85) we have

$$c_3 g^n(z) + c_4(g(z+\eta) - b) = 0 \qquad (86)$$

for all $z \in \mathbb{C}$. By noting that $|c_3| + |c_4| > 0$ and that $g(z+\eta)$ is a nonconstant meromorphic function, we have a contradiction by (86), Lemma 2.3 and the assumption $n \ge 21$. Similarly we have $|c_3| + |c_4| > 0$ by (85).

We consider the following nine subcases:

(2.2.2.1) Suppose that $c_1 \ne 0$, $c_2 = 0$, $c_3 \ne 0$, $c_4 = 0$. Then, by (85) we have $c_1 f^n = c_3 e^{\gamma} g^n$. This together with the third equation of (70) gives

$$\frac{c_1 f^n(z)}{c_3 g^n(z)} = e^{\gamma(z)} = \frac{f^n(z) + f(z+\eta) - b}{g^n(z) + g(z+\eta) - b} \qquad (87)$$

for all $z \in \mathbb{C}$. Assume that $c_1 = c_3$. Then, by (1) and (87) we deduce the conclusion of Theorem 1.6. Next we suppose that $c_1 \ne c_3$. Then, by (87) and

the first equation of (70) we have

$$(c_1 - c_3)f^n(z) + c_1 e^{n\alpha(z)}(g(z+\eta) - b) - c_3 f(z+\eta) = -bc_3 \tag{88}$$

for all $z \in \mathbb{C}$. Now we set

$$\widetilde{f}_1 = (c_1 - c_3)f^n,\ \widetilde{f}_2(z) = c_1 e^{n\alpha(z)}(g(z+\eta) - b),\ \widetilde{f}_3(z) = -c_3 f(z+\eta). \tag{89}$$

We consider the following two cases:

(2.2.2.1.1) Suppose that $\widetilde{f}_1$, $\widetilde{f}_2$, $\widetilde{f}_3$ are linearly independent. Then, by Lemma 2.5 we have a contradiction.

(2.2.2.1.2) Suppose that $\widetilde{f}_1$, $\widetilde{f}_2$, $\widetilde{f}_3$ are linearly dependent. Then, there exist three constants d_1, d_2, d_3 satisfying $\sum_{j=1}^{3} |d_j| > 0$, such that

$$d_1\widetilde{f}_1 + d_2\widetilde{f}_2 + d_3\widetilde{f}_3 = 0. \tag{90}$$

By substituting (89) into (90) we have

$$d_1(c_1 - c_3)f^n(z) + d_2 c_1 e^{n\alpha(z)}(g(z+\eta) - b) - d_3 c_3 f(z+\eta) = 0 \tag{91}$$

for all $z \in \mathbb{C}$. By (91) and Lemma 2.3 we deduce $d_2 \neq 0$. Now we prove $d_3 \neq 0$. Indeed, if $d_3 = 0$, by (91) we deduce $d_1 \neq 0$ and that (91) can be rewritten as

$$d_1(c_1 - c_3)f^n(z) + d_2 c_1 g(z+\eta)e^{n\alpha(z)} = d_2 c_1 b. \tag{92}$$

By (92), Lemmas 2.3 and 2.6 and the obtained result that $f(z+\eta)$ and $g(z+\eta)$ share 0, b, ∞ CM, we have

$$\begin{aligned} nT(r, f(z)) &= T(r, f^n(z)) + O(1) \\ &\leq \overline{N}(r, f^n(z)) + \overline{N}\left(r, \frac{1}{f^n(z)}\right) + \overline{N}\left(r, \frac{1}{g(z+\eta)e^{n\alpha(z)}}\right) + S(r, f(z)) \\ &= \overline{N}(r, f(z)) + \overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}\left(r, \frac{1}{f(z+\eta)}\right) + S(r, f(z)) \\ &\leq 3T(r, f(z)) + O(r^{\rho-1+\varepsilon}) + O(1) \end{aligned} \tag{93}$$

as $r \to \infty$. By (24), (93) and the definition of the order of a meromorphic function we deduce $n \leq 3$, which contradicts the assumption $n \geq 21$.

We consider the following two subcases:

(2.2.2.1.2.1) Suppose that $d_1 = 0$, $d_2 \neq 0$ and $d_3 \neq 0$. Then, by (91) and the first equality of (70) we deduce

$$\frac{d_2 c_1(g(z+\eta) - b)}{d_3 c_3 g(z+\eta)} = e^{\alpha(z+\eta) - n\alpha(z)} \tag{94}$$

for all $z \in \mathbb{C}$. By Lemma 2.3 and the obtained result that e^{α} is a nonconstant entire function of finite order, we deduce that $e^{\alpha(z+\eta)-n\alpha(z)}$ is a nonconstant entire function. Moreover, by the obtained result that $f(z+\eta)$ and $g(z+\eta)$

share 0, b, ∞ CM, we deduce by (94) that 0 and b are two Picard exceptional values of g. By (94) we have

$$\frac{g(z+\eta)-b}{g(z+\eta)} = \frac{d_3c_3}{d_2c_1}e^{\alpha(z+\eta)-n\alpha(z)} =: e^{\delta(z+\eta)} \tag{95}$$

for all $z \in \mathbb{C}$. This together with the first equation of (70) gives

$$g(z+\eta) = \frac{b}{1-e^{\delta(z+\eta)}}, \quad f(z+\eta) = \frac{be^{\alpha(z+\eta)}}{1-e^{\delta(z+\eta)}} \tag{96}$$

for all $z \in \mathbb{C}$. By substituting (96) into the second equation of (70) we have

$$e^{\alpha(z+\eta)} + e^{\delta(z+\eta)} - e^{\delta(z+\eta)+\beta(z+\eta)} = 1 \tag{97}$$

for all $z \in \mathbb{C}$. By (97) and Lemma 2.7, we have

$$e^{\delta(z+\eta)+\beta(z+\eta)} = -1 \quad \text{and} \quad e^{\alpha(z+\eta)} + e^{\delta(z+\eta)} = 0$$

for all $z \in \mathbb{C}$, thus $f(z+\eta)+g(z+\eta) = b$, and so $g(z+\eta)-b = -f(z+\eta)$ for all $z \in \mathbb{C}$. Combining this with (95), we have

$$\frac{d_3c_3}{d_2c_1}e^{\alpha(z+\eta)-n\alpha(z)} = e^{\delta(z+\eta)} = -1$$

for all $z \in \mathbb{C}$, which contradicts the above obtained result.

(2.2.2.1.2.2) Suppose that $d_1 \neq 0$, $d_2 \neq 0$ and $d_3 \neq 0$. Then, (91) can be rewritten as

$$\frac{d_1(c_1-c_3)f^n(z)}{f(z+\eta)} + \frac{d_2c_1e^{n\alpha(z)}(g(z+\eta)-b)}{f(z+\eta)} = d_3c_3. \tag{98}$$

By Lemma 2.3 we have

$$\begin{aligned} nT(r,f(z)) &= T(r,f^n(z)) \\ &\leq T\left(r, \frac{f^n(z)}{f(z+\eta)}\right) + T(r,f(z+\eta)) \\ &= T\left(r, \frac{f^n(z)}{f(z+\eta)}\right) + T(r,f(z)) + O(r^{\rho-1+\varepsilon}) + O(\log r), \end{aligned}$$

and so

$$(n-1)T(r,f(z)) \leq T\left(r, \frac{f^n(z)}{f(z+\eta)}\right) + O(r^{\rho-1+\varepsilon}) + O(\log r),$$

this together with (98), Lemma 2.6 and the obtained result that $f(z+\eta)$ and $g(z+\eta)$ share 0, b, ∞ CM gives

$$\begin{aligned} (n-1)T(r,f(z)) &\leq T\left(r, \frac{f^n(z)}{f(z+\eta)}\right) + O(r^{\rho-1+\varepsilon}) + O(\log r) \\ &= T\left(r, \frac{d_1(c_1-c_3)f^n(z)}{f(z+\eta)}\right) + O(r^{\rho-1+\varepsilon}) + O(\log r) + O(1) \\ &\leq \overline{N}\left(r, \frac{f(z+\eta)}{d_1(c_1-c_3)f^n(z)}\right) + \overline{N}\left(r, \frac{f(z+\eta)}{d_2c_1e^{n\alpha(z)}(g(z+\eta)-b)}\right) \end{aligned}$$

$$
\begin{aligned}
&+\overline{N}\left(r, \frac{d_1(c_1-c_3)f^n(z)}{f(z+\eta)}\right)+O(r^{\rho-1+\varepsilon})+O(\log r)\\
&=2\overline{N}(r, f(z+\eta))+\overline{N}\left(r, \frac{1}{f(z)}\right)+\overline{N}\left(r, \frac{1}{f(z+\eta)-b}\right)\\
&\quad+\overline{N}(r, f(z))+\overline{N}\left(r, \frac{1}{f(z+\eta)}\right)+O(r^{\rho-1+\varepsilon})+O(\log r)\\
&\le 2T(r, f(z))+4T(r, f(z+\eta))+O(r^{\rho-1+\varepsilon})+O(\log r)\\
&=6T(r, f(z))+O(r^{\rho-1+\varepsilon})+O(\log r),
\end{aligned}
$$

i.e.,

$$
(n-7)T(r, f(z)) \le O(r^{\rho-1+\varepsilon})+O(\log r). \tag{99}
$$

By (99), the assumption $n \ge 21$ and the definition of the order of a meromorphic function we deduce $\rho(f)=\rho \le \rho-1$, which is impossible.

(2.2.2.2) Suppose that $c_1 \ne 0$, $c_2 \ne 0$, $c_3 \ne 0$ and $c_4 = 0$. Then, (85) can be rewritten as

$$
c_1 f^n(z)+c_2 f(z+\eta)-c_3 e^{\gamma(z)} g^n(z)=0,
$$

and so

$$
\frac{c_3 e^{\gamma(z)} g^n(z)}{f(z+\eta)}-\frac{c_1 f^n(z)}{f(z+\eta)}=c_2 \tag{100}
$$

for all $z \in \mathbb{C}$. Next, in the same manner as in subcase (2.2.2.1.2.2), we get a contradiction by (100), Lemma 2.3, Lemma 2.6 and the assumption $n \ge 21$.

(2.2.2.3) Suppose that $c_1 \ne 0$, $c_2 = 0$, $c_3 = 0$ and $c_4 \ne 0$. Then, (85) can be rewritten as

$$
c_1 f^n(z)-c_4 e^{\gamma(z)}(g(z+\eta)-b)=0,
$$

and so

$$
\frac{f^n(z)}{g(z+\eta)-b}=\frac{c_4}{c_1} e^{\gamma(z)} \tag{101}
$$

for all $z \in \mathbb{C}$. By (101) we know that $f^n(z)$ and $g(z+\eta)-b$ share 0 and ∞ CM such that each zero of $g(z+\eta)-b$ is of multiplicity not less than n, and that each pole of $g(z+\eta)$ is of multiplicity not less than n. Combining this with the obtained result that $f(z+\eta)$ and $g(z+\eta)$ share 0, b, ∞ CM, we deduce that F_1 and G_1 share 0, 1, ∞ CM. Moreover, each zero of F_1 is of multiplicity not less than n, and that each 1-point of F_1 is of multiplicity not less than n. This together with the assumption $n \ge 21$ gives

$$
\delta_{1)}(0, F_1)=1-\limsup_{r\to\infty} \frac{\overline{N}_{1)}\left(r, \frac{1}{F_1}\right)}{T(r, F_1)}=1 \tag{102}
$$

and

$$\delta_{1)}(1, F_1) = 1 - \limsup_{r\to\infty} \frac{\overline{N}_{1)}\left(r, \frac{1}{F_1-1}\right)}{T(r, F_1)} = 1, \tag{103}$$

where

$$F_1(z) = \frac{b}{f(z+\eta)}, \quad G_1(z) = \frac{b}{g(z+\eta)}, \tag{104}$$

which implies that $f(z+\eta)$ and $g(z+\eta)$ are entire functions, and so $f(z)$ and $g(z)$ are entire functions. By Lemma 2.8 and (102)-(104) we deduce $F_1+G_1 = 1$, and so

$$bf(z+\eta) + bg(z+\eta) = f(z+\eta)g(z+\eta) \tag{105}$$

for all $z \in \mathbb{C}$. On the other hand, by (101) and the third equality of (70) we deduce

$$\frac{\frac{c_1}{c_4}\cdot f^n(z)}{g(z+\eta)-b} = \frac{f^n(z)+f(z+\eta)-b}{g^n(z)+g(z+\eta)-b} = \frac{(1-\frac{c_1}{c_4})f^n(z)+f(z+\eta)-b}{g^n(z)} = e^{\gamma(z)}$$

for all $z \in \mathbb{C}$, which implies that

$$\left(1-\frac{c_1}{c_4}\right)f^n(z) + f(z+\eta) - \frac{c_1}{c_4}\cdot\frac{f^n(z)g^n(z)}{g(z+\eta)-b} = b \tag{106}$$

for all $z \in \mathbb{C}$. By (105) and (106) we deduce

$$\begin{aligned}\frac{c_1}{c_4}b^{n-2}f^{2n}(z) = &\left(1-\frac{c_1}{c_4}\right)\frac{f(z)-b}{f(z+\eta)-b}\cdot f^n(z)(f(z)-b)^{n-1}\\ &+\frac{f(z)-b}{f(z+\eta)-b}\cdot(f(z)-b)^{n-1}f(z+\eta)\\ &-\frac{b(f(z)-b)}{f(z+\eta)-b}\cdot(f(z)-b)^{n-1}\end{aligned} \tag{107}$$

for all $z \in \mathbb{C}$. By (107), Lemmas 2.3 and 2.9 we deduce

$$T(r,f) = m(r,f) + O(r^{\rho-1+\varepsilon}) + O(\log r) = O(r^{\rho-1+\varepsilon}) + O(\log r). \tag{108}$$

By (108) and the definition of the order of an entire function we deduce $\rho(f) = \rho \le \rho - 1$, which is impossible.

(2.2.2.4) Suppose that $c_1 \neq 0$, $c_2 = 0$, $c_3 \neq 0$, $c_4 \neq 0$. By rewriting (85) we have

$$-c_3g^n(z) + c_1f^n(z)e^{-\gamma(z)} - c_4g(z+\eta) = -c_4b \tag{109}$$

for all $z \in \mathbb{C}$. By (109) and the first equality of (70) we have

$$(c_1e^{-\gamma(z)} - c_3e^{-n\alpha(z)})f^n(z) = c_4(g(z+\eta)-b) \tag{110}$$

for all $z \in \mathbb{C}$. By noting that $g(z+\eta)-b$ is a transcendental meromorphic function, we deduce by (110) that $c_1e^{-\gamma(z)} - c_3e^{-n\alpha(z)} \not\equiv 0$, and so (110) can be rewritten as

$$f^n(z) = \frac{c_4(g(z+\eta)-b)}{c_1e^{-\gamma(z)} - c_3e^{-n\alpha(z)}}. \tag{111}$$

Next we set (104). Then, by (104) and the obtained result that $f(z+\eta)$ and $g(z+\eta)$ share 0, b, ∞ CM we deduce that F_1 and G_1 share 0, 1, ∞ CM.

Suppose that $c_1e^{-\gamma} - c_3e^{-n\alpha}$ is a nonzero constant. Then, by (111) we deduce that each zero of $g(z+\eta)-b$ is of multiplicity not less than n, and that each pole of $g(z+\eta)$ is of multiplicity not less than n. Therefore

$$\delta_{1)}(0, F_1) = \delta_{1)}(1, F_1) = 1. \tag{112}$$

By (104), (112), Lemma 2.8 and the obtained result that F_1 and G_1 share 0, 1, ∞ CM we have $F_1 + G_1 = 1$, and so we have (105). Next, in the same manner as in (2.2.2.3) we get a contradiction from (105).

Suppose that $c_1e^{-\gamma} - c_3e^{-n\alpha}$ is a nonconstant entire function. Then, by noting that γ and α are polynomials, we deduce

$$\begin{aligned} N_{(2}\left(r, \frac{1}{c_1e^{-\gamma} - c_3e^{-n\alpha}}\right) &= N_{(2}\left(r, \frac{1}{e^{-\gamma}(c_1 - c_3e^{\gamma-n\alpha})}\right) \\ &= N_{(2}\left(r, \frac{1}{c_1 - c_3e^{\gamma-n\alpha}}\right) \\ &\le 2N\left(r, \frac{1}{(c_1 - c_3e^{\gamma-n\alpha})'}\right) = 2N\left(r, \frac{1}{\gamma' - n\alpha'}\right) \\ &\le O(\log r), \end{aligned}$$

this together with (111) implies that

$$\overline{N}_{1)}\left(r, \frac{1}{g(z+\eta)-b}\right) = O(\log r). \tag{113}$$

By (111) we can see that each pole of $g(z+\eta)$ is of multiplicity not less than $n \ge 21$. Therefore,

$$\overline{N}_{1)}\left(r, g(z+\eta)\right) = 0. \tag{114}$$

Now we set (104). Then, by the obtained result that F_1 and G_1 share 0, 1, ∞ CM, we deduce (102) and (103) by (113), (114) and the assumption that g is a transcendental meromorphic function. Next, in the same manner as in (2.2.2.3) we get a contradiction.

(2.2.2.5) Suppose that $c_1 \ne 0$, $c_2 \ne 0$, $c_3 = 0$ and $c_4 \ne 0$. Then, (85) can be rewritten as

$$\frac{c_1f^n(z)}{f(z+\eta)} + \frac{c_4e^{\gamma(z)}(b - g(z+\eta))}{f(z+\eta)} = -c_2$$

for all $z \in \mathbb{C}$. Next, in the same manner as in subcase (2.2.2.1.2.2), we get a contradiction by (99), Lemmas 2.3 and 2.6, and the assumption $n \ge 21$.

(2.2.2.6) Suppose that $c_1 = 0$, $c_2 \neq 0$, $c_3 \neq 0$ and $c_4 = 0$. Then, (85) can be rewritten as

$$\frac{c_2 f(z+\eta)}{c_3 g^n(z)} = e^{\gamma(z)}. \tag{115}$$

By (115) and the third equation of (70) we deduce

$$\frac{\frac{c_2}{c_3} \cdot f(z+\eta)}{g^n(z)} = \frac{f^n(z) + (1 - \frac{c_2}{c_3}) f(z+\eta) - b}{g(z+\eta) - b} = e^{\gamma(z)} \tag{116}$$

for all $z \in \mathbb{C}$, which implies that each zero of $f(z+\eta)$ is of multiplicity not less than $n(\geq 21)$, and each pole of $f(z+\eta)$ is of multiplicity not less than $n \geq 21$. Combining this with the obtained result that $f(z+\eta)$ and $g(z+\eta)$ share 0, b, ∞ CM, we deduce that F_2 and G_2 share 0, 1, ∞ CM such that

$$\delta_{1)}(0, F_2) = \delta_{1)}(1, F_2) = 1, \tag{117}$$

where

$$F_2(z) = \frac{b}{b - f(z+\eta)}, \quad G_2(z) = \frac{b}{b - g(z+\eta)}. \tag{118}$$

By (117), (118) and Lemma 2.8 we have $F_2 + G_2 = 1$, and so we have

$$f(z+\eta)g(z+\eta) = f(z)g(z) = b^2 \tag{119}$$

for all $z \in \mathbb{C}$. Next, in the same manner as in (2.2.2.3) we get a contradiction by (116), (119), Lemmas 2.2 and 2.9.

(2.2.2.7) Suppose that $c_1 = 0$, $c_2 \neq 0$, $c_3 = 0$ and $c_4 \neq 0$. Then, (85) can be rewritten as

$$\frac{c_2 f(z+\eta)}{c_4(g(z+\eta) - b)} = e^{\gamma(z)}. \tag{120}$$

By (120) and the obtained result that $f(z+\eta)$ and $g(z+\eta)$ share 0, b, ∞ CM, we deduce that 0 and b are two Picard exceptional values of $f(z+\eta)$ and $g(z+\eta)$. Therefore, F_3 and G_3 share 0, 1, ∞ CM and that 0 and 1 are two Picard exceptional values of F_3 and G_3, and so

$$\delta_{1)}(0, F_3) = \delta_{1)}(1, F_3) = 1, \tag{121}$$

where

$$F_3(z) = \frac{f(z+\eta)}{b}, \quad G_3(z) = \frac{g(z+\eta)}{b}. \tag{122}$$

Therefore, by (121), (122) and Lemma 2.8 we have $F_3 + G_3 = 1$, and so

$$f(z+\eta) + g(z+\eta) = b \tag{123}$$

for all $z \in \mathbb{C}$. By (123) and the obtained result that $f(z+\eta)$ and $g(z+\eta)$ share 0, b, ∞ CM, we deduce that 0 and b are two Picard exceptional values of

$f(z+\eta)$ and $g(z+\eta)$, and so of f and g. On the other hand, by (120) and the third equation of (70) we have

$$\frac{c_2 f(z+\eta)}{c_4(g(z+\eta)-b)} = \frac{f^n(z)+(1-\frac{c_2}{c_4})f(z+\eta)-b}{g^n(z)} = e^{\gamma(z)} \tag{124}$$

for all $z \in \mathbb{C}$. Therefore, by (124) we have

$$\overline{N}\left(r, \frac{1}{f^n(z)+(1-\frac{c_2}{c_4})f(z+\eta)-b}\right) = \overline{N}\left(r, \frac{1}{g(z)}\right) = 0. \tag{125}$$

Suppose that $1-\frac{c_2}{c_4} \neq 0$. Then, by (125) and Lemma 2.4 we have

$$\begin{aligned}(n-1)T(r,f(z)) &\leq \overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}\left(r, \frac{1}{f^n(z)+(1-\frac{c_2}{c_4})f(z+\eta)-b}\right) \\ &\quad + 4\overline{N}(r,f(z)) + o\left(\frac{T(r,f(z))}{r^{\delta_0}}\right) \\ &\leq 4T(r,f(z)) + o\left(\frac{T(r,f(z))}{r^{\delta_0}}\right),\end{aligned}$$

and so

$$(n-5)T(r,f) \leq o\left(\frac{T(r,f)}{r^{\delta_0}}\right) \tag{126}$$

as r runs to infinity outside of a set of finite logarithmic measure. By (126) and the assumption $n \geq 21$ we get a contradiction.

Suppose that $1-\frac{c_2}{c_4} = 0$. Then, (124) can be rewritten as

$$\frac{c_2 f(z+\eta)}{c_4(g(z+\eta)-b)} = \frac{f^n(z)-b}{g^n(z)} = e^{\gamma(z)}. \tag{127}$$

By (127) and the obtained result that 0 is a Picard exceptional value of g we deduce that f has at least $n(\geq 21)$ distinct Picard exceptional values ω_1, ω_2, $\ldots$, ω_{n-1}, ω_n, where ω_1, $\omega_2, \ldots$, ω_{n-1}, ω_n are nonzero constant satisfying $\omega_j^n = b$ for $1 \leq j \leq n$. Then we get a contradiction.

(2.2.2.8) Suppose that $c_1 = 0$, $c_2 \neq 0$, $c_3 \neq 0$ and $c_4 \neq 0$. Then, (85) can be rewritten as

$$-\frac{c_3 g^n(z)}{g(z+\eta)-b} + \frac{c_2 f(z+\eta)e^{-\gamma(z)}}{g(z+\eta)-b} = c_4. \tag{128}$$

Next, in the same manner as in subcase (2.2.2.1.2.2), we get a contradiction by (128), Lemmas 2.3 and 2.6, and the assumption $n \geq 21$.

(2.2.2.9) Suppose that $c_1 \neq 0$, $c_2 \neq 0$, $c_3 \neq 0$ and $c_4 \neq 0$. Then, (85) can be rewritten as

$$-c_3 g^n(z) + c_1 f^n(z)e^{-\gamma(z)} + c_2 f(z+\eta)e^{-\gamma(z)} - c_4 g(z+\eta) = -c_4 b. \tag{129}$$

Now we set

$$\widetilde{g}_1(z) = -c_3 g^n(z), \quad \widetilde{g}_2(z) = c_1 f^n(z)e^{-\gamma(z)} \tag{130}$$

and

$$\widetilde{g}_3(z) = c_2 f(z+\eta)e^{-\gamma(z)}, \quad \widetilde{g}_4(z) = -c_4 g(z+\eta). \tag{131}$$

Then, (129) can be rewritten as

$$\widetilde{g}_1 + \widetilde{g}_2 + \widetilde{g}_3 + \widetilde{g}_4 = -c_4 b. \tag{132}$$

Suppose that $\widetilde{g}_1, \widetilde{g}_2, \widetilde{g}_3, \widetilde{g}_4$ are linearly independent. Then, by Lemma 2.5 we have a contradiction. Next we suppose that $\widetilde{g}_1, \widetilde{g}_2, \widetilde{g}_3, \widetilde{g}_4$ are linearly dependent. Then

$$d_1\widetilde{g}_1 + d_2\widetilde{g}_2 + d_3\widetilde{g}_3 + d_4\widetilde{g}_4 = 0, \tag{133}$$

where d_1, d_2, d_3, d_4 are constants with $\sum_{j=1}^{4} |d_j| > 0$. By substituting (130) and (131) into (132) that

$$-d_1c_3g^n(z) + d_2c_1f^n(z)e^{-\gamma(z)} + d_3c_2f(z+\eta)e^{-\gamma(z)} - d_4c_4g(z+\eta) = 0 \tag{134}$$

for all $z \in \mathbb{C}$.

Noting that at least one of d_1,d_2,d_3,d_4 are nonzero constants. Firstly, we assume that $d_1 \neq 0$. It follows by (133) that

$$\widetilde{g}_1 = -\frac{d_2}{d_1} \cdot \widetilde{g}_2 - \frac{d_3}{d_1} \cdot \widetilde{g}_3 - \frac{d_4}{d_1} \cdot \widetilde{g}_4. \tag{135}$$

By substituting (130) and (131) into (135) we have

$$-c_3g^n(z) = -\frac{c_1d_2}{d_1} \cdot f^n(z)e^{-\gamma(z)} - \frac{c_2d_3}{d_1} \cdot f(z+\eta)e^{-\gamma(z)} + \frac{c_4d_4}{d_1} \cdot g(z+\eta)$$

for all $z \in \mathbb{C}$, this together with the first equality of (70) gives

$$\begin{aligned} &\left(\frac{c_1d_2}{d_1} \cdot e^{-\gamma(z)} - c_3e^{-n\alpha(z)}\right) f^n(z) \\ &= \left(\frac{c_4d_4}{d_1} \cdot e^{-\alpha(z+\eta)} - \frac{c_2d_3}{d_1} \cdot e^{-\gamma(z)}\right) f(z+\eta) \end{aligned} \tag{136}$$

for all $z \in \mathbb{C}$. By (136) and the obtained result that e^{α} is a nonconstant entire function we deduce

$$\frac{c_1d_2}{d_1} \cdot e^{-\gamma(z)} - c_3e^{-n\alpha(z)} \not\equiv 0 \text{ and } \frac{c_4d_4}{d_1} \cdot e^{-\alpha(z+\eta)} - \frac{c_2d_3}{d_1} \cdot e^{-\gamma(z)} \not\equiv 0. \tag{137}$$

By (136) and (137) we have

$$f^n(z) = \frac{c_4d_4e^{\gamma(z)-\alpha(z+\eta)} - c_2d_3}{c_1d_2 - c_3d_1e^{\gamma(z)-n\alpha(z)}} f(z+\eta) \tag{138}$$

for all $z \in \mathbb{C}$.

If $c_4d_4e^{\gamma(z)-\alpha(z+\eta)} - c_2d_3$ and $c_1d_2 - c_3d_1e^{\gamma(z)-n\alpha(z)}$ are nonconstant entire functions, then

(139)

$$N_{(2}\left(r, \frac{1}{c_1d_2 - c_3d_1e^{\gamma(z)-n\alpha(z)}}\right) + N_{(2}\left(r, \frac{1}{c_4d_4e^{\gamma(z)-\alpha(z+\eta)} - c_2d_3}\right)$$

$$\leq 2N\left(r, \frac{1}{(c_1d_2 - c_3d_1e^{\gamma(z)-n\alpha(z)})'}\right) + 2N\left(r, \frac{1}{(c_4d_4e^{\gamma(z)-\alpha(z+\eta)} - c_2d_3)'}\right)$$
$$= 2N\left(r, \frac{1}{\gamma'(z) - n\alpha'(z)}\right) + 2N\left(r, \frac{1}{\gamma'(z) - \alpha'(z+\eta)}\right) \leq O(\log r).$$

Therefore, by (138), (139), Lemma 2.1 and the assumption $n \geq 21$ we deduce

$$\begin{aligned} nN(r, f(z)) &= N(r, f^n(z)) \\ &\leq N_{(2}\left(r, \frac{1}{c_4d_4e^{\gamma(z)-\alpha(z+\eta)} - c_2d_3}\right) + N(r, f(z+\eta)) \\ &= N(r, f(z)) + o\left(\frac{T(r, f(z))}{r^{\delta_0}}\right) + O(\log r), \end{aligned}$$

and so

$$\begin{aligned} N(r, f(z)) &= N(r, f(z+\eta)) + o\left(\frac{T(r, f(z))}{r^{\delta_0}}\right) \\ &= o\left(\frac{T(r, f(z))}{r^{\delta_0}}\right) + O(\log r) \end{aligned} \tag{140}$$

as r runs to infinity outside of a set of finite logarithmic measure. Similarly, by (138), (139), Lemma 2.1 and the assumption $n \geq 21$ we deduce

$$\begin{aligned} N\left(r, \frac{1}{f(z)}\right) &= N\left(r, \frac{1}{f(z+\eta)}\right) + o\left(\frac{T(r, f(z))}{r^{\delta_0}}\right) \\ &= o\left(\frac{T(r, f(z))}{r^{\delta_0}}\right) + O(\log r) \end{aligned} \tag{141}$$

as r runs to infinity outside of a set of finite logarithmic measure.

Suppose that $c_1f^n(z)e^{-\gamma(z)}$ in the left of (129) is a constant. Then, by Lemma 2.3 we deduce that $c_2f(z+\eta)e^{-\gamma(z)}$ is a nonconstant meromorphic function. This together with (129), (140), (141) and Lemma 2.7 gives

$$c_1f^ne^{-\gamma} = -c_4b \tag{142}$$

and

$$-c_3g^n(z) + c_2f(z+\eta)e^{-\gamma(z)} - c_4g(z+\eta) = 0 \tag{143}$$

for all $z \in \mathbb{C}$. By (142) we know that 0 and ∞ are two Picard exceptional values of f and g, and of $f(z+\eta)$ and $g(z+\eta)$. Therefore

$$\delta_{1)}(0, F_4) = \delta_{1)}(1, F_4) = 1, \tag{144}$$

where

$$F_4(z) = \frac{f(z+\eta)}{f(z+\eta) - b}, \quad G_4(z) = \frac{g(z+\eta)}{g(z+\eta) - b} \tag{145}$$

for all $z \in \mathbb{C}$. By (145) and the obtained result that $f(z+\eta)$ and $g(z+\eta)$ share 0, b, ∞ CM, we deduce that F_4 and G_4 share 0, 1, ∞ CM. This together with

(144) and Lemma 2.8 gives $F_4 + G_4 = 1$, and so it follows by (145) that

$$f(z)g(z) = f(z+\eta)g(z+\eta) = b^2 \tag{146}$$

for all $z \in \mathbb{C}$. By (142), (143) and (146) we deduce

$$c_1c_4b^2f^n(z) = -c_2c_4bf^2(z+\eta) - c_1c_3b^{2n}f(z+\eta) \tag{147}$$

for all $z \in \mathbb{C}$. By (147), Lemmas 2.3 and 2.9 we deduce

$$\begin{aligned} nT(r,f(z)) &= nm(r,f(z)) + O(1) \le 2m(r,f(z+\eta)) + O(1) \\ &= 2T(r,f(z+\eta)) + O(1) \le 2T(r,f(z)) + O(r^{\rho-1+\varepsilon}) + O(\log r), \end{aligned}$$

i.e.,

$$(n-2)T(r,f) \le O(r^{\rho-1+\varepsilon}) + O(\log r). \tag{148}$$

By (148), the assumption $n \ge 21$ and the definition of the order of a meromorphic function we deduce $\rho(f) = \rho \le \rho - 1$, which is impossible.

Suppose that $c_2f(z+\eta)e^{-\gamma(z)}$ in the left of (129) is a constant. Then, by Lemma 2.3 we deduce that $c_1f^n(z)e^{-\gamma(z)}$ is a nonconstant meromorphic function. Next in the same manner as the above we get a contradiction.

Suppose that $c_1f^n(z)e^{-\gamma(z)}$ and $c_2f(z+\eta)e^{-\gamma(z)}$ in the left of (129) are nonconstant meromorphic functions. First of all, by the obtained result that $f(z+\eta)$ and $g(z+\eta)$ share 0, b, ∞ CM, we have by (24), Lemma 2.3 and the second fundamental theorem that

$$\begin{aligned} T(r,f(z+\eta)) &\le \overline{N}(r,f(z+\eta) + \overline{N}\left(r,\frac{1}{f(z+\eta)}\right) + \overline{N}\left(r,\frac{1}{f(z+\eta)-b}\right) \\ &\le 3T(r,g(z+\eta)) + O(\log r) \\ &= 3T(r,g(z)) + O(r^{\rho-1+\varepsilon}) + O(\log r) \end{aligned} \tag{149}$$

and

$$\begin{aligned} T(r,g(z+\eta)) &\le 3T(r,f(z+\eta)) + O(\log r) \\ &= 3T(r,f(z)) + O(r^{\rho-1+\varepsilon}) + O(\log r). \end{aligned} \tag{150}$$

By (129)-(132), (140), (141), (149), (150) and Lemma 2.10 we deduce that there exists some subset $I \subset (1,+\infty)$ with logarithmic measure $\int_I \frac{dt}{t} = \infty$, such that

$$\frac{1}{3}T(r,g) \le T(r,f) \le 3T(r,g), \quad T(r,f(z+\eta)) = T(r,f(z))(1+o(1)), \tag{151}$$

$$T(r,g(z+\eta)) = T(r,g(z))(1+o(1)) \tag{152}$$

and

$$N(r,f(z)) + N(r,f(z+\eta)) + N\left(r,\frac{1}{f(z)}\right) + N\left(r,\frac{1}{f(z+\eta)}\right) = o(T(r,f(z))) \tag{153}$$

as $r \in I$ and $r \to \infty$.

Suppose that there exists a subset of I with infinite logarithmic measure, say I itself, such that

$$T(r, c_1 f^n(z)e^{-\gamma(z)}) = o(T(r, f(z))) \tag{154}$$

as $r \in I$ and $r \to \infty$. Then, by Lemma 2.3 we deduce that there exists some positive constant A_1 such that

$$T(r, c_2 f(z+\eta)e^{-\gamma(z)}) \geq A_1 T(r, f(z)) \tag{155}$$

as $r \in I$ and $r \to \infty$. Next, by (149)-(155) and Lemma 2.7 we deduce (142) and (143), and so we get a contradiction.

Suppose that there exists a subset of I with infinite logarithmic measure, say I itself, such that

$$T(r, c_2 f(z+\eta)e^{-\gamma(z)}) = o(T(r, f(z))) \tag{156}$$

as $r \in I$ and $r \to \infty$. Then, by Lemma 2.3 we deduce that there exists some positive constant A_2 such that

$$T(r, c_1 f^n(z)e^{-\gamma(z)}) \geq A_2 T(r, f(z)) \tag{157}$$

as $r \in I$ and $r \to \infty$. Next, in the same manner as the above we can get a contradiction.

Suppose that there exists a subset of I with infinite logarithmic measure, say I itself, and there exist two positive constants, say A_1 and A_2 such that (155) and (157) hold. Then, by (129), (149)-(153), (155), (157) and Lemma 2.7 we get a contradiction.

Suppose that $d_2 \neq 0$, $d_3 \neq 0$, $d_4 \neq 0$ respectively. In the same manner as the above we can get a contradiction

Theorem 1.6 is thus completely proved. □

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Xiao-Min Li
Department of Mathematics
Ocean University of China
Qingdao, Shandong 266100, P. R. China
Email address: xmli01267@gmail.com

Hui Yu
Department of Mathematics
Ocean University of China
Qingdao, Shandong 266100, P. R. China
Email address: yhui2115@163.com