Honam Mathematical J. **40** (2018), No. 3, pp. 529–538
https://doi.org/10.5831/HMJ.2018.40.3.529

UNIVALENT FUNCTIONS WITH POSITIVE COEFFICIENTS INVOLVING POISSON DISTRIBUTION SERIES

Gangadharan Murugusundaramoorthy

Abstract. The purpose of the present paper is to establish connections between various subclasses of analytic univalent functions by applying certain convolution operator involving Poisson distribution series. To be more precise,we investigate such connections with the classes of analytic univalent functions with positive coefficients in the open unit disk.

1. Introduction

It is well known that the special functions (series) play an important role in geometric function theory, especially in the solution by de Branges [3] of the famous Bieberbach conjecture. The surprising use of special functions (hypergeometric functions) has prompted renewed interest in function theory in the last few decades. There is an extensive literature dealing with geometric properties of different types of special functions, especially for the generalized, Gaussian hypergeometric functions [2, 7, 8, 12, 13].

A variable x is said to be Poisson distribution if it takes the values $0, 1, 2, 3, \dots$ with probabilities $e^{-m}, m\frac{e^{-m}}{1!}, m^2\frac{e^{-m}}{2!}, m^3\frac{e^{-m}}{3!}, \dots$ respectively, where m is called the parameter. Thus

$$P(x = k) = \frac{m^k\ e^{-m}}{k!}, k = 0, 1, 2, 3, \dots$$

Received April 7, 2018. Accepted July 12, 2018.
2010 Mathematics Subject Classification. 30C45.
Key words and phrases. univalent, starlike functions, convex functions, Hadamard product, Poisson distribution.
Dedicated to my Father Prof. P.M.Gangadharan.

Recently, Porwal [9, 11] introduce a power series whose coefficients are probabilities of Poisson distribution

$$\Phi(m,z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n, \qquad z \in \mathbb{U} \tag{1.1}$$

and we note that, by ratio test the radius of convergence of above series is infinity.

Let $\mathcal{H}$ be the class of functions analytic in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A}$ be the class of functions $f \in \mathcal{H}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{U}. \tag{1.2}$$

We also let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions which are normalized by $f(0) = 0 = f'(0) - 1$ and univalent in $\mathbb{U}$.

Denote by $\mathcal{V}$ the subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0. \tag{1.3}$$

For functions $f \in \mathcal{A}$ and $g \in \mathcal{A}$ of the form $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the hadamard product (or convolution) of f and g is given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{U}. \tag{1.4}$$

Now, we define the linear operator

$$\mathcal{I}(m,z) : \mathcal{A} \to \mathcal{A}$$

defined by the hadamard product(or convolution)

$$\mathcal{I}(m,z)f = \Phi(m,z) * f(z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} a_n z^n. \tag{1.5}$$

where $\Phi(m,z)$ is the Poissons distribution series given by (1.1).

We recall the following subclasses introduced by Uralegaddi et al.[14] (see[4, 6]):

The class $\mathcal{M}(\alpha)$ of starlike functions of order $1 < \alpha \leq \frac{4}{3}$

$$\mathcal{M}(\alpha) := \left\{ f \in \mathcal{A} : \ \Re\left(\frac{zf'(z)}{f(z)}\right) < \alpha, \ z \in \mathbb{U} \right\}$$

and the class $\mathcal{N}(\alpha)$ of convex functions of order $1 < \alpha \leq \frac{4}{3}$

$$\begin{aligned}\mathcal{N}(\alpha) : &= \left\{ f \in \mathcal{A} : \ \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \alpha, \ z \in \mathbb{U} \right\} \\ &= \left\{ f \in \mathcal{A} : \ zf' \in \mathcal{M}(\alpha) \right\}\end{aligned}$$

Also let $\mathcal{M}^*(\alpha) \equiv \mathcal{M}(\alpha) \cap \mathcal{V}$ and $\mathcal{N}^*(\alpha) \equiv \mathcal{N}(\alpha) \cap \mathcal{V}$[14].

In this paper we introduce two new subclasses of $\mathcal{S}$ namely $\mathcal{M}(\lambda, \alpha)$ and $\mathcal{N}(\lambda, \alpha)$ to discuss some inclusion properties.

For some α $(1 < \alpha \leq \frac{4}{3})$ and $\lambda(0 \leq \lambda < 1)$, we let $\mathcal{M}(\lambda, \alpha)$ and $\mathcal{N}(\lambda, \alpha)$ be two new subclass of $\mathcal{S}$ consisting of functions of the form (1.2) satisfying the analytic criteria

$$\mathcal{M}(\lambda, \alpha) := \left\{ f \in \mathcal{S} : \Re\left(\frac{zf'(z)}{(1-\lambda)f(z) + \lambda z f'(z)}\right) < \alpha, \ z \in \mathbb{U} \right\}. \quad (1.6)$$

$$\mathcal{N}(\lambda, \alpha) := \left\{ f \in \mathcal{S} : \Re\left(\frac{f'(z) + zf''(z)}{f'(z) + \lambda z f''(z)}\right) < \alpha, \ z \in \mathbb{U} \right\}. \quad (1.7)$$

We also let $\mathcal{M}^*(\lambda, \alpha) \equiv \mathcal{M}(\lambda, \alpha) \cap \mathcal{V}$ and $\mathcal{N}^*(\lambda, \alpha) \equiv \mathcal{N}(\lambda, \alpha) \cap \mathcal{V}$.

Note that $\mathcal{M}(0, \alpha) = \mathcal{M}(\alpha)$, $\mathcal{N}(0, \alpha) = \mathcal{N}(\alpha)$; $\mathcal{M}^*(\alpha)$ and $\mathcal{N}^*(\alpha)$ the subclasses of studied by Uralegaddi et al.[14].

Motivated by results on connections between various subclasses of analytic univalent functions by using hypergeometric functions (see [2, 7, 8, 12, 13]) , we obtain necessary and sufficient condition for function $\Phi(m, z)$ to be in the classes $\mathcal{M}^*(\lambda, \alpha)$, $\mathcal{N}^*(\lambda, \alpha)$ and connections between $\mathcal{R}^\tau(A, B)$ by applying convolution operator.

2. Coefficient Estimate

To start with we prove the following results.

Theorem 2.1. *For some α $(1 < \alpha \leq \frac{4}{3})$ and $\lambda(0 \leq \lambda < 1)$, and if $f \in \mathcal{V}$ then $f \in \mathcal{M}^*(\lambda, \alpha)$ if and only if*

$$\sum_{n=2}^{\infty} [n - (1 + n\lambda - \lambda)\alpha]|a_n| \leq \alpha - 1. \quad (2.1)$$

Proof. To show that $f \in \mathcal{M}(\lambda, \alpha)$ it suffices to prove that

$$\left| \frac{\frac{zf'(z)}{(1-\lambda)f(z)+\lambda z f'(z)} - 1}{\frac{zf'(z)}{(1-\lambda)f(z)+\lambda z f'(z)} - (2\alpha - 1)} \right| \leq 1. \quad (2.2)$$

By (1.2) the inequality (2.2) becomes

$$\left|\frac{A}{B}-1\right| \le \left|\frac{A}{B}-(2\alpha-1)\right|, \tag{2.3}$$

Since this, the inequality (2.2) we can replace with

$$\left|\frac{A-B}{A-B(2\alpha-1)}\right| \le 1. \tag{2.4}$$

where

$$\begin{aligned}
A &= zf'(z) = z+\sum_{n=2}^{\infty} na_n z^n, \quad a_n \ge 0.\\
B &= (1-\lambda)f(z)+\lambda zf'(z) = z+\sum_{n=2}^{\infty}(1+n\lambda-\lambda)a_n z^n, \quad a_n \ge 0.
\end{aligned}$$

Now we want to show (2.4). We have

$$\begin{aligned}
\left|\frac{A-B}{A-B(2\alpha-1)}\right| &= \left|\frac{\sum_{n=2}^{\infty}[n-(1+n\lambda-\lambda)]a_n z^n}{-2(\alpha-1)z+\sum_{n=2}^{\infty}[n-(2\alpha-1)(1+n\lambda-\lambda)]a_n z^n}\right|\\
&\le \frac{\sum_{n=2}^{\infty}[n-(1+n\lambda-\lambda]a_n}{2(\alpha-1)-\sum_{n=2}^{\infty}[n-(2\alpha-1)(1+n\lambda-\lambda)]a_n}.
\end{aligned}$$

The last expression is bounded above by 1, if

$$\sum_{n=2}^{\infty}[n-(1+n\lambda-\lambda)]a_n \le 2(\alpha-1)-\sum_{n=2}^{\infty}[n-(2\alpha-1)(1+n\lambda-\lambda)]a_n$$

which is equivalent to

$$\sum_{n=2}^{\infty}[n-(1+n\lambda-\lambda)\alpha]a_n \le \alpha-1.$$

To prove converse, we assume that $f \in \mathcal{A}$ and in the class $\mathcal{M}(\lambda,\alpha)$ so the condition (1.6) readily yields

$$\Re\left(\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}\right) = \Re\left(\frac{z+\sum_{n=2}^{\infty} na_n z^n}{z+\sum_{n=2}^{\infty}(1+n\lambda-\lambda)a_n z^n}\right) < \alpha$$

Choosing values of z on the real axis and upon clearing the denominator we get the required assertion in (2.1). Thus the proof is complete. □

Theorem 2.2. *For some* α $(1 < \alpha \leq \frac{4}{3})$ *and* $\lambda(0 \leq \lambda < 1)$, *and if* $f \in \mathcal{V}$ *then* $f \in \mathcal{N}^*(\lambda, \alpha)$ *if and only if*

$$\sum_{n=2}^{\infty} n[n - (1 + n\lambda - \lambda)\alpha]a_n \leq \alpha - 1. \tag{2.5}$$

Proof. It is well known that $f \in \mathcal{M}(\lambda, \alpha)$ if and only if $zf^{'} \in \mathcal{N}(\lambda, \alpha)$ Since $zf^{'} = z + \sum_{n=2}^{\infty} na_n z^n$ we may replace a_n with na_n in Theorem 2.1. □

Corollary 2.3. *For some* α $(1 < \alpha \leq \frac{4}{3})$; $\lambda(0 \leq \lambda < 1)$ *and* $f \in \mathcal{M}^*(\lambda, \alpha)$ *then*

$$|a_n| \leq \frac{\alpha - 1}{n - (1 + n\lambda - \lambda)\alpha}. \tag{2.6}$$

Corollary 2.4. *For some* α $(1 < \alpha \leq \frac{4}{3})$; $\lambda(0 \leq \lambda < 1)$ *and* $f \in \mathcal{N}^*(\lambda, \alpha)$ *then*

$$|a_n| \leq \frac{\alpha - 1}{n[n - (1 + n\lambda - \lambda)\alpha]}. \tag{2.7}$$

In the following theorems, we determine necessary and sufficient condition for function $\Phi(m, z)$ to be in the classes $\mathcal{M}^*(\lambda, \alpha)$ and $\mathcal{N}^*(\lambda, \alpha)$.

For convenience throughout in the sequel, we use the following notations:

$$\sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} = e^m - 1 \tag{2.8}$$

$$\sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-2)!} = me^m \tag{2.9}$$

$$\sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-3)!} = m^2 e^m \tag{2.10}$$

$$\sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-4)!} = m^3 e^m. \tag{2.11}$$

In general we can state

$$\sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-j)!} = m^{j-1} e^m, \quad j \geq 2. \tag{2.12}$$

Theorem 2.5. *If $m > 0$ $(m \neq 0, -1, -2, \dots)$, then $\Phi(m,z) \in \mathcal{M}^*(\lambda,\alpha)$ if and only if*

$$\frac{(1-\lambda\alpha)m}{2e^m - 1} e^m \leq \alpha - 1. \tag{2.13}$$

Proof. Since

$$\Phi(m,z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n,$$

by virtue of Theorem 2.1, it suffices to show that

$$\sum_{n=2}^{\infty} [n - (1 + n\lambda - \lambda)\alpha] \frac{m^{n-1}}{(n-1)!} e^{-m} \leq \alpha - 1.$$

Now, by writing $n = n + 1 - 1$ we have

$$\begin{aligned}
&\sum_{n=2}^{\infty} n \frac{m^{n-1}}{(n-1)!} e^{-m} - \lambda\alpha \sum_{n=2}^{\infty} (n-1) \frac{m^{n-1}}{(n-1)!} e^{-m} - \alpha \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} \\
&\quad = \sum_{n=2}^{\infty} (n-1) \frac{m^{n-1}}{(n-1)!} e^{-m} - \lambda\alpha \sum_{n=2}^{\infty} (n-1) \frac{m^{n-1}}{(n-1)!} e^{-m} \\
&\qquad\qquad + (1-\alpha) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} \\
&\quad \leq (1-\lambda\alpha) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-2)!} e^{-m} + (1-\alpha) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} \\
&\quad = (1-\lambda\alpha)m + (1-\alpha)(1 - e^{-m}) \\
&\quad = (1-\lambda\alpha)m - (\alpha - 1)(1 - e^{-m}).
\end{aligned}$$

But this expression is bounded above by $\alpha - 1$ if and only if (2.13) holds. Thus the proof is completed. □

Theorem 2.6. *If $m > 0 (m \neq 0, -1, -2, \dots)$, then $\Phi(m,z)$ is in $\mathcal{N}^*(\lambda,\alpha)$ if and only if*

$$\frac{(1-\lambda\alpha)m^2 + (3 - 2\lambda\alpha - \alpha)m}{2e^m - 1} e^m \leq \alpha - 1. \tag{2.14}$$

Proof. Let f be of the form (1.2) belong to the class $\mathcal{S}$. By virtue of Theorem 2.2, it suffices to show that

$$\sum_{n=2}^{\infty} n[n - (1 + n\lambda - \lambda)\alpha] \frac{m^{n-1}}{(n-1)!} e^{-m} \leq \alpha - 1. \tag{2.15}$$

Let

$$L(\lambda,\alpha)=\sum_{n=2}^{\infty}\left[(1-\lambda\alpha)n^2-\alpha(1-\lambda)n\right]\frac{m^{n-1}}{(n-1)!}e^{-m}.$$

Writing $n=(n-1)+1$, and $n^2=(n-1)(n-2)+3(n-1)+1$, we can rewrite the above term as

$$\begin{aligned}L(\lambda,\alpha)&=(1-\lambda\alpha)\sum_{n=2}^{\infty}(n-1)(n-2)\frac{m^{n-1}}{(n-1)!}e^{-m}\\&\quad+(3-2\lambda\alpha-\alpha)\sum_{n=2}^{\infty}(n-1)\frac{m^{n-1}}{(n-1)!}e^{-m}+(1-\alpha)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}\\&=(1-\lambda\alpha)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-3)!}e^{-m}+(3-2\lambda\alpha-\alpha)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-2)!}e^{-m}\\&\quad+(1-\alpha)\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}\\&=(1-\lambda\alpha)m^2+(3-2\lambda\alpha-\alpha)m+(1-\alpha)(1-e^{-m})\\&=(1-\lambda\alpha)m^2+(3-2\lambda\alpha-\alpha)m-(\alpha-1)(1-e^{-m}).\end{aligned}$$

But the last expression is bounded above by $\alpha-1$ if and only if (2.15) holds. Thus the proof is complete. □

By taking $\lambda=0$ we state the following corollary,

Corollary 2.7. *Let $m>0$ $(m\neq 0,-1,-2,\dots)$. Then the following are true.*

(1) $\Phi(m,z)\in\mathcal{M}^*(\alpha)$ *if and only if*

$$\frac{me^m}{2e^m-1}\leq\alpha-1.$$

(2) $\Phi(m,z)\in\mathcal{N}^*(\alpha)$ *if and only if*

$$\frac{m^2+(3-\alpha)m}{2e^m-1}e^m\leq\alpha-1.$$

3. Inclusion Properties

A function $f\in\mathcal{A}$ is said to be in the class $\mathcal{R}^\tau(A,B)$, $(\tau\in\mathbb{C}\backslash\{0\},\ -1\leq B<A\leq 1)$, if it satisfies the inequality

$$\left|\frac{f'(z)-1}{(A-B)\tau-B[f'(z)-1]}\right|<1\quad(z\in\mathbb{U}).$$

The class $\mathcal{R}^\tau(A,B)$ was introduced earlier by Dixit and Pal [5]. It is of interest to note that if

$$\tau = 1,\ A = \beta \text{ and } \ B = -\beta\ (0 < \beta \leq 1),$$

we obtain the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$\left|\frac{f'(z)-1}{f'(z)+1}\right| < \beta \quad (z \in \mathbb{U})$$

which was studied by (among others) Padmanabhan [10] and Caplinger and Causey [1].

Lemma 3.1. [5] *If $f \in \mathcal{R}^\tau(A,B)$ is of form* (1.2)*, then*

$$|a_n| \leq (A-B)\frac{|\tau|}{n}, \quad n \in \mathbb{N}\setminus\{1\}. \tag{3.1}$$

The result is sharp.

Making use of the Lemma 3.1 we will study the action of the Poissons distribution series on the class $\mathcal{M}(\lambda,\alpha)$.

Theorem 3.2. *Let $m > 0 (m \neq 0,-1,-2,\dots)$. If $f \in \mathcal{R}^\tau(A,B)$,then $\mathcal{I}(m,z)f \in \mathcal{N}^*(\lambda,\alpha)$ if and only if*

$$\frac{(A-B)|\tau|\ (1-\lambda\alpha)m}{1+(A-B)|\tau|(1-e^{-m})} \leq \alpha - 1. \tag{3.2}$$

Proof. Let f be of the form (1.2) belong to the class $\mathcal{R}^\tau(A,B)$. By virtue of Theorem 2.2, it suffices to show that

$$\sum_{n=2}^{\infty} n[n-(1+n\lambda-\lambda)\alpha]\frac{m^{n-1}}{(n-1)!}e^{-m}|a_n| \leq \alpha - 1.$$

Let

$$L(n,m,\alpha) = \sum_{n=2}^{\infty} n[n-(1+n\lambda-\lambda)\alpha]\frac{m^{n-1}}{(n-1)!}e^{-m}|a_n|.$$

Since $f \in \mathcal{R}^\tau(A,B)$ then by Lemma 3.1 we have

$$|a_n| \leq (A-B)\frac{|\tau|}{n}.$$

Hence

$$L(n,m,\alpha) \leq \quad (A-B)|\tau|\sum_{n=2}^{\infty}[n-(1+n\lambda-\lambda)\alpha]\frac{m^{n-1}}{(n-1)!}e^{-m}.$$

Proceeding as in Theorem 2.5 we get

$$L(n,m,\alpha) \le (A-B)|\tau|\ [(1-\lambda\alpha)m-(\alpha-1)(1-e^{-m})].$$

But this last expression is bounded above by $\alpha-1$ if and only if (3.2) holds. □

By taking $\lambda=0$ we state the following:

Corollary 3.3. *Let $m>0(m\neq 0,-1,-2,\dots)$. If $f\in\mathcal{R}^\tau(A,B)$, then $\mathcal{I}(m,z)f\in\mathcal{N}^*(\alpha)$ if and only if*

$$\frac{(A-B)|\tau|\ m}{1+(A-B)|\tau|(1-e^{-m})}\le\alpha-1.$$

Theorem 3.4. *Let $m>0(m\neq 0,-1,-2,\dots)$, then $\mathcal{L}(m,z)=\int_0^z\frac{\mathcal{I}(m,t)}{t}dt$ is in $\mathcal{N}^*(\lambda,\alpha)$ if and only if*

$$\frac{(1-\lambda\alpha)m}{2e^m-1}e^m\le\alpha-1. \tag{3.3}$$

Proof. Since

$$\mathcal{L}(m,z)=z+\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}\ \frac{z^n}{n}.$$

By virtue of Theorem 2.5, it suffices to show that

$$\sum_{n=2}^{\infty}n[n-(1+n\lambda-\lambda)\alpha]\frac{m^{n-1}}{n(n-1)!}e^{-m}\le\alpha-1.$$

Now,

$$\sum_{n=2}^{\infty}n[n-(1+n\lambda-\lambda)\alpha]\frac{m^{n-1}}{n(n-1)!}e^{-m}=\sum_{n=2}^{\infty}[n-(1+n\lambda-\lambda)\alpha]\frac{m^{n-1}}{(n-1)!}e^{-m}.$$

Proceeding as in Theorem 2.5 we get

$$\sum_{n=2}^{\infty}n[n-(1+n\lambda-\lambda)\alpha]\frac{m^{n-1}}{(n-1)!}e^{-m}=(1-\lambda\alpha)m-(\alpha-1)(1-e^{-m}),$$

which is bounded above by $\alpha-1$ if and only if (3.3) holds. □

By taking $\lambda=0$ we state the following:

Corollary 3.5. *Let $m>0(m\neq 0,-1,-2,\dots)$, then $\mathcal{L}(m,z)=\int_0^z\frac{\mathcal{I}(m,t)}{t}dt$ is in $\mathcal{N}^*(\alpha)$ if and only if*

$$\frac{m\ e^m}{2e^m-1}\le\alpha-1.$$

Acknowledgement I record my sincere thanks to the referees for their suggestions.

References

[1] T. R. Caplinger and W. M. Causey,*A class of univalent functions*, Proc. Amer. Math. Soc. **39** (1973), 357–361.

[2] N. E. Cho, S. Y. Woo and S. Owa,*Uniform convexity properties for hypergeometric functions*, Fract. Cal. Appl. Anal., **5(3)** (2002), 303-313.

[3] L. de Branges, *A proof of the Bieberbach conjecture*, Acta Math. **154** (1985), 137–152.

[4] K. K.Dixit and V.Chandra,*On subclass of univalent functions with positive coefficients*, Aligarh Bull. Math., **27(2)** 2008, 87–93.

[5] K. K. Dixit and S. K. Pal, *On a class of univalent functions related to complex order*, Indian J. Pure Appl. Math. **26(9)**(1995), 889-896.

[6] K. K.Dixit and A. L. Pathak, *A new class of analytic functions with positive coefficients*, Indian J. Pure Appl. Math., **34 (2)** (2003),209–218.

[7] E. Merkes and B. T. Scott, *Starlike hypergeometric functions*, Proc. Amer. Math. Soc., **12** (1961), 885-888.

[8] A. O. Mostafa,*A study on starlike and convex properties for hypergeometric functions*, J. Inequal. Pure Appl. Math., **10(3)** (2009), Art., 87, 1-16.

[9] G. Murugusundaramoorthy , K. Vijaya and S. Porwal, *Some inclusion results of certain subclass of analytic functions associated with Poisson distribution series*, Hacettepe J. Math. Stat. **45(4)**, 2016, 1101-1107.

[10] K. S. Padmanabhan, *On a certain class of functions whose derivatives have a positive real part in the unit disc*, Ann. Polon. Math. **23** (1970), 73–81.

[11] S. Porwal, *An application of a Poisson distribution series on certain analytic functions*, J. Complex Anal., Vol.(2014), Art. ID 984135, 1-3.

[12] H. Silverman, *Starlike and convexity properties for hypergeometric functions*, J. Math. Anal. Appl., **172** (1993), 574-581.

[13] H. M. Srivastava, G. Murugusundaramoorthy and S. Sivasubramanian, *Hypergeometric functions in the parabolic starlike and uniformly convex domains*, Integral Transform Spec. Funct. **18**(2007), 511–520.

[14] B. A.Uralegaddi, M. D Ganigi and S. M.Sarangi*, Univalent functions with positive coefficients,* Tamkang J. Math., **25** (1994), 225–230.

Gangadharan Murugusundaramoorthy
Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology,
Vellore - 632014, India.
Email:gmsmoorthy@yahoo.com