

SYMMETRIC IDENTITIES INVOLVING THE MODIFIED (p, q) -HURWITZ EULER ZETA FUNCTION

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ABSTRACT. The main subject of this paper is to introduce the (p, q) -Euler polynomials and obtain several interesting symmetric properties of the modified (p, q) -Hurwitz Euler Zeta function with regard to (p, q) Euler polynomials. In order to get symmetric properties, we introduce the new (p, q) -analogue of Euler polynomials $E_{n,p,q}(x)$ and numbers $E_{n,p,q}$.

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1. Introduction

Many mathematicians have studied in the area of the Euler polynomials and numbers. The history of the Euler numbers E_n can be traced back to the Switzerland mathematician, Leonhard Euler(1707-1783). Until now, the Euler polynomials and numbers have been extensively studied in many different contexts in such branches of mathematics as, for example, number theory, analytic number theory, geometry, combinatorial analysis and so on(see [7, 8, 11-14]). Also, Euler polynomials and numbers are associated with Zeta function. The Zeta function play a crucial role in analytic number theory and have applications the field of applied statistics, probability theory, complex analysis, physics, mathematical physics, p -adic analysis and other related fields. Especially, the Zeta function happen within the concept of knot theory, quantum theory, number theory and applied analysis(see [1-10, 12-15]).

In this paper, we define the new (p, q) -analogue of Euler polynomials and numbers. Furthermore, we introduce the new (p, q) -modified Euler polynomials and numbers. We give some interesting properties of the new (p, q) -analogue of Euler polynomials and numbers. In the last section, we define the new modified

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(p, q) -Hurwitz Euler Zeta function and investigate the symmetric property of the new modified (p, q) - Hurwitz Euler Zeta function.

Throughout, we use the following notations: $\mathbb{N} = \{1, 2, 3, \dots\}$ for the natural numbers, $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\} = \mathbb{N} \cup \{0\}$ for the set of nonnegative integers, \mathbb{Z} for the set of integers, $\mathbb{Z}_0^- = \{0, -1, -2, -3, \dots\} = \mathbb{Z}^- \cup \{0\}$ for the set of nonpositive integers, \mathbb{R} for the set of real numbers and \mathbb{C} for the set of complex numbers. The q -number and (p, q) -number are defined as below:

$$[\lambda]_q = \frac{1 - q^\lambda}{1 - q} \quad \text{and} \quad [\lambda]_{p,q} = \frac{p^\lambda - q^\lambda}{p - q}$$

where $0 < q < p \leq 1$ and $\lambda \in \mathbb{C}$. Note that $\lim_{q \rightarrow 1} [\lambda]_q = \lambda$ for any λ .

From now on, we introduce well-known definitions and theorems about Euler polynomials, Euler numbers and Euler Zeta function.

Firstly, the Euler polynomials $E_n(x)$ and the Euler numbers $E_n = E_n(0)$ are defined by the following generating functions(see [1, 7, 8, 11, 12, 14]):

$$\sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = \frac{2}{e^t + 1} e^{xt} \quad (|t| < \pi) \tag{1.1}$$

and

$$\sum_{n=0}^{\infty} E_n \frac{t^n}{n!} = \frac{2}{e^t + 1} \quad (|t| < \pi). \tag{1.2}$$

Definition 1.1. For $0 < q < p \leq 1$, the Carlitz's type (p, q) -Euler polynomials $\mathcal{E}_{n,p,q}(x)$ and (p, q) -Euler numbers $\mathcal{E}_{n,p,q}$ are defined by the following generating functions(see [12]):

$$F_{p,q}(x, t) = \sum_{n=0}^{\infty} \mathcal{E}_{n,p,q}(x) \frac{t^n}{n!} = [2]_q \sum_{n=0}^{\infty} (-q)^n e^{[x+n]_{p,q}t} \tag{1.3}$$

and

$$F_{p,q}(t) = \sum_{n=0}^{\infty} \mathcal{E}_{n,p,q} \frac{t^n}{n!} = [2]_q \sum_{n=0}^{\infty} (-q)^n e^{[n]_{p,q}t}. \tag{1.4}$$

The Hurwitz Zeta function was discovered by Adolf Hurwitz in 1881. The Hurwitz(or generalized) Zeta function defined by

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s},$$

where a is a real parameter satisfying $0 < a \leq 1$. It was a simple but important generalization(see [4, 5, 6, 7, 10, 15]).

Definition 1.2. For $a > 0$ and $\Re(s) > 0$, The Hurwitz-Euler Zeta funtion $\zeta_E(s, a)$ is defined by(see [1, 5, 7, 8, 14])

$$\zeta_E(s, a) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n + a)^s}.$$

2. Properties of the (p, q) -analogue of Euler polynomials and numbers

In this section, we define the new (p, q) -analogue of Euler polynomials and numbers. Also, we define the modified (p, q) -Euler polynomials. Furthermore, we provide some of their relevant properties.

Definition 2.1. For $0 < q < p \leq 1$, the (p, q) -analogue of Euler polynomials $E_{n,p,q}(x)$ and (p, q) -analogue of Euler numbers $E_{n,p,q}$ are defined by the following generating functions

$$F_{p,q}(x, t) = \sum_{n=0}^{\infty} E_{n,p,q}(x) \frac{t^n}{n!} = [2]_q \sum_{n=0}^{\infty} (-q)^n e^{(x+[n]_{p,q})t} \tag{2.1}$$

and

$$F_{p,q}(t) = \sum_{n=0}^{\infty} E_{n,p,q} \frac{t^n}{n!} = [2]_q \sum_{n=0}^{\infty} (-q)^n e^{[n]_{p,q}t}. \tag{2.2}$$

If we take $p = 1$ and let $q \rightarrow 1$ in Definition 2.1, then $E_{n,p,q}(x)$ reduces to the Euler polynomials $E_n(x)$ and $E_{n,p,q}$ reduces to the Euler numbers E_n :

$$\lim_{q \rightarrow 1} E_{n,p,q}(x) = E_n(x) \quad \text{and} \quad \lim_{q \rightarrow 1} E_{n,p,q} = E_n.$$

From (2.1), we find that

$$\begin{aligned} \sum_{l=0}^{\infty} E_{l,p,q}(x) \frac{t^l}{l!} &= [2]_q \sum_{n=0}^{\infty} (-q)^n e^{(x+[n]_{p,q})t} \\ &= \sum_{l=0}^{\infty} \left([2]_q \sum_{n=0}^{\infty} (-q)^n (x + [n]_{p,q})^l \right) \frac{t^l}{l!}. \end{aligned}$$

By comparing coefficients of $\frac{t^l}{l!}$ in the above equation, we have the following theorem.

Theorem 2.2. For $l \in \mathbb{N}_0$ and $0 < q < p \leq 1$, we have

$$E_{l,p,q}(x) = [2]_q \sum_{n=0}^{\infty} (-q)^n (x + [n]_{p,q})^l.$$

Definition 2.3. For $0 < q < p \leq 1$, the modified (p, q) -Euler polynomials are defined by the following generating functions

$$F_{p,q}^*(x, t) = \sum_{n=0}^{\infty} E_{n,p,q}^*(x) \frac{t^n}{n!} = [2]_q \sum_{n=0}^{\infty} (-q)^n e^{(p^n x + [n]_{p,q})t}.$$

From (2.1) and Definition 2.3, we get the following theorem.

Theorem 2.4. For $l \in \mathbb{N}_0$ and $0 < q < p \leq 1$, we have

$$\begin{aligned} E_{l,p,q}^*(x) &= E_{l,p,q}(p^n x) \\ &= [2]_q \sum_{n=0}^{\infty} (-q)^n (p^n x + [n]_{p,q})^l. \end{aligned}$$

Remark 2.1. Setting $p = 1$ and letting $q \rightarrow 1$ in Definition 2.3, we get

$$\sum_{n=0}^{\infty} E_n^*(x) \frac{t^n}{n!} = 2 \sum_{n=0}^{\infty} (-1)^n e^{(x+n)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

From (2.1), we find that

$$\begin{aligned} \sum_{n=0}^{\infty} E_{n,p,q}(x+y) \frac{t^n}{n!} &= [2]_q \sum_{n=0}^{\infty} (-q)^n e^{(x+y+[n]_{p,q})t} \\ &= [2]_q \sum_{n=0}^{\infty} (-q)^n e^{(x+[n]_{p,q})t} e^{yt} \\ &= \left(\sum_{n=0}^{\infty} E_{n,p,q}(x) \frac{t^n}{n!} \right) \left(\sum_{k=0}^{\infty} y^k \frac{t^k}{k!} \right), \end{aligned}$$

which, by applying the Cauchy product, yields

$$\sum_{n=0}^{\infty} E_{n,p,q}(x+y) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \binom{n}{k} E_{k,p,q}(x) y^{n-k} \right) \frac{t^n}{n!}. \tag{2.3}$$

Thus, we have the following theorem by comparing the coefficients of $\frac{t^n}{n!}$ on both sides of this last equation (2.3).

Theorem 2.5. (Addition formula) For $n \in \mathbb{N}_0$, we obtain

$$E_{n,p,q}(x+y) = \sum_{k=0}^n \binom{n}{k} E_{k,p,q}(x) y^{n-k}.$$

Let us take $\frac{d}{dx}$, which is a differential operator, on both sides of expression in (2.1), we get

$$\begin{aligned} \frac{d}{dx} \left(\sum_{n=0}^{\infty} E_{n,p,q}(x) \frac{t^n}{n!} \right) &= \frac{d}{dx} \left([2]_q \sum_{n=0}^{\infty} (-q)^n e^{(x+[n]_{p,q})t} \right) \\ &= [2]_q \sum_{n=0}^{\infty} (-q)^n t e^{(x+[n]_{p,q})t} \\ &= t \sum_{n=0}^{\infty} E_{n,p,q}(x) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} n E_{n-1,p,q}(x) \frac{t^n}{n!}. \end{aligned}$$

Therefore, we have the following theorem by comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation.

Theorem 2.6. (Difference relation) For $n \in \mathbb{N}$, we have

$$\frac{d}{dx} E_{n,p,q}(x) = n E_{n-1,p,q}(x).$$

Let us take \int_a^b , which is a integrated operations, on both sides of expression in Theorem 2.6, we get

$$\int_a^b \frac{d}{dx} E_{n,p,q}(x) dx = \int_a^b n E_{n-1,p,q}(x) dx. \quad (2.4)$$

Calculating the left-hand side of the equation (2.4), we have

$$\int_a^b \frac{d}{dx} E_{n,p,q}(x) dx = E_{n,p,q}(b) - E_{n,p,q}(a). \quad (2.5)$$

Consequently, we obtain the following theorem from (2.4) and (2.5).

Theorem 2.7. (Integral relation) For $n \in \mathbb{N}$, we have

$$\int_a^b E_{n-1,p,q}(x) dx = \frac{E_{n,p,q}(b) - E_{n,p,q}(a)}{n}.$$

By using (2.1), we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} E_{n,p,q}(x) \frac{t^n}{n!} \\ &= [2]_q e^{xt} \sum_{n=0}^{\infty} (-q)^n \sum_{k=0}^{\infty} [n]_{p,q}^k \frac{t^k}{k!} \\ &= [2]_q e^{xt} \sum_{k=0}^{\infty} \left(\frac{1}{p-q} \right)^k \sum_{n=0}^{\infty} \sum_{l=0}^k \binom{k}{l} (-1)^{n+l} q^{n(1+l)} p^{n(k-l)} \frac{t^k}{k!} \\ &= [2]_q \left(\sum_{n=0}^{\infty} x^n \frac{t^n}{n!} \right) \left(\sum_{k=0}^{\infty} \left(\frac{1}{p-q} \right)^k \sum_{l=0}^k \binom{k}{l} \frac{(-1)^l}{1+q^{1+l} p^{k-l}} \frac{t^k}{k!} \right) \end{aligned}$$

which, by applying the Cauchy product, yields

$$\begin{aligned} & \sum_{n=0}^{\infty} E_{n,p,q}(x) \frac{t^n}{n!} \\ &= [2]_q \sum_{n=0}^{\infty} \sum_{k=0}^n x^{n-k} \frac{t^{n-k}}{(n-k)!} \left(\frac{1}{p-q} \right)^k \sum_{l=0}^k \binom{k}{l} \frac{(-1)^l}{1+q^{1+l} p^{k-l}} \frac{t^k}{k!} \\ &= \sum_{n=0}^{\infty} \left([2]_q \sum_{k=0}^n \sum_{l=0}^k \binom{n}{k} \binom{k}{l} \left(\frac{1}{p-q} \right)^k \frac{x^{n-k} (-1)^l}{1+q^{1+l} p^{k-l}} \right) \frac{t^n}{n!}. \end{aligned} \quad (2.6)$$

Thus, we have the following theorem by comparing the coefficients of $\frac{t^n}{n!}$ on both sides of this last equation (2.6).

Theorem 2.8. (Explicit formula) For $n \in \mathbb{N}_0$, we have

$$E_{n,p,q}(x) = [2]_q \sum_{k=0}^n \sum_{l=0}^k \binom{n}{k} \binom{k}{l} \left(\frac{1}{p-q} \right)^k \frac{x^{n-k} (-1)^l}{1+q^{1+l} p^{k-l}}.$$

3. Symmetric Identities involving the modified (p, q) -Hurwitz Euler Zeta function

In this section, we introduce Hurwitz (p, q) -Euler Zeta function and define the new (p, q) -analogue of Hurwitz Euler Zeta function. Furthermore, we define the modified (p, q) -Hurwitz Euler Zeta function and investigate the symmetric property of the modified (p, q) -Hurwitz Euler Zeta function.

In 2017, C.S. Ryoo defined the Hurwitz (p, q) -Euler Zeta function as follows(see [12]):

$$\zeta_{p,q}(s, x) = [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{[x+n]_{p,q}^s} \quad (s \in \mathbb{C}; x \in \mathbb{C} \setminus \mathbb{Z}_0^-). \tag{3.1}$$

The new (p, q) -analogue of Hurwitz Euler Zeta function is slightly different from $\zeta_{p,q}(s, x)$ defined by (3.1).

Definition 3.1. For $s \in \mathbb{C}$ and $0 < q < p \leq 1$, the (p, q) -analogue of Hurwitz Euler Zeta function is defined by the following generating functions

$$\begin{aligned} \tilde{\zeta}_{p,q,E}(s, x) &= \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \{F_{p,q}(x, -t)\} dt \\ &= [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{(x + [n]_{p,q})^s} \end{aligned}$$

where $F_{p,q}(x, -t)$ is given by (2.1).

By using Theorem 2.2 and Definition 3.1, we get

$$\begin{aligned} \tilde{\zeta}_{p,q,E}(-l, x) &= [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{(x + [n]_{p,q})^{-l}} \\ &= [2]_q \sum_{n=0}^{\infty} (-1)^n q^n (x + [n]_{p,q})^l \\ &= E_{l,p,q}(x). \end{aligned} \tag{3.2}$$

Hence, we obtain the following theorem from (3.2).

Theorem 3.2. For $l \in \mathbb{N}_0$ and $0 < q < p \leq 1$, we have

$$\tilde{\zeta}_{p,q,E}(-l, x) = E_{l,p,q}(x).$$

Definition 3.3. For $s \in \mathbb{C}$ and $0 < q < p \leq 1$, the modified (p, q) -Hurwitz Euler Zeta function is defined by the following generating functions

$$\tilde{\zeta}_{p,q,E}^*(s, x) = [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{(p^n x + [n]_{p,q})^s}.$$

From Definition 3.1 and Definition 3.3, we get the following theorem.

Theorem 3.4. For $s \in \mathbb{C}$ and $0 < q < p \leq 1$, we get

$$\begin{aligned} \tilde{\zeta}_{p,q,E}^*(s, x) &= \tilde{\zeta}_{p,q,E}(s, p^n x) \\ &= [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{(p^n x + [n]_{p,q})^s}. \end{aligned}$$

By using Theorem 2.4 and Theorem 3.4, we obtain the following theorem.

Theorem 3.5. For $l \in \mathbb{N}_0$ and $0 < q < p \leq 1$, we have

$$\tilde{\zeta}_{p,q,E}^*(-l, x) = E_{l,p,q}^*(x).$$

Remark 3.1. Setting $p = 1$ and letting $q \rightarrow 1$ in Definition 3.3, we get

$$\tilde{\zeta}_E^*(s, x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(x+n)^s} = 2\zeta_E(s, x).$$

Moreover, by using (3.1) and Definition 3.3, we have

$$\begin{aligned} \zeta_{p,q}(s, x) &= [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{[x+n]_{p,q}^s} \\ &= [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{\left(\frac{p^{x+n} - q^{x+n}}{p-q}\right)^s} \\ &= [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{\left(\frac{q^x(p^n - q^n) - p^{x+n}q^x(p^{-x} - q^{-x})}{p-q}\right)^s} \tag{3.3} \\ &= [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{q^{sx} \left(-p^{x+n} \frac{p^{-x} - q^{-x}}{p-q} + [n]_{p,q}\right)^s} \\ &= q^{-sx} [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{(p^{x+n} p^{-1} q^{-1} [x]_{p^{-1}, q^{-1}} + [n]_{p,q})^s} \\ &= q^{-sx} \tilde{\zeta}_{p,q,E}^*(s, p^x p^{-1} q^{-1} [x]_{p^{-1}, q^{-1}}). \end{aligned}$$

Consequently, we obtain the following theorem from (3.3).

Theorem 3.6. For $s \in \mathbb{C}$ and $0 < q < p \leq 1$, we have

$$q^{-sx} \tilde{\zeta}_{p,q,E}^*(s, p^x p^{-1} q^{-1} [x]_{p^{-1}, q^{-1}}) = \zeta_{p,q}(s, x).$$

In view of Theorem 3.6, we consider Definition 3.3 in the following form:

$$\begin{aligned}
 & q^{-absx-sbj} \tilde{\zeta}_{p^a, q^a, E}^* \left(s, p^{abx+bj} p^{-a} q^{-a} \left[bx + \frac{bj}{a} \right]_{p^{-a}, q^{-a}} \right) \\
 &= q^{-absx-sbj} [2]_{q^a} \sum_{n=0}^{\infty} \frac{(-1)^n q^{na}}{\left(p^{na+abx+bj} p^{-a} q^{-a} \left[bx + \frac{bj}{a} \right]_{p^{-a}, q^{-a}} + [n]_{p^a, q^a} \right)^s} \\
 &= [2]_{q^a} \sum_{n=0}^{\infty} \frac{(-1)^n q^{na}}{\left(p^{abx+bj} q^{abx+bj} \right)^s \left(p^{-abx-bj} \frac{p^{na}-q^{na}}{p^a-q^a} - p^{na} \frac{p^{-abx-bj}-q^{-abx-bj}}{p^a-q^a} \right)^s} \\
 &= [2]_{q^a} \sum_{n=0}^{\infty} \frac{(-1)^n q^{na}}{\left(\frac{p^{abx+bj} q^{abx+bj} (p^{na} q^{-abx-bj} - p^{-abx-bj} q^{na})}{p^a - q^a} \right)^s} \\
 &= [2]_{q^a} \sum_{n=0}^{\infty} \frac{(-1)^n q^{na}}{\left(\frac{p^{na+abx+bj} - q^{na+abx+bj}}{p^a - q^a} \right)^s} \\
 &= [2]_{q^a} \sum_{n=0}^{\infty} \frac{(-1)^n q^{na}}{\left[n + bx + \frac{bj}{a} \right]_{p^a, q^a}^s}.
 \end{aligned}$$

For non-negative integers k and i such that $n = kb + i$ with $0 \leq i \leq b - 1$, if we suppose that $a \equiv 1 \pmod{2}$ and $b \equiv 1 \pmod{2}$, then we have

$$\begin{aligned}
 & q^{-absx-sbj} \tilde{\zeta}_{p^a, q^a, E}^* \left(s, p^{abx+bj} p^{-a} q^{-a} \left[bx + \frac{bj}{a} \right]_{p^{-a}, q^{-a}} \right) \\
 &= [2]_{q^a} \sum_{n=0}^{\infty} \frac{(-1)^n q^{na}}{\left(\frac{p^{na+abx+bj} - q^{na+abx+bj}}{p^a - q^a} \right)^s} \tag{3.4} \\
 &= [a]_{p, q}^s [2]_{q^a} \sum_{n=0}^{\infty} \frac{(-1)^n q^{na}}{[na + abx + bj]_{p, q}^s}.
 \end{aligned}$$

Putting $n = kb + i$ in above equation (3.4). Then we get the following that

$$\begin{aligned}
 & q^{-absx-sbj} \tilde{\zeta}_{p^a, q^a, E}^* \left(s, p^{abx+bj} p^{-a} q^{-a} \left[bx + \frac{bj}{a} \right]_{p^{-a}, q^{-a}} \right) \\
 &= [a]_{p, q}^s [2]_{q^a} \sum_{k=0}^{\infty} \sum_{i=0}^{b-1} \frac{(-1)^{kb+i} q^{(kb+i)a}}{[(kb + i)a + abx + bj]_{p, q}^s} \tag{3.5} \\
 &= [a]_{p, q}^s [2]_{q^a} \sum_{i=0}^{b-1} (-1)^i q^{ai} \sum_{k=0}^{\infty} \frac{(-1)^k q^{abk}}{[ab(x + k) + ai + bj]_{p, q}^s}.
 \end{aligned}$$

Let us take $\sum_{j=0}^{a-1} (-1)^j q^{bj}$ in (3.5), we obtain

$$\begin{aligned} & \sum_{j=0}^{a-1} (-1)^j q^{bj} q^{-absx-sbj} \tilde{\zeta}_{p^a, q^a, E}^* \left(s, p^{abx+bj} p^{-a} q^{-a} \left[bx + \frac{bj}{a} \right]_{p^{-a}, q^{-a}} \right) \\ &= [a]_{p, q}^s [2]_{q^a} \sum_{j=0}^{a-1} (-1)^j q^{bj} \sum_{i=0}^{b-1} (-1)^i q^{ai} \sum_{k=0}^{\infty} \frac{(-1)^k q^{abk}}{[ab(x+k) + ai + bj]_{p, q}^s}. \end{aligned} \tag{3.6}$$

Exchanging a with b and j with i in (3.5), we get

$$\begin{aligned} & q^{-absx-sai} \tilde{\zeta}_{p^b, q^b, E}^* \left(s, p^{abx+ai} p^{-b} q^{-b} \left[ax + \frac{ai}{b} \right]_{p^{-b}, q^{-b}} \right) \\ &= [b]_{p, q}^s [2]_{q^b} \sum_{j=0}^{a-1} (-1)^j q^{bj} \sum_{k=0}^{\infty} \frac{(-1)^k q^{abk}}{[ab(x+k) + ai + bj]_{p, q}^s}. \end{aligned} \tag{3.7}$$

Therefore, we get the following theorem by applying (3.6) in (3.7).

Theorem 3.7. *For any odd integers a and b , we have*

$$\begin{aligned} & \frac{[2]_{q^b}}{[a]_{p, q}^s} \sum_{i=0}^{a-1} (-1)^i q^{bi(1-s)} \tilde{\zeta}_{p^a, q^a, E}^* \left(s, p^{abx+bi} p^{-a} q^{-a} \left[bx + \frac{bi}{a} \right]_{p^{-a}, q^{-a}} \right) \\ &= \frac{[2]_{q^a}}{[b]_{p, q}^s} \sum_{i=0}^{b-1} (-1)^i q^{ai(1-s)} \tilde{\zeta}_{p^b, q^b, E}^* \left(s, p^{abx+ai} p^{-b} q^{-b} \left[ax + \frac{ai}{b} \right]_{p^{-b}, q^{-b}} \right). \end{aligned}$$

Setting $b = 1$ in Theorem 3.7, we get

$$\begin{aligned} & \frac{[2]_q}{[a]_{p, q}^s} \sum_{i=0}^{a-1} (-1)^i q^{i(1-s)} \tilde{\zeta}_{p^a, q^a, E}^* \left(s, p^{ax+i} p^{-a} q^{-a} \left[x + \frac{i}{a} \right]_{p^{-a}, q^{-a}} \right) \\ &= [2]_{q^a} \tilde{\zeta}_{p, q, E}^* \left(s, p^{ax} p^{-1} q^{-1} [ax]_{p^{-1}, q^{-1}} \right). \end{aligned} \tag{3.8}$$

Hence, we easily deduce the following corollary by applying (3.8).

Corollary 3.8. *For any odd integers a , we have*

$$\begin{aligned} & \tilde{\zeta}_{p, q, E}^* \left(s, p^{ax} p^{-1} q^{-1} [ax]_{p^{-1}, q^{-1}} \right) \\ &= \frac{[2]_q}{[2]_{q^a} [a]_{p, q}^s} \sum_{i=0}^{a-1} (-1)^i q^{i(1-s)} \tilde{\zeta}_{p^a, q^a, E}^* \left(s, p^{ax+i} p^{-a} q^{-a} \left[x + \frac{i}{a} \right]_{p^{-a}, q^{-a}} \right). \end{aligned}$$

If $p = 1$ and $q \rightarrow 1$ in Theorem 3.7, then we get the following corollary.

Corollary 3.9. *For any odd integers a and b , we get*

$$b \sum_{i=0}^{a-1} (-1)^i \zeta_E \left(s, bx + \frac{bi}{a} \right) = a \sum_{i=0}^{b-1} (-1)^i \zeta_E \left(s, ax + \frac{ai}{b} \right).$$

If we take $s = -l$ in Theorem 3.7, then we have the following symmetric property of the modified (p, q) -Euler polynomials.

Theorem 3.10. *(symmetric property) For any odd integers a and b , we have*

$$\begin{aligned}
 & [2]_{q^b} [a]_{p,q}^l \sum_{i=0}^{a-1} (-1)^i q^{bi(1+l)} E_{l,p^a,q^a}^* \left(p^{abx+bi} p^{-a} q^{-a} \left[bx + \frac{bi}{a} \right]_{p^{-a},q^{-a}} \right) \\
 &= [2]_{q^a} [b]_{p,q}^l \sum_{i=0}^{b-1} (-1)^i q^{ai(1+l)} E_{l,p^b,q^b}^* \left(p^{abx+ai} p^{-b} q^{-b} \left[ax + \frac{ai}{b} \right]_{p^{-b},q^{-b}} \right).
 \end{aligned}$$

Taking $b = 1$ and replacing x by $\frac{x}{a}$ in Theorem 3.10, we get

$$\begin{aligned}
 & [2]_q [a]_{p,q}^l \sum_{i=0}^{a-1} (-1)^i q^{i(1+l)} E_{l,p^a,q^a}^* \left(p^{ax+i} p^{-a} q^{-a} \left[\frac{x}{a} + \frac{i}{a} \right]_{p^{-a},q^{-a}} \right) \tag{3.9} \\
 &= [2]_{q^a} E_{l,p,q}^* \left(p^{ax} p^{-1} q^{-1} [x]_{p^{-1},q^{-1}} \right).
 \end{aligned}$$

Therefore, we easily deduce the following corollary by applying (3.9).

Corollary 3.11. *For any odd integers a , we have the distribution formula for the modified (p, q) -Euler polynomials as follows*

$$\begin{aligned}
 & E_{l,p,q}^* (-p^{ax} [-x]_{p,q}) \\
 &= \frac{[2]_q [a]_{p,q}^l}{[2]_{q^a}} \sum_{i=0}^{a-1} (-1)^i q^{i(1+l)} E_{l,p^a,q^a}^* \left(p^{ax+i} p^{-a} q^{-a} \left[\frac{x}{a} + \frac{i}{a} \right]_{p^{-a},q^{-a}} \right).
 \end{aligned}$$

If $p = 1$ and $q \rightarrow 1$ in Theorem 3.10, then we get the following corollary.

Corollary 3.12. *For any odd integers a and b , we have*

$$a \sum_{i=0}^{a-1} (-1)^i E_l \left(bx + \frac{bi}{a} \right) = b \sum_{i=0}^{b-1} (-1)^i E_l \left(ax + \frac{ai}{b} \right).$$

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