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A STUDY ON q-SPECIAL NUMBERS AND POLYNOMIALS WITH q-EXPONENTIAL DISTRIBUTION†

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Abstract. We introduce q-special numbers and polynomials with q-exponential distribution. From these numbers and polynomials we derive some properties and identites. We also find approximated zeros of q-special polynomials and investigate property of two parameters λ, q.

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1. Introduction

The exponential distribution is one of the widely used continuous distributions. It is often used to model the time elapsed between events(see [1,2,4,5,6,7]).

Definition 1.1. For $\lambda > 0$, the probability density function of an exponential distribution is given by

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } \ x > 0 \\ 0 & \text{if } \ x \leq 0, \end{cases}$$

where X is a continuous random variable which is said to have an exponential distribution with parameter $\lambda > 0$. An interesting property of the exponential distribution is that it can be viewed as a continuous analogue of the geometric distribution. The most important property of the exponential distribution is that it is memoryless, so we can state this formally as follows:

$$P(X > a + b | X > a) = P(X > b), \qquad a, b \geq 0.$$

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Properties 1.2. For $\lambda > 0$, an exponential distribution has

(i) $(Mean)\quad E(X) = \dfrac{1}{\lambda},$

(ii) $(Variance)\quad V(X) = \dfrac{1}{\lambda^2},$

(iii) $(Moments)\quad E(X^n) = \dfrac{n!}{\lambda^n}, \qquad \text{for} \quad n = 1, 2, \cdots,$

(iv) $(Median)\quad m(X) = \dfrac{\ln(2)}{\lambda} < E(X).$

where X is a continuous random variable which is said to have an exponential distribution with parameter $\lambda > 0$.

Since the 1950s, many mathematicians have tried to find the properties of the exponential distribution using various perspectives and methods(see [1-2,4-7]) and they have found some theorems relevant to life testing using an exponential distribution. Nowadays, those who study life testing concentrate on the prediction of future records(see [3,8,13]) and mathematicians have studied expanded exponential function(see [1-15]).

Definition 1.2. The distribution function $F_n(t) = P(T_n \le t), -\infty < t < \infty$, of the q-Erlang distribution of the first kind, with parameters n, λ, and q, is given by

$$F_n(t) = 1 - \sum_{x=0}^{n-1} e_q(-\lambda t)\frac{q^{\binom{x}{2}}(\lambda t)^x}{[x]_q!}, \quad 0 < t < \infty,$$

and $F_n(t) = 0, \infty < t < 0$, where n is a positive integer, $0 < \lambda < \infty$, and $0 < q < 1$. Its q-density function $f_n(t) = d_qF_n(t)/d_qt$ is given by

$$f_n(t) = \frac{q^{\binom{n}{2}}\lambda^n}{[n-1]_q!}t^{n-1}e_q(-\lambda t), \quad 0 < t < \infty.$$

The q-density function and q-moments of the q-exponential distribution of the first kind are deduced in the following definition of Definition 2.2.

Definition 1.3. The q-density function of the q-exponential distribution of the first kind, with parameter λ and q, is given by

$$f(t) = \frac{\lambda}{e_{q^{-1}}(\lambda t)}, \quad 0 < t < \infty,$$

where $0 < \lambda < \infty$ and $0 < q < 1$. Also, its jth q-moment is given by

$$\mu'_{j,q} = E(T_q^j) = \frac{[j]_q!}{\lambda^j q^{\binom{j+1}{2}}}, \quad j = 1, 2, \cdots.$$

Since an q-exponential distribution is very basic and important in the q-distribution, we feel that we need to study q-special polynomials including this distribution in detail. We hypothesized that q-special polynomials would have some characteristic properties when we combine the q-probability denseity function which is related to the q-exponential distribution.

Based on this idea, the main concern of this paper is to define q-special polynomials and study some of their formulae. Our paper is organised as follows: in Section 2, we define q-special polynomials with q-distribution which is related to the q-exponential distribution . From this definition, we investigate some interesting identities of q-special polynomials and derive some relations.

2. Some properties of q-special polynomials with q-exponential distribution

In this section, we define q-special numbers and polynomials with q-exponential distribution. From these polynomials, we find some identities and obtain properties by using q-numbers. In addition, we will find some algebra properties of q-exponential distribution when we choose $\lambda > 0$ and $0 < q < 1$.

Definition 2.1. Let λ be a real number and $|q| < 1$. Then we define

$$\sum_{n=0}^{\infty} \mathfrak{E}_{n.q}(\lambda : x)\frac{t^n}{[n]_q!} = \frac{\lambda}{e_{q^{-1}}(\lambda t)}e_q(tx).$$

For $x = 0$ q-special numbers are defined by

$$\sum_{n=0}^{\infty} \mathfrak{E}_{n.q}(\lambda : 0)\frac{t^n}{[n]_q!} = \frac{\lambda}{e_{q^{-1}}(\lambda t)} = \lambda e_q(-\lambda t) = \sum_{n=0}^{\infty} \mathfrak{E}_{n.q}(\lambda)\frac{t^n}{[n]q!}.$$

For $\lambda > 0$ and and $0 < q < 1$, we can note that q-special numbers are q-exponential distribution. For two parameters x, y we define $\mathfrak{E}_{n.q}(\lambda : x, y)$ as

$$\sum_{n=0}^{\infty} \mathfrak{E}_{n.q}(\lambda : x, y)\frac{t^n}{[n]_q!} = \frac{\lambda}{e_{q^{-1}}(\lambda t)}e_q(tx)e_q(ty).$$

Theorem 2.2. For a real number λ, we have

$$\text{(i)} \qquad \mathfrak{E}_{n.q}(\lambda : x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_{k.q}(\lambda)x^{n-k}$$

$$\text{(ii)} \qquad \mathfrak{E}_{n.q}(\lambda : x, y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_{k.q}(\lambda : x)y^{n-k}.$$

Proof. (i) From generating function of q-special polynomials we find

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{E}_{n.q}(\lambda:x)\frac{t^n}{[n]_q!} &= \frac{\lambda}{e_{q^{-1}}(\lambda t)}e_q(tx)\\ &= \sum_{n=0}^{\infty}\mathfrak{E}_{n.q}(\lambda)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}x^n\frac{t^n}{[n]_q!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\mathfrak{E}_{k.q}(\lambda)x^{n-k}\right)\frac{t^n}{[n]_q!}.\end{aligned}$$

Comparing coefficients of the both sides, we obtain the required result.
(ii) We omit the proof of Theorem 2.2.(ii) since we can find it by the similiar method.

□

Corollary 2.3. In commutative algebra, we have

$$\mathfrak{E}_{n.q}(\lambda:x,y)=\mathfrak{E}_{n.q}(\lambda:x+y).$$

Theorem 2.4. Let λ be a real number and $|q|<1$. Then we find

$$\sum_{l=0}^{n}\sum_{k=0}^{l}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}l\\k\end{bmatrix}_q(-x)^{n-l}\lambda^{l-k}q^{\binom{n-l}{2}+\binom{l-k}{2}}\mathfrak{E}_{n.q}(\lambda:x)=\begin{cases}\lambda & \text{if } n=0\\ 0 & \text{if } n\neq 0\end{cases}.$$

Proof. From $e_{q^{-1}}(\lambda t)\neq 0$ and $e_q(tx)e_{q^{-1}}(-tx)=1$, we can turn the generating function of q-special polynomials with q-exponential distribution to

$$\sum_{n=0}^{\infty}\mathfrak{E}_{n.q}(\lambda:x)\frac{t^n}{[n]_q!}e_{q^{-1}}(\lambda t)e_{q^{-1}}(-tx)=\lambda.$$

The left hand side on the above equation is changed to

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathfrak{E}_{n.q}(\lambda:x)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\lambda^n q^{\binom{n}{2}}\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}(-x)^n q^{\binom{n}{2}}\frac{t^n}{[n]_q!}\\ &=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}_q\mathfrak{E}_{k.q}(\lambda:x)\lambda^{n-k}q^{\binom{n-k}{2}}\right]\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\left[(1-x)^n q^{\binom{n}{2}}\right]\frac{t^n}{[n]_q!}\\ &=\sum_{n=0}^{\infty}\left[\sum_{l=0}^{n}\sum_{k=0}^{l}\begin{bmatrix}n\\l\end{bmatrix}_q\begin{bmatrix}l\\k\end{bmatrix}_q(-x)^{n-l}\lambda^{l-k}q^{\binom{n-l}{2}+\binom{l-k}{2}}\mathfrak{E}_{k.q}(\lambda:x)\right]\frac{t^n}{[n]_q!}.\end{aligned}$$

From coparison of coefficients on both sides, we obtain the required result.

□

Corollary 2.5. From Theorem 2.4, we hold

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \lambda^{n-k} q^{\binom{n-k}{2}} \mathfrak{E}_{n.q}(\lambda) = \begin{cases} \lambda & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases} .$$

Theorem 2.6. For a real number λ, we derive

(i) $\quad (-1)^{n-1}\mathfrak{E}_{n.q}(-\lambda : -x) = \mathfrak{E}_{n.q}(\lambda : x),$

(ii) $\quad \displaystyle\sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \lambda^l \prod_{i=1}^{l}(1+q^{i-1})\mathfrak{E}_{n-l.q}(\lambda : -x) = (-1)^n \mathfrak{E}_{n.q}(\lambda : x).$

Proof. (i) putting $t \to -t$, $x \to -x$, and $\lambda \to -\lambda$ we can transform the generating function of q-special polynomials with q-exponential distribution as

$$\sum_{n=0}^{\infty} \mathfrak{E}_{n.q}(-\lambda : -x)\frac{(-t)^n}{[n]_q!} = \frac{\lambda}{e_{q^{-1}}(\lambda t)} e_q(tx) = -\sum_{n=0}^{\infty} \mathfrak{E}_{n.q}(\lambda : x)\frac{t^n}{[n]_q!}.$$

Hence, we have

$$(-1)^n \mathfrak{E}_{n.q}(-\lambda : -x) = -\mathfrak{E}_{n.q}(\lambda : x).$$

(ii) Substituting $-t, -x$ instead of t, x, respectively, on the generating function of q-special polynomials we can find

$$\sum_{n=0}^{\infty} \mathfrak{E}_{n.q}(\lambda : -x)\frac{(-t)^n}{[n]_q!} = \frac{e_{q^{-1}}(\lambda t)}{e_{q^{-1}}(-\lambda t)} \frac{\lambda}{e_{q^{-1}}(\lambda t)} e_q(tx).$$

We can transform the above equation as

$$\sum_{n=0}^{\infty} \mathfrak{E}_{n.q}(\lambda : -x)\frac{(-t)^n}{[n]q!} e_{q^{-1}}(-\lambda t) e_q(-\lambda t) = \sum_{n=0}^{\infty} \mathfrak{E}_{n.q}(\lambda : x)\frac{(t)^n}{[n]_q!}.$$

The left hand side is transformed as

$$\begin{aligned}
&\sum_{n=0}^{\infty}(-1)^n \mathfrak{E}_{n.q}(\lambda : -x)\frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} (-\lambda)^n q^{\binom{n}{2}} \frac{t^n}{[n]q!} \sum_{n=0}^{\infty} (-1)^n \lambda^n \frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty} ((-1)^n \mathfrak{E}_{n.q}(\lambda : -x)) \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} (-\lambda)^n \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} \right) \frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty} ((-1)^n \mathfrak{E}_{n.q}(\lambda : -x)) \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} \left((-\lambda)^n \prod_{i=1}^{n}(1+q^{i-1}) \right) \frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q (-1)^n \lambda^l \prod_{i=1}^{l}(1+q^{i-1}) \mathfrak{E}_{n-1.q}(\lambda : -x) \right) \frac{t^n}{[n]_q!}.
\end{aligned}$$

Therefore, we complete proof of the Theorem 2.6.

$\square$

Theorem 2.7. Let $|q|<1$. Then we derive

$$\text{(i)}\qquad (-1)^n\lambda\mathfrak{E}_{n.q^{-1}}(x:\lambda)=x\mathfrak{E}_{n.q^{-1}}(\lambda:x)$$

$$\begin{aligned}\text{(ii)}\qquad &\sum_{l=0}^{n}\sum_{k=0}^{l}\begin{bmatrix}n\\l\end{bmatrix}_q\begin{bmatrix}l\\k\end{bmatrix}_q\lambda^{l-k}q^{\binom{k}{2}}\mathfrak{E}_{k.q^{-1}}(\lambda:x)x^{n-l}\\ &=\sum_{l=0}^{n}\sum_{k=0}^{l}\begin{bmatrix}n\\l\end{bmatrix}_q\begin{bmatrix}l\\k\end{bmatrix}_q\lambda^{l-k}q^{\binom{l-k}{2}+\binom{n-l}{2}}\mathfrak{E}_{k.q}(\lambda:x)x^{n-l}.\end{aligned}$$

Proof. (i) Setting q^{-1}, $-t$ instead of q, t, respectively, and $x\leftrightarrow\lambda$, we get

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{E}_{n.q^{-1}}(x:\lambda)\frac{(-t)^n}{[n]_{q^{-1}}!}&=\frac{x}{e_q(-tx)}e_{q^{-1}}(-\lambda t)\\ &=\frac{x}{\lambda}\frac{\lambda}{e_q(\lambda t)}e_{q^{-1}}(tx)\\ &=\frac{x}{\lambda}\sum_{n=0}^{\infty}\mathfrak{E}_{n.q^{-1}}(\lambda:x)\frac{t^n}{[n]_{q^{-1}}!}.\end{aligned}$$

Therefore, we have

$$(-1)^n\mathfrak{E}_{n.q^{-1}}(x:\lambda)=\frac{x}{\lambda}\mathfrak{E}_{n.q^{-1}}(\lambda:x).$$

The required relation now follows immediately.

(ii) Substituting q^{-1} instead of q, we lead to

$$\sum_{n=0}^{\infty}\mathfrak{E}_{n.q^{-1}}(x:\lambda)\frac{t^n}{[n]_{q^{-1}}!}=\left(\frac{e_{q^{-1}}(tx)e_{q^{-1}}(\lambda t)}{e_q(\lambda t)e_q(tx)}\right)\frac{\lambda}{e_{q^{-1}(\lambda t)}}e_q(tx).$$

We can turn the left hand side on above equation as

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathfrak{E}_{n.q^{-1}}(x:\lambda)\frac{t^n}{[n]_{q^{-1}}!}e_q(\lambda t)e_q(tx)\\ &=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\lambda^{n-k}q^{\binom{k}{2}}\mathfrak{E}_{k.q^{-1}}(\lambda:x)\right)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}x^n\frac{t^n}{[n]_q!}\\ &=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{k=0}^{l}\begin{bmatrix}n\\l\end{bmatrix}_q\begin{bmatrix}l\\k\end{bmatrix}_q\lambda^{l-k}q^{\binom{k}{2}}\mathfrak{E}_{n.q^{-1}}(\lambda:x)x^{n-l}\right)\frac{t^n}{[n]_q!},\end{aligned}$$

and the right hand side is turn to

$$\begin{aligned}
&e_{q^{-1}}(tx)e_{q^{-1}}(\lambda t)\sum_{n=0}^{\infty}\mathfrak{E}_{n.q}(\lambda : x)\frac{t^n}{[n]_q!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \lambda^{n-k}q^{\binom{n-k}{2}}\mathfrak{E}_{k.q}(\lambda : x)\right)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}q^{\binom{n}{2}}x^n\frac{t^n}{[n]_q!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{k=0}^{l}\begin{bmatrix} n \\ l \end{bmatrix}_q\begin{bmatrix} l \\ k \end{bmatrix}_q \lambda^{l-k}q^{\binom{l-k}{2}+\binom{n-l}{2}}\mathfrak{E}_{k.q}(\lambda : x)x^{n-l}\right)\frac{t^n}{[n]_q!}.
\end{aligned}$$

Comparing the coefficients ar once gives the required relation.

□

Corollary 2.8. From Theorem 2.7.(ii), we hold

$$\mathfrak{E}_{0.q^{-1}}(\lambda : x) = \mathfrak{E}_{0.q}(\lambda : x).$$

Theorem 2.9. For a real number λ, we investigate

$$\text{(i)}\qquad x^n = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \lambda^{n-k-1}q^{\binom{n-k}{2}}\mathfrak{E}_{k.q}(\lambda : x)$$

$$\text{(ii)}\qquad q^{\binom{n}{2}}x^n = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \lambda^{n-k-1}q^{\binom{n}{2}}\mathfrak{E}_{n.q^{-1}}(\lambda : x)$$

$$\text{(iii)}\qquad \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{n-l}\lambda^{n-k+1}x^k = \mathfrak{E}_{n.q}(\lambda : x).$$

Proof. From the generating function of q-special polynomials, we have

$$\sum_{k=0}^{n}\mathfrak{E}_{n.q}(\lambda : x)\frac{t^n}{[n]q!}e_{q^{-1}}(\lambda t) = \lambda\mathfrak{e}_q(tx).$$

We can turn the left hand side on the above equation to

$$\sum_{k=0}^{n}\mathfrak{E}_{n.q}(\lambda : x)\frac{t^n}{[n]q!}\sum_{k=0}^{n}q^{\binom{n}{2}}\lambda^n\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \lambda^{n-k}q^{\binom{n-k}{2}}\mathfrak{E}_{k.q}(\lambda : x)\right)\frac{t^n}{[n]q!},$$

and the right hand side is turn to

$$\lambda e_q(tx) = \lambda\sum_{n=0}^{\infty}x^n\frac{t^n}{[n]_q!}.$$

Hence, we obtain

$$\lambda x^n = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \lambda^{n-k}q^{\binom{n-k}{2}}\mathfrak{E}_{k.q}(\lambda : x),$$

and complete the proof of Theorem 2.9.(i).

(ii) Substituting q^{-1} instead of q on the generating function of q-special polynomials, we can make

$$\sum_{n=0}^{\infty} \mathfrak{E}_{n.q^{-1}}(\lambda : x)\frac{t^n}{[n]q^{-1}!}e_q(\lambda t) = \lambda e_{q^{-1}}(tx).$$

We omit the proof of Theorem 2.9.(ii) because the required relation now follows immediately by the same method of (i).

(iii) Using a property of q-numbers, $e_q(-\lambda t)e_q(\lambda t) = 1$, we get

$$\begin{aligned}\sum_{n=0}^{\infty} \mathfrak{E}_{n.q}(\lambda : x)\frac{t^n}{[n]q!} &= \lambda e_q(-\lambda t)e_q(tx)\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{n-k}\lambda^{n-k+1}x^k\right)\frac{t^n}{[n]_q!}.\end{aligned}$$

The required relation now follows at once.

□

3. The observation of scattering zeros of the q-special polynomials

This section aims to find an approximate root of q-special polynomials. Using the Mathematica program, the structure of the accumulation of the root in three-dimensional space is verified and the characteristics of approximate roots due to changes in q and a are discussed. For this, we use Theorem 2.9.(iii) to calculate some elements of q-special numbers and polynomials. The first few q-special numbers are

$$\begin{aligned}
&\mathfrak{E}_{0.q}(\lambda) = \lambda,\\
&\mathfrak{E}_{1.q}(\lambda) = -\lambda(\lambda - 1),\\
&\mathfrak{E}_{2.q}(\lambda) = \lambda\{1 + \lambda^2 - \lambda(1 + q)\},\\
&\mathfrak{E}_{3.q}(\lambda) = -\lambda(\lambda - 1)\{1 + \lambda^2 - \lambda q(1 + q)\},\\
&\mathfrak{E}_{4.q}(\lambda) = \lambda\{1 + \lambda^4 - \lambda(1 + q + q^2 + q^3) - \lambda^3(1 + q + q^2 + q^3)\\
&\qquad + \lambda^2(1 + q + 2q^2 + q^3 + q^4)\},\\
&\mathfrak{E}_{5.q}(\lambda) = -\lambda\{1 + \lambda^5 - \lambda^4(1 + q + q^2 + q^3 + q^4) + \lambda^3(1 + q^2)(1 + q + q^2 + q^3 + q^4)\\
&\qquad - \lambda^2(1 + q^2)(1 + q + q^2 + q^3 + q^4) + \lambda(1 + q + q^2 + q^3 + q^4),\\
&\qquad \cdots,
\end{aligned}$$

and the first six q-special polynomials are:

$$\begin{aligned}
&\mathfrak{E}_{0.q}(\lambda : x) = \lambda,\\
&\mathfrak{E}_{1.q}(\lambda : x) = -\lambda(\lambda - x),\\
&\mathfrak{E}_{2.q}(\lambda : x) = \lambda\{\lambda^2 - \lambda(1+q)x + x^2\},\\
&\mathfrak{E}_{3.q}(\lambda : x) = -\lambda(\lambda - x)\{\lambda^2 - \lambda q(1+q)x + x^2\},\\
&\mathfrak{E}_{4.q}(\lambda : x) = \lambda\{\lambda^4 - \lambda^3(1+q+q^2+q^3)x + \lambda^2(1+q+2q^2+q^3+q^4)x^2\\
&\qquad - \lambda(1+q+q^2+q^3)x^3 + x^4\},\\
&\mathfrak{E}_{5.q}(\lambda : x) = -\lambda\{\lambda^5 - \lambda^4(1+q+q^2+q^3+q^4)x + \lambda^3(1+q^2)(1+q+q^2+q^3+q^4)x^2\\
&\qquad - \lambda^2(1+q^2)(1+q+q^2+q^3+q^4)x^3 + \lambda(1+q+q^2+q^3+q^4)x^4 + x^5\},\\
&\qquad \cdots .
\end{aligned}$$

Given that values of λ and q have $0.1, 0.5$, and 0.9, respectively, the approximate root of each of the q-special polynomials when $n = 1, 2, 3, 4$, and 5 can be checked by the following table. Here, the results show that the value of the real root out of the approximate root values was also increased as λ and q were increased. In addition, this study found that only a single real root exists when $n = 2k+1, k > 0$. This phenomenon is a rather special phenomenon even if n is increased.

n	$\lambda = q = 0.1$	$\lambda = q = 0.5$	$\lambda = q = 0.9$
1	0.1	0.5	0.9
2	0.055 - 0.0835165i 0.055 + 0.0835165i	0.375 - 0.330719i 0.375 + 0.330719i	0.855 - 0.281025i 0.855 + 0.281025i
3	0.0055 - 0.0998486i 0.1 0.0055 + 0.0998486i	0.1875 - 0.463512i 0.5 0.1875 + 0.463512i	0.7695 - 0.466765i 0.9 0.7695 + 0.466765i
4	-0.0267108 - 0.0963667i -0.0267108 + 0.0963667i 0.0822608 - 0.0568609i 0.0822608 + 0.0568609i	0.0264979 - 0.499297i 0.0264979 + 0.499297i 0.442252 - 0.233266i 0.442252 + 0.233266i	0.671574 - 0.599156i 0.671574 + 0.599156i 0.875976 - 0.206559i 0.875976 + 0.206559i
5	-0.0470215 - 0.0882552i -0.0470215 + 0.0882552i 0.1 0.0525765 - 0.085063i 0.0525765 + 0.085063i	-0.0971941 - 0.490462i -0.0971941 + 0.490462i 0.5 0.331569 - 0.374248i 0.331569 + 0.374248i	0.570816 - 0.695823i 0.570816 + 0.695823i 0.9 0.821979 - 0.366539i 0.821979 + 0.366539i
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Let us assume that $n = 100$. The figure when λ and q are given as $0.1, 0.5$, and 0.9 can be found in Figure 1. In Figure 1, this study shows that the position of the roots appears as a circle, which becomes larger as the radius increases. In addition, as λ and q increase, the root is further away from the imaginary number axis when the real number is negative. Here, this study can assume that

λ and q affect the radius size and distancing from the imaginary number axis in the negative real numbers, respectively.

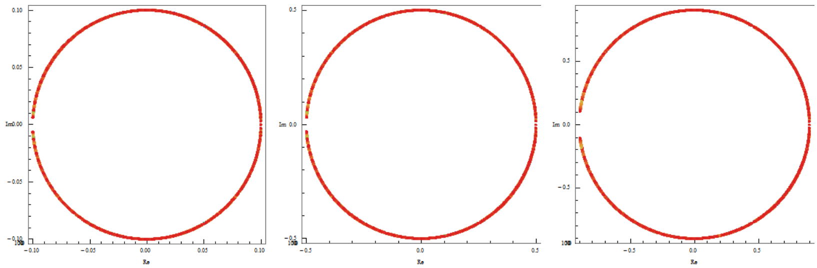

FIGURE 1. Zeros of $\mathfrak{E}_{n.q}(x)$ for $q = \lambda = 0.1$, $q = \lambda = 0.5$, $q = \lambda = 0.9$

Figure 2 shows a graph in which the value of λ changes ($\lambda = 1, 10$, and 100) when $n = 100$. Here, the approximated root maintains a circle shape, and the level of the gap where the negative real number and imaginary number axis come into contact is not changed, but the radius is matched with the value of λ as 1 on the left, 10 in the middle, and 100 on the right.

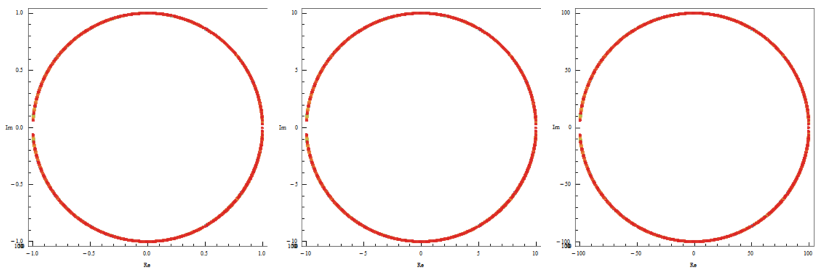

FIGURE 2. Zeros of $\mathfrak{E}_{n.q}(x)$ for $\lambda = 1$, $\lambda = 10$, $\lambda = 100$, and $q = 0.1$

The next table is configured using $q = 0.1$ while changing λ. It reveals the approximate root value of the q-special polynomials given that $\lambda = 1, 10$, and 100. Here, when n is an odd number term, it has only a single real root and the value is matched with λ.

n	$\lambda = 1, q = 0.1$	$\lambda = 10, q = 0.1$	$\lambda = 100, q = 0.1$
1	1	10	100
2	0.55 - 0.835165i 0.55 + 0.835165i	5.5 - 8.35165i 5.5 + 8.35165i	55 - 83.5165 i 55 + 83.5165 i
3	0.055 - 0.998486i 1 0.055 + 0.998486i	0.55 - 9.98486i 10 0.55 + 9.98486i	5.5 - 99.8486 i 100 5.5 + 99.8486 i
4	-0.267108 - 0.963667i -0.267108 + 0.963667i 0.822608 - 0.568609i 0.822608 + 0.568609i	-2.67108 - 9.63667i -2.67108 + 9.63667i 8.22608 - 5.68609i 8.22608 + 5.68609i	-26.7108 - 96.3667i -26.7108 + 96.3667i 82.2608 - 56.8609i 82.2608 + 56.8609i
5	-0.470215 - 0.882552i -0.470215 + 0.882552i 1 0.525765 - 0.85063i 0.525765 + 0.85063i	-4.70215 - 8.82552i -4.70215 + 8.82552i 10 5.25765 - 8.5063i 5.25765 + 8.5063i	-47.0215 - 88.2552i -47.0215 + 88.2552i 100 52.5765 - 85.063i 52.5765 + 85.063i
⋮	⋮	⋮	⋮

Conjecture 3.1. The value of λ in the q-special polynomials determines the real number root and radius when n is an odd number.

Figure 3 shows a graph when changing $q = 0.1, 0.5$, and 0.9 while fixing $\lambda = 10$ given that $n = 100$. Here, a real number value does not exist in the negative real number section, and the gap of the approximate root value becomes larger when q increases.

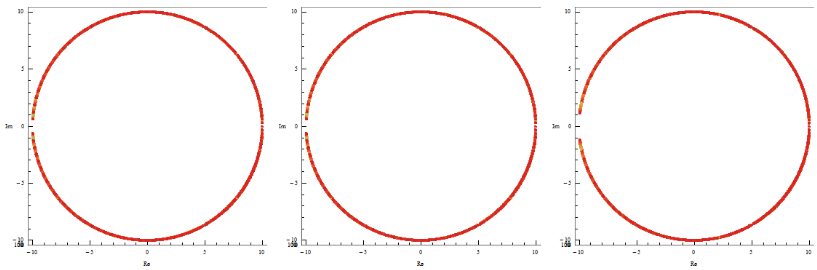

FIGURE 3. Zeros of $\mathfrak{E}_{n.q}(x)$ for $q = 0.1$, $q = 0.5$, $q = 0.9$, and $\lambda = 10$

The next table presents approximate roots when $q = 0.1, 0.5$, and 0.9 while fixing $\lambda = 10$. Since λ is fixed to 10, it always has a value of 10 when a real root exists.

n	$\lambda = 10, q = 0.1$	$\lambda = 10, q = 0.5$	$\lambda = 10, q = 0.9$
1	10	10	10
2	5.5 - 8.35165i	7.5 - 6.61438i	9.5 - 3.1225i
	5.5 + 8.35165i	7.5 + 6.61438i	9.5 + 3.1225i
3	0.55 - 9.98486i	3.75 - 9.27025i	8.55 - 5.18628i
	10	10	10
	0.55 + 9.98486i	3.75 + 9.27025i	8.55 + 5.18628i
4	-2.67108 - 9.63667i	0.529958 - 9.98595i	7.46194 - 6.65729i
	-2.67108 + 9.63667i	0.529958 + 9.98595i	7.46194 + 6.65729
	8.22608 - 5.68609i	8.84504 - 4.66532i	9.73306 - 2.29509i
	8.22608 + 5.68609i	8.84504 + 4.66532i	9.73306 + 2.29509i
5	-4.70215 - 8.82552i	-1.94388 - 9.80925i	6.3424 - 7.73136i
	-4.70215 + 8.82552i	-1.94388 + 9.80925i	6.3424 + 7.73136i
	10	10	10
	5.25765 - 8.5063i	6.63138 - 7.48497i	9.1331 - 4.07265i
	5.25765 + 8.5063i	6.63138 + 7.48497i	9.1331 + 4.07265i
6	-6.02349 - 7.98233i	-3.78131 - 9.25752i	5.23912 - 8.51772i
	-6.02349 + 7.98233i	-3.78131 + 9.25752i	5.23912 + 8.51772i
	2.51625 - 9.67825i	4.29942 - 9.02856i	8.36729 - 5.47617i
	2.51625 + 9.67825i	4.29942 + 9.02856i	8.36729 + 5.47617i
	9.06279 - 4.2268i	9.32564 - 3.61004i	9.82154 - 1.8808i
	9.06279 + 4.2268i	9.32564 + 3.61004i	9.82154 + 1.8808i
⋮	⋮	⋮	⋮

Conjecture 3.2. The value of q in the q-special polynomials determines the degree of the gap of the approximated root in the negative real number section.

4. Conclusions

Two variables, λ and q, in the q-special polynomials determine the distribution of the approximated roots. This study verifies the special rule when n is an odd number. That is, the value of λ determines the location of the approximated root, and q represents the degree of the gap in the negative real number section.

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