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EXISTENCE OF PERIODIC SOLUTIONS WITH PRESCRIBED MINIMAL PERIOD FOR A FOURTH ORDER NONLINEAR DIFFERENCE SYSTEM†

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Abstract. In this article, we consider a fourth order nonlinear difference system. By making use of the critical point theory, we obtain some new existence theorems of at least one periodic solution with minimal period. Our main approach used in this article is the variational technique and the Saddle Point Theorem.

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1. Introduction

In this article, we are interested in the existence of periodic solutions with prescribed minimal period for the fourth order nonlinear difference system

$$\Delta^4 u_{n-2} = f(n, u_n),\ n \in \mathbb{Z}, \tag{1.1}$$

where $\Delta^j u_n = \Delta(\Delta^{j-1} u_n)$ $(j = 2, 3, 4)$, $\Delta u_n = u_{n+1} - u_n$, $f(s, u) \in C^1(\mathbb{R}^2, \mathbb{R})$, $f(s+T, u) = f(s, u), \forall (s, u) \in \mathbb{R}^2$. Here, $\mathbb{Z}$ denotes the sets of integers, $\mathbb{R}$ denotes the sets of real numbers. Given $a \le b$ in $\mathbb{Z}$, let $\mathbb{Z}[a, b] := \mathbb{Z} \cap [a, b]$ and $T \ge 3$ be a given integer.

We may regard (1.1) as being a discrete analogue of the following fourth order differential system

$$u^{(4)}(s) = f(s, u(s)),\ s \in \mathbb{R}, \tag{1.2}$$

which is used to describe the stationary states of the deflection of an elastic beam [29]. Equations similar to (1.2) arise in the study of the existence of solutions to differential equations [6, 9–12, 14, 15, 25, 26].

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The study of nonlinear difference equations [1–5, 7, 8, 13, 16–19, 21, 24, 27, 30, 33, 35] is growing up in the last years, not only as a fundamental tool in the discrete analogue of a differential equation [22, 23, 28, 31, 32, 34], but also as a useful model for several economical and population problems.

If $f(n, u_n) = 0$, Domshlak and Matakaev [7] considered the delay difference equation

$$u_{n+1} - u_n + b_n u_{n-k} = 0, b_n > 0, n \geq 1. \tag{1.3}$$

Conditions for the existence and for the non-existence of eventually positive solution of (1.3) are obtained.

Cabada and Dimitrov [3] devoted to the study of nonlinear singular and nonsingular fourth order difference equations

$$u_{n+4} + Mu_n = \lambda g_n f(u_n) + c_n, n \in \{0, 1, \cdots, T-1\}$$

coupled with periodic boundary value conditions. Some existence and nonexistence results are given by using Krasnoselskii's fixed point theorems in cones.

By using the critical point method, some new criteria are obtained for the existence and multiplicity of periodic solutions for the following fourth-order nonlinear functional difference equation [18]

$$\Delta^2\left(p_{n-2}\Delta^2 u_{n-2}\right) - \Delta\left(q_{n-1}\Delta u_{n-1}\right) = f(n, u_{n+1}, u_n, u_{n-1}),\ n \in \mathbb{Z}.$$

The proof is based on the linking theorem in combination with variational technique.

A fourth-order nonlinear difference equation in the form

$$\Delta a_n\left(\Delta b_n\left(\Delta c_n\left(\Delta u_n\right)^{\gamma}\right)^{\beta}\right)^{\alpha} + d_n u_{n+\tau}^{\lambda} = 0$$

is considered. *Došlá*, Krej*čová* and Marini [8] presented a classification of nonoscillatory solutions which is based on their asymptotic behaviour and gave necessary and sufficient conditions for the existence of the so-called minimal and maximal solutions.

In [22], a class of fourth-order nonlinear difference equation

$$\Delta^2\left(p_{n-1}\Delta^2 u_{n-2}\right) - \Delta\left(q_n\Delta u_{n-1}\right) + r_n u_n = f(n, u_{n+1}, u_n, u_{n-1}),\ n \in \mathbb{Z},$$

is studied. By making use of the critical point theory, various sets of sufficient conditions for the existence of homoclinic solutions are established and some new results are obtained.

Raafat [21] introduced an explicit formula and discussed the global behavior of solutions of the fourth order difference equation

$$u_{n+1} = \frac{a u_{n-3}}{b - c u_{n-1} u_{n-3}}, n = 0, 1, 2, \cdots,$$

where a, b, c are positive real numbers and the initial conditions u_3, u_2, u_1, u_0 are real numbers.

Motivated by the papers [6, 11, 13], here we deal with the existence of periodic solution with minimal period for fourth order nonlinear difference system (1.1).

Our main approach used in this article is the variational technique and the Saddle Point Theorem.

Throughout this article, let

$$\omega = \frac{2\pi}{T}.$$

Assume that there is a function $F(s,u)$ with $F(s+T,u) = F(s,u)$, $F(-s,-u) = F(s,u)$ and

$$F(s,u) = \int_0^u f(s,t)dt,$$

for any $(s,u) \in \mathbb{R}^2$. As usual, a solution $\{u_n\}$ of (1.1) is said to be periodic of period T if

$$u_{j+T} = u_j, \ \forall j \in \mathbb{Z}.$$

Our main results are the following theorems.

Theorem 1.1. *Assume that the function $F(s,u)$ satisfies the following assumptions.*

(F_1) *There exist constants $a_1 > 0$, $a_2 > 0$ and $0 \le \theta < 1$ such that*

$$f(s,u) \le a_1|u|^\theta + a_2, \ \forall (s,u) \in \mathbb{R}^2.$$

(F_2) $\lim\limits_{|u|\to+\infty} \frac{\int_0^u f(s,t)dt}{|u|^{2\theta}} = \infty$ *for $s \in \mathbb{R}$.*

(F_3) *There exist three constants $\sigma > 0$ and $\rho > \varrho > 0$ such that*

$$\left(\frac{\partial^2 F(s,u)}{\partial u^2}\varsigma, \varsigma\right) \le \rho\varsigma^2, \forall (s,u,\varsigma) \in \mathbb{R}^3$$

and

$$\left(\frac{\partial^2 F(s,u)}{\partial u^2}\varsigma, \varsigma\right) \ge \varrho\varsigma^2, \forall |u| \le \sigma, (s,u) \in \mathbb{R}^2.$$

(F_4) *If β is a rational number, u is a solution of (1.1) with a minimal period βT, and $f(s,u)$ also has a minimal period βT, then β must be an integer.*

(F_5) *For any integer $q > 1$ and above two constants ρ and ϱ,*

$$\sin^4 \frac{\omega\chi_q}{2q} > \frac{\rho}{16}, \ \sin^4 \frac{\omega}{2q} < \frac{\varrho}{16}$$

and

$$\sum_{n=1}^{qT} f^2(n,0) < \frac{4\pi\delta^2 \left(16\sin^4 \frac{\omega\chi_q}{2q} - \rho\right)\left(\varrho - 16\sin^4 \frac{\omega}{2q}\right)}{\omega},$$

where χ_q is a prime number of q.

Then (1.1) admits at least one periodic solution with minimal period qT.

Corollary 1.2. *Assume that the function $F(s,u)$ satisfies the assumptions (F_1)–(F_4) and*

$$\sin^4 \frac{\omega}{2} > \frac{\rho}{16}, \ \sin^4 \frac{\omega}{2q} < \frac{\varrho}{16}.$$

If

$$\sum_{n=1}^{qT} f^2(n,0) < \frac{4\pi\delta^2\left(16\sin^4\frac{\omega}{2}-\rho\right)\varrho}{\omega}.$$

Then there exists a positive constant C such that for any prime integer $q > C$, (1.1) admits at least one periodic solution with minimal period qT.

Theorem 1.3. *Assume that the function $F(s,u)$ satisfies the assumptions (F_1)–(F_4) and*

$$\sin^4\frac{\omega\chi_q}{2q} > \frac{\rho}{16},\ \sin^4\frac{\omega}{2q} < \frac{\varrho}{16}.$$

For any $s \in \mathbb{R}$, $f(s,0) = 0$. Then (1.1) admits at least one periodic solution with minimal period qT.

Corollary 1.4. *Assume that the function $F(s,u)$ satisfies the assumptions (F_1)–(F_4) and*

$$\sin^4\frac{\omega}{2} > \frac{\rho}{16},\ \sin^4\frac{\omega}{2q} < \frac{\varrho}{16}.$$

For any $s \in \mathbb{R}$, $f(s,0) = 0$. Then there exists a positive constant C such that for any prime integer $q > C$, (1.1) admits at least one periodic solution with minimal period qT.

2. Variational framework

To apply the critical point theory, we define the finite dimensional Hilbert space H_{qT} as

$$H_{qT} := \{u : \mathbb{Z} \to \mathbb{R} | u_{n+qT} = u_n,\ \forall n \in \mathbb{Z}\}$$

and equip it with the inner product

$$(u,v) := \sum_{j=1}^{qT} u_j v_j,\ \forall u,v \in H_{qT},$$

and the induced norm

$$\|u\| := \left(\sum_{j=1}^{qT} u_j^2\right)^{\frac{1}{2}},\ \forall u \in H_{qT},$$

where $(\cdot,\cdot)$ and $\|\cdot\|$ denote the usual inner product and the usual inner norm in $\mathbb{R}$.

We define a functional J on H_{qT} by

$$J(u) := \frac{1}{2}\sum_{n=1}^{qT}\left(\Delta^2 u_{n-1}\right)^2 - \sum_{n=1}^{qT} F(n,u_n). \tag{2.1}$$

After a careful computation, by (2.1), we get that

$$\frac{\partial J}{\partial u_n} = \Delta^4 u_{n-2} - f(n,u_n),\ \forall n \in \mathbb{Z}[1,qT].$$

Therefore, $J'(u) = 0$ if and only if

$$\Delta^4 u_{n-2} = f(n, u_n),\ \forall n \in \mathbb{Z}[1, qT].$$

Note that $\{u_n\} \in H_{qT}$ is qT-periodic in n, and $f(s, u)$ is T-periodic in s. Thus, we reduce the problem of finding a qT-periodic solution of (1.1) to that of seeking a critical point of the functional J on H_{qT}.

We rewrite $J(u)$ as

$$\begin{aligned} J(u) &= \frac{1}{2}\sum_{n=1}^{qT} \left(\Delta^2 u_n\right)^2 - \sum_{n=1}^{qT} F(n, u_n) \\ &= \frac{1}{2} u^* X u - G(u), \end{aligned} \tag{2.2}$$

where u^* means the transpose of a vector u, X is the corresponding $qT \times qT$ matrix to the quadratic form

$$\sum_{n=1}^{qT} \left(\Delta^2 u_n\right)^2$$

with $u_{n+T} = u_n$ for $n \in \mathbb{Z}$, and $G(u) = \sum\limits_{n=1}^{qT} F(n, u_n)$.

It is easy to see that X is positive semi-definite. Define X^0 and X^+ by the eigenspaces associated with the 0 eigenvalue, all positive eigenvalues, respectively. Then $H_{qT} = X^0 \oplus X^+$. Let $E^+(X)$ be the sets of all the positive eigenvalues of X, $\underline{\lambda} = \min\{\lambda | \lambda \in E^+(X)\}$, $\bar{\lambda} = \max\{\lambda | \lambda \in E^+(X)\}$. Thus, by (2.2),

$$\frac{\underline{\lambda}}{2} - G(u) \le J(u) \le \frac{\bar{\lambda}}{2} - G(u). \tag{2.3}$$

Set

$$E = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}_{qT \times qT}.$$

It is obvious that the eigenvalues of E are

$$\lambda_k = 4\sin^2 \frac{k\pi}{qT}, \forall k \in \mathbb{Z}[0, qT-1], \tag{2.4}$$

0 is an eigenvalue of E and $(1, 1, \cdots, 1)^*$ is eigenvector associated to 0.

Set

$$\lambda_{\min} = \min\{\lambda_1, \lambda_2, \cdots, \lambda_{qT-1}\},$$

and

$$\lambda_{\max} = \max\{\lambda_1, \lambda_2, \cdots, \lambda_{qT-1}\}.$$

Then by (2.4), $\lambda_{\min} = 4\sin^2 \frac{\pi}{qT}$ and $\lambda_{\max} \le 4$.

For $1 \le i \le \left[\frac{qT-1}{2}\right]$, let

$$\epsilon_i = \left(\cos\frac{2i\pi}{qT}, \cos\frac{2i\pi\cdot 2}{qT}, \cdots, \cos\frac{2i\pi\cdot qT}{qT}\right)^*,$$

and

$$\varepsilon_i = \left(\sin\frac{2i\pi}{qT}, \sin\frac{2i\pi\cdot 2}{qT}, \cdots, \sin\frac{2i\pi\cdot qT}{qT}\right)^*.$$

where $[\cdot]$ means the greatest-integer function. It is easy to see that ϵ_i and ε_i are the eigenvectors of E corresponding to the eigenvalues λ_i.

Define

$$R = \text{span}\left\{\epsilon_i, i \in \mathbb{Z}\left[1, \left[\frac{qT-1}{2}\right]\right]\right\},$$

and

$$S = \text{span}\left\{\varepsilon_i, i \in \mathbb{Z}\left[1, \left[\frac{qT-1}{2}\right]\right]\right\}.$$

We need to consider the following two cases.

On one hand, if qT is even, clearly $\alpha = (-1, 1, -1, 1, \cdots, -1, 1)^*$ is the eigenvector corresponding to 4. Taking $T = \text{span}\,\{\alpha\}$, we know that $H_{qT} = X^0 \oplus T \oplus R \oplus S$. Thus, for any $u \in H_{qT}$,

$$u_n = a + \sum_{i=1}^{\left[\frac{qT-1}{2}\right]}\left(a_i\cos\frac{\omega i}{q}n + b_i\sin\frac{\omega i}{q}n\right), \forall n \in \mathbb{Z},$$

where a, a_i and b_i are constants.

On the other hand, if qT is odd, we know that $H_{qT} = X^0 \oplus R \oplus S$. Therefore, for any $u \in H_{qT}$,

$$u_n = a + (-1)^n b + \sum_{i=1}^{\left[\frac{qT-1}{2}\right]}\left(a_i\cos\frac{\omega i}{q}n + b_i\sin\frac{\omega i}{q}n\right), \forall n \in \mathbb{Z},$$

where a, b, a_i and b_i are constants.

3. Some basic lemmas

Let D be a real Banach space. Define the symbol B_r as the open ball in D about 0 of radius r, ∂B_r as its boundary, and $\bar{B}_r$ as its closure.

Lemma 3.1. *(Saddle Point Theorem [20]). Let D be a real Banach space, $D = D_1 \oplus D_2$, where $D_1 \neq \{0\}$ and is finite dimensional. Assume that $J \in C^1(D, \mathbb{R})$ satisfies the Palais-Smale condition and*
(J_1) there are constants $\gamma,\ \delta > 0$ such that $J|_{\partial B_\delta \cap D_1} \le \gamma$;
(J_2) there is $e \in B_\delta \cap D_1$ and a constant $\vartheta \ge \gamma$ such that $J_{e+D_2} \ge \vartheta$.

Then J has a critical value $d \ge \vartheta$, where

$$d = \inf_{g\in\Omega}\max_{u\in B_\delta\cap D_1} J(g(u)),\ \Omega = \{g \in C(\bar{B}_\delta \cap D_1, D) \mid g|_{\partial B_\delta \cap D_1} = I\}$$

and I denotes the identity operator.

Set

$$\bar{H}_{qT} = \{u \in H_{qT} | u_{-n} = u_n, \forall n \in \mathbb{Z}\}.$$

Therefore, $\bar{H}_{qT} = S$ and

$$u_n = \sum_{i=1}^{\left[\frac{qT-1}{2}\right]} b_i \sin \frac{\omega i}{q} n, \forall n \in \mathbb{Z}.$$

In order to prove Theorem 1.1, we need the following three lemmas.

Lemma 3.2. *Assume that the assumptions* (F_1) *and* (F_2) *are satisfied. Then* J *satisfies the Palais-Smale condition.*

Proof. Let $\left\{u^{(i)}\right\} \subset H_{qT}$ be a sequence such that $\left\{J\left(u^{(i)}\right)\right\}$ is bounded and

$$\lim_{i\to\infty} J'\left(u^{(i)}\right) = 0.$$

Then there exist constants $K > 0$ and $i_0 \in \mathbb{N}$, where $\mathbb{N}$ denotes the sets of integers, such that $\left|J\left(u^{(i)}\right)\right| \le K$ and $\left|J'\left(u^{(i)}\right)\right| \le 1$ for $i \ge i_0$.

Indeed, suppose that $\left\{u^{(i)}\right\}$ is unbounded. Without loss of generality (otherwise, choose a subsequence), we suppose that $\left\|u^{(i)}\right\| \ge 1$ for $i \in \mathbb{N}$ and

$$\lim_{i\to\infty} \left\|u^{(i)}\right\| = \infty.$$

Let $u^{(i)} = v^{(i)} + w^{(i)}$ where $v^{(i)} \in X^0$ and $w^{(i)} \in X^+$. By (F_1),

$$\begin{aligned}\|\nabla G(u)\| &= \left\|(f(1,u_1), f(2,u_2), \cdots, f(qT, u_{qT}))^*\right\| \\ &\le \sum_{i=1}^{qT} |f(i,u_i)| \\ &\le \sum_{i=1}^{qT} \left(a_1|u_i|^\theta + a_2\right) \\ &\le qTa_1\|u\|^\theta + aTb_2,\end{aligned}$$

where ∇ is defined by the gradient. Combining with $J'\left(u^{(i)}\right) = Xu^{(i)} - \nabla G\left(u^{(i)}\right)$, we have

$$\begin{aligned}\left\|Xu^{(i)}\right\| &\le \left\|J'\left(u^{(i)}\right)\right\| + \left\|\nabla G(u^{(i)})\right\| \\ &\le 1 + qTa_1\left\|u^{(i)}\right\|^\theta + qTa_2.\end{aligned}$$

It comes from (2.3) and $\left\|Xu^{(i)}\right\| = \left\|Xw^{(i)}\right\| \ge \underline{\lambda}\left\|w^{(i)}\right\|$ that

$$\begin{aligned}\underline{\lambda}\left\|w^{(i)}\right\| &\le 1 + qTa_1\left\|u^{(i)}\right\|^\theta + qTa_2 \\ &\le (1 + qTa_1 + qTa_2)\left\|u^{(i)}\right\|^\theta.\end{aligned}$$

That is,

$$\left\|w^{(i)}\right\| \leq \frac{(1+qTa_1+qTa_2)\left\|u^{(i)}\right\|^{\theta}}{\underline{\lambda}}. \tag{3.1}$$

Therefore, by (3.1),

$$\lim_{i\to\infty}\frac{\left\|w^{(i)}\right\|}{\left\|u^{(i)}\right\|}=0 \tag{3.2}$$

and

$$\lim_{i\to\infty}\frac{\left\|v^{(i)}\right\|}{\left\|u^{(i)}\right\|}=1. \tag{3.3}$$

Since

$$\begin{aligned}
G\left(v^{(i)}\right) &= \frac{1}{2}\left(u^{(i)}\right)^{*}Xu^{(i)}-\left[G\left(u^{(i)}\right)-G\left(v^{(i)}\right)\right]-J\left(u^{(i)}\right)\\
&= \frac{1}{2}\left(w^{(i)}\right)^{*}Xw^{(i)}-\left(\nabla G\left(v^{(i)}+\xi_i w^{(i)}\right),w^{(i)}\right)-J\left(u^{(i)}\right)\\
&\leq \frac{\bar{\lambda}}{2}\left\|w^{(i)}\right\|^2+\left(qTa_1\left\|v^{(i)}+\xi_i w^{(i)}\right\|^{\theta}+qTa_2\right)\left\|w^{(i)}\right\|+K\\
&\leq \frac{\bar{\lambda}}{2}\left\|w^{(i)}\right\|^2+\left(qTa_1\left\|u^{(i)}\right\|^{\theta}+qTa_2\right)\left\|w^{(i)}\right\|+K,
\end{aligned} \tag{3.4}$$

where $\xi_i \in (0,1)$. By (3.2) and (3.4), we have that there exists a constant $K_1>0$ such that

$$\frac{G\left(v^{(i)}\right)}{\left\|u^{(i)}\right\|^{2\theta}} \leq K_1, \forall i\in\mathbb{N}. \tag{3.5}$$

If $v=\{v_n\}\in X^0$, then

$$\sum_{n=1}^{qT}\left(\Delta^2 v_n\right)^2=0.$$

We have $\Delta^2 v_n=0, n\in\mathbb{Z}[1,qT]$. Thus, $v_1=v_2=\cdots=v_{qT}$. It comes from the assumption (F_2) that

$$\lim_{\|v\|\to\infty}\frac{G(v)}{\|v\|^{2\theta}}=\lim_{\|v_1\|\to\infty}\frac{\sum_{n=1}^{qT}\frac{\int_0^{v_1}f(n,t)dt}{|v_1|^{2\theta}}}{(qT)^{\theta}}=\infty. \tag{3.6}$$

(3.6) combining with (3.3), we have

$$\lim_{i\to\infty}\frac{G\left(v^{(i)}\right)}{\left\|v^{(i)}\right\|^{2\theta}}=\lim_{i\to\infty}\frac{G\left(v^{(i)}\right)}{\left\|u^{(i)}\right\|^{2\theta}}=\infty,$$

which is a contradiction to (3.5). Therefore, the desired result is obtained. □

Lemma 3.3. *If u is a critical point of $J(u)$ on $\bar{H}_{qT}$, then u is a critical point of $J(u)$ on H_{qT}.*

According to the proof of Lemma 2.2 in [30], it is easy to prove the Lemma 3.3 and so we omit it.

Let

$$\Gamma_\beta = -\frac{q}{2\left(16\sin^4\frac{\omega\beta}{2q} - \rho\right)}\sum_{n=1}^{qT} f^2(n,0).$$

Lemma 3.4. *Assume that $F(s,u)$ satisfies the assumptions (F_4) and (F_5). If u is a critical point of $J(u)$ on $\bar{H}_{qT}$, and $J(u) < \Gamma_{\chi_q}$, then u has a minimal period qT.*

Proof. For the contradiction, assume that u has a minimal period $\frac{qT}{\beta}$. Due to (F_4), if u is a solution of (1.1) with a minimal period $\frac{qT}{\beta}$, then $f(s,u)$ has a minimal period $\frac{qT}{\beta}$. Hence $\frac{q}{\beta}$ must be an integer. Therefore, $\beta \geq \chi_q$. By the assumption (F_5),

$$16\sin^4\frac{\omega\beta}{2q} \geq 16\sin^4\frac{\omega\chi_q}{2q} > \rho,$$

then

$$\Gamma_{\chi_q} \leq \Gamma_\beta.$$

For any $u \in \bar{H}_{qT}$,

$$u_n = \sum_{k=1}^{\left[\frac{qT-\beta}{2\beta}\right]} b_k \sin\frac{\omega\beta k}{q} n.$$

Consequently,

$$\begin{aligned} J(u) &= \frac{1}{2}u^* X u - \sum_{n=1}^{qT} F(n,u_n) \\ &\geq 2\sin^2\frac{\omega\gamma}{2q}\|x\|^2 - \sum_{n=1}^{qT} F(n,u_n), \end{aligned} \tag{3.7}$$

where $x = (\Delta u_1, \Delta u_2, \cdots, \Delta u_{qT})^*$.

Since

$$4\sin^2\frac{\omega\gamma}{2q}\|u\|^2 \leq \|x\|^2 = \sum_{n=1}^{qT}(u_{n+1}-u_n, u_{n+1}-u_n) = u^*Eu \leq 4\|u\|^2,$$

we get

$$\left(4\sin^2\frac{\omega\gamma}{2q}\right)^2\|u\|^2 \leq u^*Xu \leq 16\|u\|^2. \tag{3.8}$$

By (3.7) and (3.8), we have

$$J(u) \geq 8\sin^4\frac{\omega\beta}{2q}\|u\|^2 - \left(\sum_{n=1}^{qT} f^2(n,0)\right)^{\frac{1}{2}}\|u\| - \frac{\rho}{2}\|u\|^2$$

$$\geq -\frac{1}{2\left(16\sin^4\frac{\omega\beta}{2q}-\rho\right)}\sum_{n=1}^{qT}f^2(n,0)$$

$$\geq -\frac{q}{2\left(16\sin^4\frac{\omega\beta}{2q}-\rho\right)}\sum_{n=1}^{qT}f^2(n,0)$$

$$=\Gamma_\beta. \tag{3.9}$$

(3.9) contradicts to the assumption $J(u) < \Gamma_{\chi_q}$. The proof is finished. □

4. Proofs of theorems

Proof of Theorem 1.1. It comes from Lemma 3.2 that $J(u)$ satisfies the Palais-Smale condition. For the sake of proving Theorem 1.1 by using the Saddle Theorem, we shall prove the assumptions (J_1) and (J_2).

Let $D_1 = X^0$ and $D_2 = X^+$. On one hand, for any $v \in X^0 = D_1$, we have

$$J(v) = \frac{1}{2}v^*Xv - G(v) = -G(v).$$

By (F_2),

$$\lim_{\|v\|\to\infty,\ v\in D_1} J(v) = -\infty.$$

Consequently, there is a constant $\delta > 0$ such that $J|_{\partial B_\delta\cap D_1} \leq \gamma$. This is clear that the assumption (J_1) of Lemma 3.2 is satisfied.

On the other hand, from (F_1), there are constants $a_3 > 0$ and $a_4 > 0$ such that

$$G(u) \leq a_3\|u\|^{1+\theta} + a_4\|u\|^\theta + |G(0)|.$$

For any $u \in X^+$, we have

$$\begin{aligned} J(u) &= \frac{1}{2}u^*Xu - G(u) \\ &\geq \frac{\lambda}{2}\|u\|^2 - a_3\|u\|^{1+\theta} - a_4\|u\| - |F(0)|. \end{aligned}$$

On account of $0 \leq \theta < 1$, it is easy to see that there is a constant $\vartheta > 0$ such that $J(u) \geq \vartheta$ for any $u \in X^+$. Thus, taking $e = 0$, the assumption (J_2) in Lemma 3.2 is satisfied.

The assumptions of (J_1) and (J_2) in Lemma 3.2 hold. Therefore, (1.1) admits at least one qT-periodic solution.

Next, by Lemma 3.4, it sufficient to show that

$$J(u) < \Gamma_{\chi_q}, \forall u \in \bar{H}_{qT}. \tag{4.1}$$

Owing to (F_3), for $|u| \leq \sigma$, we have

$$F(n,u) = f(n,0)u + \frac{1}{2}\times\frac{\partial^2 F(n,\xi u)}{\partial u^2}u^2 \geq f(n,0)u + \frac{\varrho}{2}u^2.$$

Then

$$J(u) = \frac{1}{2}u^* X u - \sum_{n=1}^{qT} F(n, u_n)$$
$$\leq \frac{1}{2}u^* X u - \frac{\varrho}{2}\sum_{n=1}^{qT} u_n^2 - \sum_{n=1}^{qT} f(n,0)u_n. \tag{4.2}$$

We choose

$$u_n = \sigma \sin \frac{\omega}{q} n.$$

Since $f(n,0)$ is T-periodic and $f(-n,0) = f(n,0)$, we get

$$f(n,0) = \sum_{k=1}^{[\frac{T-1}{2}]} a_k \sin \frac{2k\pi}{T} n = \sum_{k=1}^{[\frac{T-1}{2}]} a_k \sin \frac{2k\pi}{qT} qn,$$

where a_k is a constant. In virtue of $q > 1$ and $T \geq 3$,

$$\sum_{n=1}^{qT} f(n,0)u_n = \sum_{k=1}^{[\frac{T-1}{2}]} \sigma a_k \sum_{n=1}^{qT} \sin \frac{2k\pi}{qT} qn \cdot \sin \frac{2\pi}{qT} n = 0.$$

Accordingly, by (4.2),

$$J(u) \leq 2\left(16 \sin^4 \frac{\omega}{2q} - \varrho\right) \|u\|^2. \tag{4.3}$$

It is easy to see that

$$\|u\| = \sigma \left(\frac{q\pi}{\omega}\right)^{\frac{1}{2}}.$$

Combining with (4.3),

$$J(u) = \frac{2\left(16 \sin^4 \frac{\omega}{2q} - \varrho\right)\sigma^2 q\pi}{\omega} < \Gamma_{\chi_q}.$$

(4.1) holds and this finishes the proof. □

Proof of Corollary 1.1. For any prime integer $q > 0$, it is evident that $\chi_q = q$. For this reason,

$$\sum_{n=1}^{qT} f^2(n,0) < \frac{4\pi\sigma^2\left(16\sin^4 \frac{\omega}{2} - \rho\right)\left(\varrho - 16 \sin^4 \frac{\omega}{2q}\right)}{\omega}.$$

As a result of Theorem 1.1, (1.1) admits at least one periodic solution with minimal period qT. □

Remark 4.1. The techniques of the proof of Theorem 1.2 are similar to those carried out in the proof of Theorem 1.1. For simplicity, we omit its proof.

Remark 4.2. Thanks to Theorem 1.2, similar to the proof of Corollary 1.1, the conclusion of Corollary 1.2 is clearly right.

Remark 4.3. As an application of Theorem 1.1, we give a example to illustrate our main result.

Example 4.1. For $q = 4$, assume that

$$\Delta^4 u_{n-2} = \frac{1}{9} \sin\left(\frac{2n\pi}{3} u_n\right) u_n^2, n \in \mathbb{Z}. \tag{4.4}$$

We have

$$\omega = \frac{2\pi}{3},\ \chi_q = 2,\ T = 3,$$

and

$$f(s, u) = \frac{1}{9} \sin\left(\frac{2s\pi}{3} u\right) u^2.$$

It is easy to verify all the assumptions of Theorem 1.1 are satisfied. Consequently, (4.4) admits at least one periodic solution with minimal period 12.

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