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A RESEARCH ON THE GENERALIZED POLY-BERNOULLI POLYNOMIALS WITH VARIABLE $a^{\dagger}$

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ABSTRACT. In this paper, by using the polylogarithm function, we introduce a generalized poly-Bernoulli numbers and polynomials with variable a. We find several combinatorial identities and properties of the polynomials. We give some properties that is connected with the Stirling numbers of second kind. Symmetric properties can be proved by new configured special functions. We display the zeros of the generalized poly-Bernoulli polynomials with variable a and investigate their structure.

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1. Introduction

Recently, one of the researchers' interests in many fields is the applications of Bernoulli numbers and polynomials. Many researchers have studied Bernoulli numbers and polynomials and focus on expansion and generalization of theirs. Specially, it is being studied actively about poly-Bernoulli numbers and polynomials is concerned with polylogarithm function(cf. [1-9]).

In this paper, we use the following notations. $\mathbb{N} = \{1, 2, 3, \dots\}$ denotes the set of natural numbers, $\mathbb{Z}_+$ denotes the set of nonnegative integers, $\mathbb{Z}$ denotes the set of integers, and $\mathbb{C}$ denotes the set of complex numbers, respectively. The classical Bernoulli polynomials $B_n(x)$ are given by the generating function

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad \text{(cf. [1, 2, 3, 4, 5])}. \tag{1.1}$$

When $x = 0, B_{n,q}^{(k)} = B_{n,q}^{(k)}(0)$ are called poly-Bernoulli numbers.

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Definition 1.1. For $a \in \mathbb{C} \setminus \{0\}$, we define a generalized Bernoulli polynomials $B_n(x;a)$ with variable a by the following generating function

$$\frac{t}{e^{at}-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x;a)\frac{t^n}{n!}, \quad |t| < \frac{2\pi}{|a|}. \tag{1.2}$$

When $a = 1$, it is equal to the classical Bernoulli polynomials. The polylogarithm function $Li_k(x)$ is defined by

$$Li_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}, \quad (k \in \mathbb{Z}), \text{ (cf. [1, 2, 3, 5, 7, 8, 9, 12])}. \tag{1.3}$$

For some k, the polylogarithm functions $Li_k(x)$ are as follows:

$$Li_1(x) = -log(1-x), \quad Li_0(x) = \frac{x}{1-x},$$
$$Li_{-1}(x) = \frac{x}{(1-x)^2}, \quad Li_{-2}(x) = \frac{x^2+x}{(1-x)^3}, \quad \ldots .$$

For $n \in \mathbb{Z}_+, k \in \mathbb{Z}$, the poly-Bernoulli polynomials is defined by

$$\frac{Li_k(1-e^{-t})}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x)\frac{t^n}{n!}, \tag{1.4}$$

where

$$Li_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}$$

is k-th polylogarithm function(cf. [1, 2, 3, 4, 5, 7, 12]). When $k = 1$, $Li_1(x) = -log(1-x)$ and $Li_1(1-e^{-t}) = t$. Using the result of polylogarithm function, we deduce that the poly Bernoulli polynomials is identical to the Bernoulli polynomials when $k = 1$.

The classical Stirling numbers of the second kind $S_2(n,m)$ are defined by the relations

$$x^n = \sum_{m=0}^{n} S_2(n,m)(x)_m,$$

where $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$ is falling factorial.

The Stirling numbers of the second kind is defined by

$$\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} = \frac{(e^t-1)^m}{m!}, \text{ (cf. [4, 6, 9, 10, 11])}. \tag{1.5}$$

In this paper, we introduce a generalized poly-Bernoulli numbers and polynomials with variable a. The properties of the Bernoulli polynomials with parameters were studied in [5, 8]. We construct a generalized poly-Bernoulli polynomials with variable a and give some relations between the generalized poly-Bernoulli polynomials and the classical Bernoulli polynomials. We also investigate several identities that are connected with the Stirling numbers of the second kind.

Furthermore, we find some symmetric identities by using special functions and power sum polynomials.

2. Generalized poly-Bernoulli polynomials with variable a

In this section, we introduce a generalized poly-Bernoulli numbers $B_n^{(k)}(a)$ and polynomials $B_n^{(k)}(x;a)$ with variable a by the generating functions. We give some identities of the polynomials, and find a relation that is connected with classical Bernoulli polynomials.

Definition 2.1. For $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$, the generalized poly-Bernoulli polynomials $B_n^{(k)}(x;a)$ with variable a are defined by means of the following generating function

$$\frac{Li_k(1-e^{-t})}{e^{at}-1}e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x;a)\frac{t^n}{n!} \tag{2.1}$$

where

$$Li_k(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^k}$$

is the k-th polylogarithm function. When $x = 0$, $B_n^{(k)}(a) = B_n^{(k)}(0;a)$ are called the generalized poly-Bernoulli numbers with variable a. When the condition allow $a = 1$, it is trivial that the generalized poly-Bernoulli polynomials is reduced to poly-Bernoulli polynomials.

From (2.1), we have a relation between the generalized poly-Bernoulli numbers and polynomials.

Theorem 2.2. *Let n, m be a nonnegative integers and $k \in \mathbb{Z}$. We have*

$$B_n^{(k)}(mx;a) = \sum_{l=0}^{n} \binom{n}{l}(m-1)^{n-1}B_l^{(k)}(x;a)x^{n-l}. \tag{2.2}$$

Proof. For $n, m \in \mathbb{Z}_+$, $k \in \mathbb{Z}$, we get

$$\sum_{n=0}^{\infty} B_n^{(k)}(mx;a)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}B_l^{(k)}(x;a)(m-1)^{n-1}x^{n-l}\right)\frac{t^n}{n!}.$$

Therefore, we obtain above result. □

Corollary 2.3. *Let $m > 0$, $n \geq 0$, $k \in \mathbb{Z}$. We get*

$$B_n^{(k)}(mx;a) = \sum_{l=0}^{n}\binom{n}{l}m^{n-l}B_l^{(k)}(a)x^{n-l}.$$

When $m = 1$, it is satisfies

$$B_n^{(k)}(x;a) = \sum_{l=0}^{n}\binom{n}{l}B_l^{(k)}(a)x^{n-l}.$$

If x is replaced $x+y$ in Corollary 2.3, we get the next addition theorem.

Theorem 2.4. *For $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$, we have*

$$B_n^{(k)}(x+y;a) = \sum_{l=0}^{n} \binom{n}{l} B_l^{(k)}(x;a) y^{n-l}.$$

Proof. Let $n \in \mathbb{Z}_+$, $k \in \mathbb{Z}$. Then we get

$$\begin{aligned}\sum_{n=0}^{\infty} B_n^{(k)}(x+y;a)\frac{t^n}{n!} &= \frac{Li_k(1-e^{-t})}{e^{at}-1} e^{(x+y)t} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} B_l^{(k)}(x;a) y^{n-l} \right) \frac{t^n}{n!}.\end{aligned}$$

Thus, we get the explicit result. □

Theorem 2.5. *For $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$, we derive*

$$B_n^{(k)}(x+1;a) - B_n^{(k)}(x;a) = \sum_{l=0}^{n-1} \binom{n}{l} B_l^{(k)}(x;a).$$

Proof. Let $n \in \mathbb{Z}_+, k \in \mathbb{Z}$. From (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_n^{(k)}(x+1;a)\frac{t^n}{n!} &- \sum_{n=0}^{\infty} B_n^{(k)}(x;a)\frac{t^n}{n!} \\ &= \frac{Li_k(1-e^{-t})}{e^{at}-1} e^{xt}(e^t - 1) \\ &= \sum_{n=0}^{\infty} \sum_{l=0}^{n-1} \binom{n}{l} B_l^{(k)}(x;a)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficient on both sides, we obtain the desired result. □

By using the binomials series and the definition of polylogarithm function, we derive the result as below.

Theorem 2.6. *For $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$, we have*

$$B_n^{(k)}(x;a) = \sum_{l=0}^{\infty} \sum_{m=0}^{l} \sum_{r=0}^{m+1} \binom{m+1}{r} \frac{(-1)^{r+1}(x-r+al-am)^n}{(m+1)^n}.$$

Proof. Let $n \in \mathbb{Z}_+, k \in \mathbb{Z}$. From (1.3), we obtain

$$\begin{aligned}\frac{Li_k(1-e^{-t})}{e^{at}-1}e^{xt} &= \left((-1)\sum_{m=0}^{\infty} e^{mat}\right)\left(\sum_{l=0}^{\infty}\frac{(1-e^{-t})^{l+1}}{(l+1)^k}\right)e^{xt} \\ &= (-1)\sum_{l=0}^{\infty}\sum_{m=0}^{l} e^{(l-m)at}\frac{(1-e^{-t})^{m+1}}{(m+1)^k}e^{xt} \\ &= \left(\sum_{l=0}^{\infty}\sum_{m=0}^{l}(-1)\frac{e^{(l-m)at}}{(m+1)^k}\right)\left(\sum_{r=0}^{m+1}\binom{m+1}{r}(-1)^r e^{(x-r)t}\right) \\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{r=0}^{m+1}\binom{m+1}{r}\frac{(-1)^{r+1}(x-r+al-am)^n}{(m+1)^k}\frac{t^n}{n!}\end{aligned}$$

Therefore, we get

$$B_n^{(k)}(x;a) = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{r=0}^{m+1}\binom{m+1}{r}\frac{(-1)^{r+1}(x-r+al-am)^n}{(m+1)^k}.$$

□

Similarly, we find next result that is related with the generalized Bernoulli polynomials with variable a.

Theorem 2.7. *Let $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$. Then we have*

$$B_n^{(k)}(x;a) = \sum_{l=0}^{\infty}\frac{1}{(l+1)^k}\sum_{r=0}^{l+1}\binom{l+1}{r}(-1)^r\frac{B_{n+1}(x-r;a)}{n+1}.$$

Proof. For $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$, we have

$$\begin{aligned}\frac{Li_k(1-e^{-t})}{e^{at}-1}e^{xt} &= \sum_{l=1}^{\infty}\frac{(1-e^{-t})^l}{l^k}\frac{e^{xt}}{e^{at}-1} \\ &= \sum_{n=0}^{\infty}\frac{1}{(l+1)^k}\sum_{r=0}^{l+1}\binom{l+1}{r}(-1)^r\frac{e^{(x-r)t}}{e^{at}-1} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\frac{1}{(l+1)^k}\sum_{r=0}^{l+1}\binom{l+1}{r}(-1)^r\frac{B_{n+1}(x-r;a)}{n+1}\right)\frac{t^n}{n!}.\end{aligned}$$

So, we easily get

$$B_n^{(k)}(x;a) = \sum_{l=0}^{\infty}\frac{1}{(l+1)^k}\sum_{r=0}^{l+1}\binom{l+1}{r}(-1)^r\frac{B_{n+1}(x-r;a)}{n+1}.$$

□

3. Relations with the Stirling numbers of the second kind

In this section, by using the generationg function of the Stirling numbers of the second kind, we derive some interesting relations that is associated with the generalized poly-Bernoulli polynomials with variable a. Recall that the Stirling numbers of the second kind are given by

$$\frac{(e^t-1)^m}{m!} = \sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!}.$$

By the definitions of the polylogarithm function $Li_k(x)$ and the Stirling numbers of the second kind, we get the following result.

$$\begin{aligned} Li_k(1-e^{-t}) &= \sum_{l=1}^{\infty} \frac{(1-e^{-t})^l}{l^k} \\ &= \sum_{n=1}^{\infty}\sum_{l=1}^{n} \frac{(-1)^{l+n}}{l^k} l! S_2(n,l)\frac{t^n}{n!} \end{aligned} \tag{3.1}$$

From the Equation (3.1), we have the next theorem which is connected with the Stirling numbers.

Theorem 3.1. *For $n \in \mathbb{Z}_+, k \in \mathbb{Z}$, we get*

$$B_n^{(k)}(x;a) = \sum_{r=0}^{n} \binom{n}{r} \sum_{l=1}^{r+1} \frac{(-1)^{l+r+1} l! S_2(r+1,l)}{l^k(r+1)} B_{n-r}(x;a).$$

Proof. Let $n \in \mathbb{Z}_+, k \in \mathbb{Z}$. By the definition of polylogarithm function, the Equation (3.1) is recomposed as follows,

$$\frac{Li_k(1-e^{-t})}{t} = \sum_{n=0}^{\infty}\sum_{l=1}^{n+1} \frac{(-1)^{l+n+1}}{l^k} l! \frac{S_2(n+1,l)}{n+1}\frac{t^n}{n!}. \tag{3.2}$$

By using the Equation (3.2), the generalized poly Bernoulli polynomials $B_n^{(k)}(x;a)$ is indicated with the Stirling numbers and the generalized Bernoulli polynomials with variable a. By Equation (3.2) and Definition 1.1, we have

$$\begin{aligned} \sum_{n=0}^{\infty} B_n^{(k)}(x;a)\frac{t^n}{n!} &= \frac{Li_k(1-e^{-t})}{e^{at}-1} e^{xt} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=1}^{n+1} \frac{(-1)^{l+n+1}}{l^k} l! \frac{S_2(n+1,l)}{n+1} \right) \frac{t^n}{n!} \sum_{n=0}^{\infty} B_n(x;a)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{r=0}^{n} \binom{n}{r} \sum_{l=1}^{r+1} \frac{(-1)^{l+r+1} l! S_2(r+1,l)}{l^k(r+1)} B_{n-r}(x;a)\frac{t^n}{n!}. \end{aligned}$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides, the proof of Theorem 3.1 is now complete. □

Theorem 3.2. *Let $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$. From the Equation (2.1), we get*

$$B_n^{(k)}(x;a) = \sum_{m=0}^{\infty}\sum_{l=0}^{m}\binom{n}{m}(x)_l S_2(m,l) B_{n-a}^{(k)}(a),$$

where $(x)_l = x(x-1)(x-2)\cdots(x-l+1)$ is falling factorial.

Proof. For $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$, the generalized poly-Bernoulli numbers and polynomials can be indicated by the formula that is concerned with the Stirling numbers. By Equation (1.5) and (2.1), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} B_n^{(k)}(x;a)\frac{t^n}{n!} &= \frac{Li_k(1-e^{-t})}{e^{at}-1}((e^t-1)+1)^x \\
&= \frac{Li_k(1-e^{-t})}{e^{at}-1}\sum_{l=0}^{\infty}(x)_l\frac{(e^t-1)^l}{l!} \\
&= \sum_{n=0}^{\infty} B_n^{(k)}(a)\frac{t^n}{n!}\sum_{l=0}^{\infty}(x)_l\sum_{r=l}^{\infty}S_2(r,l)\frac{t^r}{r!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\sum_{l=0}^{m}\binom{n}{m}(x)_l S_2(m,l)B_{n-m}^{(k)}(a)\right)\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficient on both sides, we get

$$B_n^{(k)}(x;a) = \sum_{m=0}^{\infty}\sum_{l=0}^{m}\binom{n}{m}(x)_l S_2(m,l)B_{n-m}^{(k)}(a).$$

□

From Definition 2.1 and Equation (3.2), we have the recurrence formula that is another one with the result of Theorem 2.5.

Theorem 3.3. *For $n \geq 1, k \in \mathbb{Z}$, we get*

$$\begin{aligned}
&B_n^{(k)}(x+a;a) - B_n^{(k)}(x;a) \\
&= \sum_{r=1}^{n}\binom{n}{r}\left(\sum_{l=0}^{r-1}\frac{(-1)^{l+1+r}}{(l+1)^k}(l+1)!S_2(r,l+1)\right)x^{n-r}.
\end{aligned}$$

Proof. Let $n \geq 1, k \in \mathbb{Z}$. Using the Definition 2.1, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} B_n^{(k)}(x+a;a)\frac{t^n}{n!} - \sum_{n=0}^{\infty} B_n^{(k)}(x;a)\frac{t^n}{n!} \\
&\qquad = \frac{Li_k(1-e^{-t})}{e^{at}-1}e^{(x+a)t} - \frac{Li_k(1-e^{-t})}{e^{at}-1}e^{xt} \\
&\qquad = \sum_{n=0}^{\infty}\sum_{l=0}^{n-1} \frac{(-1)^{n+l+1}}{(l+1)^k}(l+1)!S_2(n,l+1)\frac{t^n}{n!}\sum_{m=0}^{\infty} x^m \frac{t^m}{m!} \\
&\qquad = \sum_{n=0}^{\infty}\sum_{r=1}^{n}\binom{n}{r}\left(\sum_{l=0}^{r-1}\frac{(-1)^{r+l+1}}{(l+1)^k}(l+1)!S_2(r,l+1)\right)x^{n-r}\frac{t^n}{n!}.
\end{aligned}$$

Hence, the recurrence formula is indicated by

$$\begin{aligned}
&B_n^{(k)}(x+a;a) - B_n^{(k)}(x;a) \\
&\qquad = \sum_{r=1}^{n}\binom{n}{r}\left(\sum_{l=0}^{r-1}\frac{(-1)^{r+l+1}}{(l+1)^k}(l+1)!S_2(r,l+1)\right)x^{n-r}.
\end{aligned}$$

□

4. Symmetric properties of the generalized poly-Bernoulli polynomials involving special functions

In this section, we consider several special functions and investigate some symmetric properties of the generalized poly-Bernoulli polynomials with variable a.

Theorem 4.1. *Let $n \in \mathbb{Z}_+, k \in \mathbb{Z}$, $m_1, m_2 > 0$ and $m_1 \neq m_2$. Then we obtain*

$$\begin{aligned}
&\sum_{r=0}^{n}\binom{n}{r} m_1^{n-r}m_2^{r}B_{n-r}^{(k)}(m_2x;a)B_r^{(k)}(m_1x;a) \\
&\qquad\qquad = \sum_{r=0}^{n}\binom{n}{r} m_1^{r}m_2^{n-r}B_r^{(k)}(m_2x;a)B_{n-r}^{(k)}(m_1x;a).
\end{aligned}$$

Proof. For $n \in \mathbb{Z}_+, k \in \mathbb{Z}$ and $m_1, m_2 > 0 (m_1 \neq m_2)$, we consider a special function as follows

$$F(t) = \frac{Li_k(1-e^{-m_1t})Li_k(1-e^{-m_2t})}{(e^{am_1t}-1)(e^{am_2t}-1)}e^{2m_1m_2xt}. \tag{4.1}$$

The Equation (4.1) is appeared by

$$
\begin{aligned}
F(t) &= \frac{Li_k(1-e^{-m_1t})}{(e^{am_1t}-1)}e^{m_1m_2xt}\frac{Li_k(1-e^{-m_2t})}{(e^{am_2t}-1)}e^{m_1m_2xt} \\
&= \sum_{n=0}^{\infty} B_n^{(k)}(m_2x;a)\frac{(m_1t)^n}{n!}\sum_{r=0}^{\infty} B_r^{(k)}(m_1x;a)\frac{(m_2t)^r}{r!} \qquad (4.2)\\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r} m_1^{n-r}m_2^{r}B_r^{(k)}(m_1x;a)B_{n-r}^{(k)}(m_2x;a)\frac{t^n}{n!}.
\end{aligned}
$$

Similarly, we can see that

$$
F(t) = \sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r} m_1^{r}m_2^{n-r}B_{n-r}^{(k)}(m_1x;a)B_r^{(k)}(m_2x;a)\frac{t^n}{n!}. \qquad (4.3)
$$

Comparing the coefficient of Equation (4.2) and (4.3), it is clear to Theorem 4.1. □

Note that $\widetilde{S}_k(m) = \sum_{i=1}^{m} i^k$ is a power sum polynomials(cf [4, 6, 11]). The exponential generating function of the power sum polynomials are expressed by

$$
\frac{e^{(m+1)t}-1}{e^t-1} = \sum_{m=0}^{\infty}\widetilde{S}_k(m)\frac{t^k}{k!}. \qquad (4.4)
$$

Using the Equation (4.4), we have the symmetric identity of the generalized poly-Bernoulli polynomials.

Theorem 4.2. *Let $n \in \mathbb{Z}_+, k \in \mathbb{Z}$, $m_1, m_2 > 0$ and $m_1 \neq m_2$. Then we get*

$$
\begin{aligned}
&Li_k(1-e^{-m_2t})\sum_{r=0}^{n}\binom{n}{r} a^{n-r}m_1^{r}m_2^{n-r}B_r^{(k)}(m_2x;a)\widetilde{S}_{n-r}(m_1-1) \\
&\qquad = Li_k(1-e^{-m_1t})\sum_{r=0}^{n}\binom{n}{r} a^{n-r}m_1^{n-r}m_2^{r}B_r^{(k)}(m_1x;a)\widetilde{S}_{n-r}(m_2-1)
\end{aligned}
$$

Proof. Let $n \in \mathbb{Z}_+$, $m_1, m_2 > 0$ and $m_1 \neq m_2$. If we start a special function that is given below

$$
\begin{aligned}
F(t) &= \frac{Li_k(1-e^{-m_1t})Li_k(1-e^{-m_2t})(e^{m_1m_2t}-1)(e^{am_1m_2xt})}{(e^{am_1t}-1)(e^{am_2t}-1)} \\
&= Li_k(1-e^{-m_2t})\sum_{n=0}^{\infty} B_n^{(k)}(m_2x;a)\frac{(m_1t)^n}{n!}\sum_{r=0}^{\infty}\widetilde{S}_r(m_1-1)\frac{(am_2t)^r}{r!} \\
&= \sum_{n=0}^{\infty} Li_k(1-e^{-m_2t})\sum_{r=0}^{n}\binom{n}{r} a^{n-r}m_1^{r}m_2^{n-r}B_r^{(k)}(m_2x;a)\widetilde{S}_{n-r}(m_1-1)\frac{t^n}{n!}.
\end{aligned}
$$

In analogous method, we get

$$
\begin{aligned}
F(t) &= Li_k(1-e^{-m_1t})\sum_{n=0}^{\infty}B_n^{(k)}(m_1x)\frac{(m_2t)^n}{n!}\sum_{r=0}^{\infty}\widetilde{S}_r(m_2-1)\frac{(am_1t)^r}{r!}\\
&=\sum_{n=0}^{\infty}Li_k(1-e^{-m_1t})\sum_{r=0}^{n}\binom{n}{r}a^{n-r}m_1^{n-r}m_2^rB_r^{(k)}(m_1x;a)\widetilde{S}_{n-r}(m_2-1)\frac{t^n}{n!}.
\end{aligned}
$$

Comparing the coefficient of both sides, then it gives the symmetric identity. □

Theorem 4.3. *For $n\in\mathbb{Z}_+, k\in\mathbb{Z}$ and $m_1,m_2>0$ $(m_1\neq m_2)$, we have*

$$
\begin{aligned}
&\sum_{r=0}^{n}\binom{n}{r}a^{n-r}m_1^{n-r}m_2^{r-1}B_r(m_1x)\widetilde{S}_{n-r}(m_2-1)\\
&\qquad=\sum_{r=0}^{n}\binom{n}{r}a^{n-r}m_1^{r-1}m_2^{n-r}B_r(m_2x)\widetilde{S}_{n-r}(m_1-1).
\end{aligned}
$$

Proof. Let $n\in\mathbb{Z}_+, k\in\mathbb{Z}$ and $m_1,m_2>0$ $(m_1\neq m_2)$. Then we consider the generating function as follows

$$
F(t)=\frac{Li_k(1-e^{-m_1t})Li_k(1-e^{-m_2t})(e^{am_1m_2t}-1)(e^{am_1m_2xt})t}{(e^{am_1t}-1)^2(e^{am_2t}-1)^2}.
$$

From the generating function $F(t)$ and the Equation (4.4), we get

$$
\begin{aligned}
F(t)&=\frac{Li_k(1-e^{-m_1t})Li_k(1-e^{-m_2t})(e^{am_1m_2t}-1)(e^{am_1m_2xt})t}{(e^{am_1t}-1)^2(e^{am_2t}-1)^2}\\
&=\sum_{n=0}^{\infty}B_n^{(k)}(a)\frac{(m_1t)^n}{n!}\sum_{n=0}^{\infty}B_n^{(k)}(a)\frac{(m_2t)^n}{n!}\\
&\qquad\times\sum_{r=0}^{\infty}\widetilde{S}_r(m_2-1)\frac{(am_1t)^r}{r!}a^{-1}m_2^{-1}\sum_{n=0}^{\infty}B_n(m_1x)\frac{(am_2t)^n}{n!}\\
&=\sum_{n=0}^{\infty}B_n^{(k)}(a)\frac{(m_1t)^n}{n!}\sum_{n=0}^{\infty}B_n^{(k)}(a)\frac{(m_2t)^n}{n!}\\
&\qquad\times\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}a^{n-1}m_1^{n-r}m_2^{r-1}B_r(m_1x)\widetilde{S}_{n-r}(m_2-1)\frac{t^n}{n!}.
\end{aligned}
$$

In similar method, $F(t)$ is expressed by

$$
\begin{aligned}
F(t)&=\sum_{n=0}^{\infty}B_n^{(k)}(a)\frac{(m_1t)^n}{n!}\sum_{n=0}^{\infty}B_n^{(k)}(a)\frac{(m_2t)^n}{n!}\\
&\qquad\times\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}a^{n-1}m_2^{n-r}m_1^{r-1}B_r(m_2x)\widetilde{S}_{n-r}(m_1-1)\frac{t^n}{n!}.
\end{aligned}
$$

Comparing the coefficient of $\frac{t^n}{n!}$, we find the symmetric identity:

$$\sum_{r=0}^{n}\binom{n}{r}a^{n-l}m_1^{n-r}m_2^{r-1}B_r(m_1x)\widetilde{S}_{n-r}(m_2-1)$$
$$=\sum_{r=0}^{n}\binom{n}{r}a^{n-l}m_1^{r-1}m_2^{n-l}B_r(m_2x)\widetilde{S}_{n-r}(m_1-1).$$

□

5. Distribution of zeros of the generalized poly-Bernoulli polynomials

In this section, we discover new interesting pattern of the zeros of the generalized poly-Bernoulli polynomials $B_n^{(k)}(x;a)$. We propose some conjectures by numerical experiments. The generalized poly-Bernoulli polynomials $B_n^{(k)}(x;a)$ can be determined explicitly. A few of them are

$$\begin{aligned}
B_0^{(k)}(x;a) &= \frac{1}{a},\\
B_1^{(k)}(x;a) &= -\frac{1}{2}-\frac{1}{2a}+\frac{2^{-k}}{a}+\frac{x}{a},\\
B_2^{(k)}(x;a) &= \frac{1}{2}-2^{-k}+\frac{1}{3a}-\frac{2^{1-k}}{a}+\frac{2\cdot 3^{-k}}{a}+\frac{a}{6}-x-\frac{x}{a}+\frac{2^{1-k}x}{a}+\frac{x^2}{a},\\
B_3^{(k)}(x;a) &= -\frac{1}{2}+3\cdot 2^{-k}-3^{1-k}-\frac{1}{4a}+\frac{3\cdot 2^{1-2k}}{a}+\frac{7\cdot 2^{-1-k}}{a}-\frac{3^{2-k}}{a}-\frac{a}{4}\\
&\quad +2^{-1-k}a+\frac{3x}{2}-3\cdot 2^{-k}x+\frac{x}{a}-\frac{3\cdot 2^{1-k}x}{a}+\frac{2\cdot 3^{1-k}x}{a}+\frac{ax}{2}\\
&\quad -\frac{3x^2}{2}-\frac{3x^2}{2a}+\frac{3\cdot 2^{-k}x^2}{a}+\frac{x^3}{a}.
\end{aligned}$$

We investigate the beautiful zeros of the generalized poly-Bernoulli polynomials $B_n^{(k)}(x;a)$ by using a computer. We plot the zeros of the generalized poly-Bernoulli polynomials $B_n^{(k)}(x;a)$ for $n=40, k=-4,-2,2,4$ and $x\in\mathbb{C}$(Figure 1). In Figure 1(top-left), we choose $n=30, k=-4$ and $a=3$. In Figure 1(top-right), we choose $n=30, k=-2$ and $a=3$. In Figure 1(bottom-left), we choose $n=30, k=2$ and $a=3$. In Figure 1(bottom-right), we choose $n=30, k=4$ and $a=3$. Stacks of zeros of $B_n^{(k)}(x:a)$ for $1\le n\le 40$ from a 3-D structure are presented(Figure 2). In Figure 2(left), we choose $k=-2$ and $a=3$. In Figure 2(middle), we choose $k=2$ and $a=3$. Our numerical results for approximate solutions of real zeros of $B_n^{(k)}(x;a)$ are displayed(Tables 1, 2).

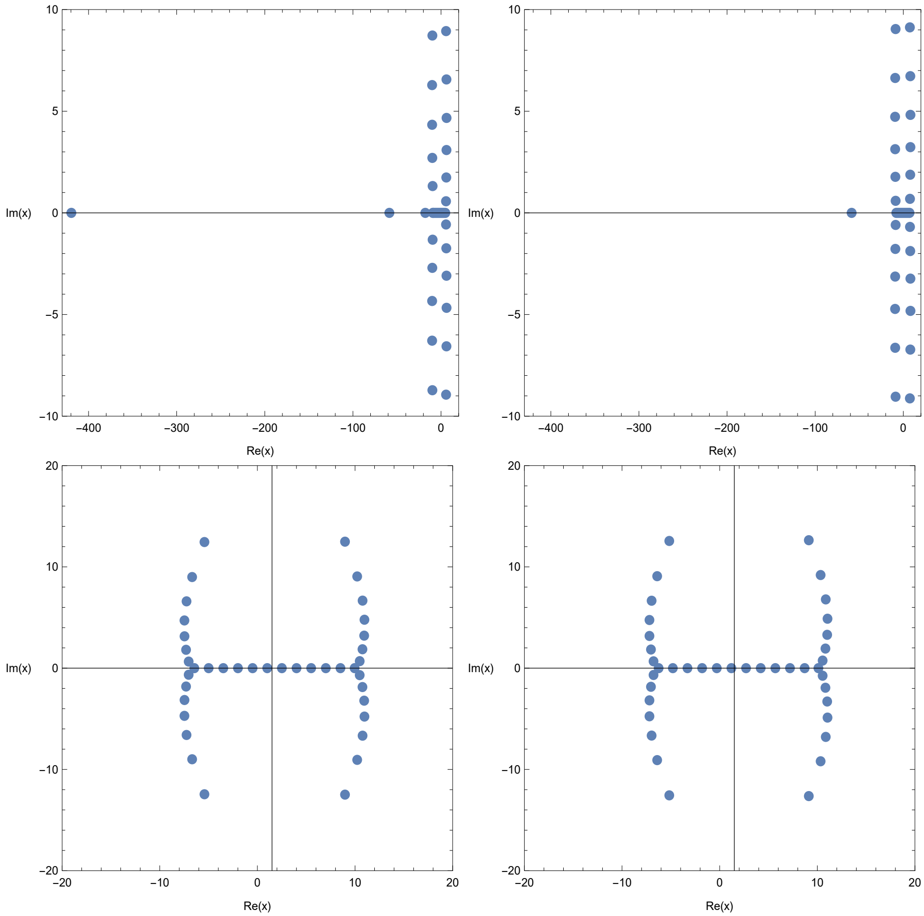

FIGURE 1. Zeros of $B_n^{(k)}(x;a)$

Table 1. Numbers of real and complex zeros of $B_n^{(k)}(x;a)$

degree n	$k=-2$		$k=2$	
	real zeros	complex zeros	real zeros	complex zeros
1	1	0	1	0
2	2	0	2	0
3	3	0	3	0
4	4	0	4	0
5	5	0	5	0
6	4	2	2	4
7	3	4	3	4
8	4	4	4	4
9	5	4	5	4
10	6	4	6	4
11	7	4	7	4

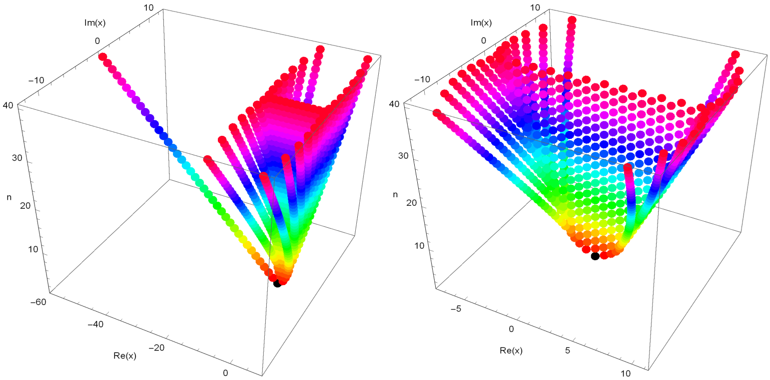

FIGURE 2. Stacks of zeros of $B_n^{(k)}(x;a)$ for $1 \leq n \leq 40$

The plot of real zeros of $B_n^{(k)}(x;a)$ for $1 \leq n \leq 40$ structure are presented(Figure 3). In Figure 3(left), we choose $k = -2$ and $a = 3$. In Figure 3(right), we choose

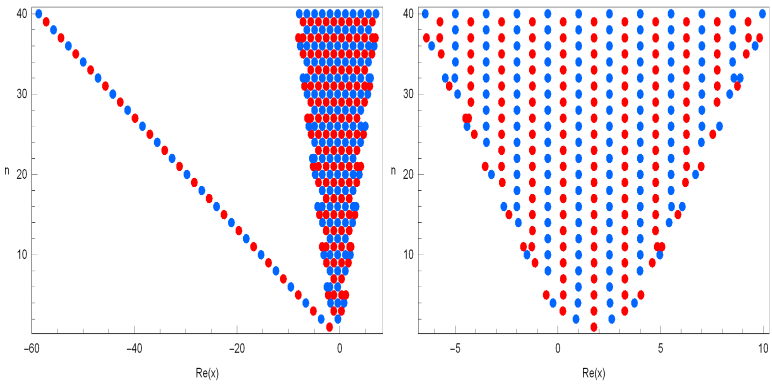

FIGURE 3. Real zeros of $B_n^{(k)}(x;a)$ for $1 \leq n \leq 40$

$k = 2$ and $a = 3$. We observe a remarkable regular structure of the complex roots of the generalized poly-Bernoulli polynomials $B_n^{(k)}(x;a)$. We also hope to verify a remarkable regular structure of the complex roots of the generalized poly-Bernoulli polynomials $B_n^{(k)}(x;a)$(Table 1). Next, we calculated an approximate solution satisfying generalized poly-Bernoulli polynomials $B_n^{(k)}(x;a) = 0$ for $x \in$

$\mathbb{R}$. The results are given in Table 2 and Table 3.

Table 2. Approximate solutions of $B_n^{(k)}(x;a) = 0, k = -2$

degree n	x
1	-2.0000
2	-3.6330, -0.36701
3	-5.1474, -1.1814, 0.32887
4	-6.6236, -1.8091, -0.40059, 0.83330
5	-8.0847, -2.2649, -1.1655, 0.34794, 1.1671
6	-9.5387, -2.5133, -1.9368, -0.40717

Table 3. Approximate solutions of $B_n^{(k)}(x;a) = 0, k = 2$

degree n	x
1	1.7500
2	0.87997, 2.6200
3	0.24078, 1.7546, 3.2546
4	-0.23924, 0.97561, 2.5369, 3.7267
5	-0.56980, 0.25952, 1.7568, 3.2627, 4.0408
6	0.99983, 2.5157

By numerical computations, we will make a series of the following conjectures:

Conjecture 4.1. Prove that $B_n^{(1)}(x;a), x \in \mathbb{C}$, has $Re(x) = \frac{a}{2}$ and $Im(x) = 0$ reflection symmetry analytic complex functions. However, $B_n^{(k)}(x;a), k \neq 1$, has not $Re(x) = \frac{a}{2}$ reflection symmetry for $a \in \mathbb{R}$.

Using computers, many more values of n have been checked. It still remains unknown if the conjecture fails or holds for any value n(see Figures 1, 2, 3). We are able to decide if $B_n^{(k)}(x;a) = 0$ has n distinct solutions(see Tables 1, 2, 3).

Conjecture 4.2. Prove that $B_n^{(k)}(x;a) = 0$ has n distinct solutions.

The authors expect that investigations along these directions will lead to a new approach employing numerical method in the research field of the generalized poly-Bernoulli polynomials $B_n^{(k)}(x;a)$ which appear in applied mathematics and mathematical physics.

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