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A STUDY OF SUM OF DIVISOR FUNCTIONS AND STIRLING NUMBER OF THE FIRST KIND DERIVED FROM LIOUVILLE FUNCTIONS†

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Abstract. Using the theory of combinatoric convolution sums, we establish some arithmetic identities involving Liouville functions and restricted divisor functions. We also prove some relations involving restricted divisor functions and Stirling numbers of the first kind for divisor functions.

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1. Introduction

The study of arithmetic identities is classical in number theory and such investigations have been carried out by several mathematicians including Euler and Gauss.

A function $f : \mathbb{N} \to \mathbb{C}$ is called an arithmetic function. If f is an arithmetic function and $n \notin \mathbb{N}$ we set $f(n) = 0$. The arithmetic function f is said to be multiplicative if $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}$ with $(m, n) = 1$.

It d is positive integer, the *Liouville function* denoted by $\lambda(d)$ is defined as

$$\lambda(d) = \begin{cases} 1 & \text{if } d = 1, \\ (-1)^l & \text{if } d = p_1^{s_1} \cdots p_r^{s_r} \text{ and } s_1 + \cdots + s_r = l. \end{cases} \tag{1}$$

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We know the arithmetic function $\sigma_k(n)$ defined for all $k \in \mathbb{Z}$ by

$$\sigma_k(n) := \sum_{\substack{n\in\mathbb{N} \\ d|n}} d^k,$$

where d runs through the positive integers dividing n. We set $d(n) := \sigma_0(n)$ and $\sigma(n) := \sigma_1(n)$. We also make use of the following convention:

$$\sigma_s(N) = 0 \text{ if } N \notin \mathbb{Z} \text{ or } N < 0, \qquad \sigma(N) := \sigma_1(N) = \sum_{d|N} d.$$

Ramanujan [14] wrote several formulas for

$$\sigma_r(0)\sigma_s(N) + \sigma_r(1)\sigma_s(N-1) + \cdots + \sigma_r(N)\sigma_s(0).$$

Here, $\sigma_r(0) = \frac{1}{2}\zeta(-r)$.

The history of the convolution sums involving the divisor functions $\sigma_s(N)$ goes back to Glaisher [8, 9, 10]. Many recent works on convolution formulas for divisor functions can be found in B.C. Berndt [5]; H. Hahn [11]; J.G. Huard, Z.M. Ou, B.K. Spearman, and K.S. Williams [12]; G. Melfi [13]; B. Cho, D. Kim, and J.-K. Koo [6, 7]; and A. Alaca, S. Alaca, and K.S. Williams [3, 4].

In this paper, instead of $\sigma_k(n)$ we study the arithmetic function $S_k(n)$ that we define as follows for $k \in \mathbb{N} \cup \{0\}$

$$S_k(n) := \sum_{\substack{n\in\mathbb{N} \\ d|n}} \lambda(d)\, d^k.$$

Moreover, we study

$$\bar{S}_r(n) := \left[\sqrt[r]{|S_r(n)|}\right] \quad \text{and} \quad \hat{\mathrm{S_k}}(\mathrm{n}) := \sum_{\mathrm{d|n}} \lambda(\mathrm{d})\, \mathrm{S_k(n/d)},$$

for $r \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$ where $[x]$ is the greatest integer that is less than or equal to x.

If p is a prime and s is a nonnegative integer, we have

$$S_k(p^s) = 1 - p^k + p^{2k} - \cdots + (-1)^s p^{sk} = \frac{1 + (-1)^s p^{(s+1)k}}{p^k + 1}\ . \tag{2}$$

In this paper, we will prove the following theorems and corollary.

Theorem 1.1. *Let* $\sigma_k^+(n) = \sum_{d|n} \lambda(d)\, \sigma_k(d)$. *Then* $\hat{S_k}(n) = \sigma_k^+(n)$.

Theorem 1.2. *If* $n = {p_1}^{s_1} \cdots p_r^{s_r}$ *is a positive integer and* $k \in \mathbb{N}$, *then we get*

$$\hat{S_k}(n) = \lambda(n) \prod_{i=1}^{r} p_i^{\epsilon(s_i)k} \prod \frac{(p^{2k([\frac{s_i}{2}]+1)} - 1)}{(p_i^{2k} - 1)}.$$

Here,

$$\epsilon(s_i) = \begin{cases} 0 & if\ \ s \equiv 0 \pmod 2, \\ 1 & otherwise. \end{cases}$$

Corollary 1.3. *For a given positive integer* $n = p_1^{s_1}\cdots p_r^{s_r}$,

$$\sum_{d|n}\lambda(d)\,\sigma_0(d) = \lambda(n)\left\{\left(\left[\frac{s_1}{2}\right]+1\right)\cdots\left(\left[\frac{s_r}{2}\right]+1\right)\right\}.$$

In particular, if n *is square-free, then*

$$\sum_{d|n}\lambda(d)\,\sigma_0(d) = \lambda(n).$$

The (unsigned) Stirling number of the first kind $\begin{bmatrix} n \\ i \end{bmatrix}$ is defined by

$$n!x(1+x)(1+\frac{x}{2})\cdots(1+\frac{x}{n}) = \sum_{i=0}^{n}\begin{bmatrix} n+1 \\ i+1 \end{bmatrix}x^{i+1},$$

with $\begin{bmatrix} n \\ i \end{bmatrix}$ =0, if $n < i$ and $\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ i \end{bmatrix} = 0$, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ =1.

And we know

$$\begin{bmatrix} n+1 \\ i \end{bmatrix} = n\begin{bmatrix} n \\ i \end{bmatrix} + \begin{bmatrix} n \\ i-1 \end{bmatrix}.$$

Let $\bar{S}_k^{(0)}(n) := n, \bar{S}_k^{(1)}(n) := \bar{S}_k(n)$ and $\bar{S}_k^{(l)}(n) := \bar{S}_k(\bar{S}_k^{(l-1)}(n))$. We define

$$(x+n)(x+\bar{S}_k(n))(x+\bar{S}_k^{(2)}(n))\cdots(x+\bar{S}_k^{(l)}(n)) = \sum_{i=0}^{n}\begin{Bmatrix} n \\ i \end{Bmatrix}_k x^i,$$

where $\bar{S}_k^{(l-1)}(n) \neq 1,\ \ \bar{S}_k^{(l)}(n) = 1$ and $\left\{ {n \atop i} \right\}_k = 0$ if $i \geq l+2$ or $i < 0$.

For convenience, we denote as $\left\{ {n \atop i} \right\}_1 = \left\{ {n \atop i} \right\}$.

Now, we present the following theorem for the recurrence formula of the Stirling numbers of divisor functions.

Theorem 1.4. *Let* $n \in \mathbb{N}$. *Then we obtain*

$$\begin{Bmatrix} n \\ i \end{Bmatrix}_k = n\begin{Bmatrix} \bar{S}_k(n) \\ i \end{Bmatrix}_k + \begin{Bmatrix} \bar{S}_k(n) \\ i-1 \end{Bmatrix}_k.$$

2. Some Properties of Arithmetic functions $\sigma_k(n)$ and $S_k(n)$

Theorem 2.1. *(a)* S_k *is multiplicative.*

(b) Let p *be a prime. For* $k,\ s \in \mathbb{N}$

$$S_k(p^{s+1}) + (p^k-1)S_k(p^s) - p^kS_k(p^{s-1}) = 0$$

and

$$S_k(pn) + (p^k-1)S_k(n) - p^kS_k(n/p) = 0.$$

Proof. (a) Let $m,\, n \in \mathbb{N}$ with $(m,\, n) = 1$. Let p_i and q_j be primes for each i, j. Let $m = {p_1}^{e_1} \cdots p_i^{e_i}$, $n = q_1^{r_1} \cdots q_j^{r_j}$ be factorizations of $m, n \in \mathbb{N}$ into distinct prime powers, respectively. Then we obtain

$$\begin{aligned}
&S_k(m)S_k(n)\\
&= [1 + (-1)p_1^k + \cdots + (-1)^2(p_1p_2)^k + \cdots + (-1)^3(p_1p_2p_3)^k + \cdots + \lambda(m)({p_1}^{e_1} \cdots p_i^{e_i})^k]\\
&\quad \times [1 + (-1)q_1^k + \cdots + (-1)^2(q_1q_2)^k + \cdots + (-1)^3(q_1q_2q_3)^k + \cdots + \lambda(n)(q_1^{r_1} \cdots q_j^{r_j})^k]\\
&= 1 + (-1)p_1^k + \cdots + (-1)q_j^k + (-1)^2(p_1p_2)^k + \cdots + (-1)^2(p_1q_1)^k + \cdots\\
&\quad + (-1)^3(p_1p_2q_1)^k + \cdots + (-1)^4(p_1p_2q_1q_2)^k + \cdots + \lambda(m)\lambda(n)(p_1^{e_1} \cdots p_i^{e_i} q_1^{r_1} \cdots q_j^{r_j})^k\\
&= S_k(mn).
\end{aligned}$$

(b) If s is odd, then from (2)

$$S_k(p^{s+1}) = \frac{1 + p^{(s+2)k}}{p^k + 1}, \quad S_k(p^s) = \frac{1 - p^{(s+1)k}}{p^k + 1} \text{ and } S_k(p^{s-1}) = \frac{1 + p^{sk}}{p^k + 1}.$$

Hence, we have

$$S_k(p^{s+1}) + (p^k - 1)S_k(p^s) - p^k S_k(p^{s-1}) = 0.$$

Similarly, if s is even, then we also obtained same result.

Let $n = p^s N$, where $s \in \mathbb{N} \cup \{0\}$ and $\mathrm{N} \in \mathbb{N}$ with $(N,\, p) = 1$. If $s = 0$ then we are done. Suppose $s \in \mathbb{N}$. Then multiplying the above theorem by $S_k(N)$ and using part (a), we obtain the assertion of part (b). □

Remark 2.1. We set $S_1(n) = S(n)$. The first ten values of $S_i(n)$ $(i = 1, 2, 3)$ and $\sigma_j(n)$ $(j = 1, 2, 3,)$ are given Table 1 and 2.

n	2	3	4	5	6	7	8	9	10
$S_1(n)$	-1	-2	3	-4	2	-6	-5	7	4
$S_2(n)$	-3	-8	13	-24	24	-48	-51	73	72
$S_3(n)$	-7	-26	53	-124	182	-342	-459	703	868

Table 1. Values of $S_i(n)$ $(i = 1, 2, 3, n = 2, \ldots, 9)$

n	2	3	4	5	6	7	8	9	10
$\sigma_1(n)$	3	4	7	6	12	8	15	13	18
$\sigma_2(n)$	5	10	21	26	60	50	85	91	130
$\sigma_3(n)$	9	28	73	126	252	344	585	757	1134

Table 2. Values of $\sigma_j(n)$ $(i = 1, 2, 3, n = 2, \ldots, 9)$

Lemma 2.2. *Let n be a positive integer. Then*

$\#\{d|\ d|n\}$ is odd if and only if n is a perfect square and $S_0(n) = 1$. Here, $\#A$ means the cardinality of the set A.

Proof. Let $n = {p_1}^{s_1} \cdots p_r^{s_r}$ be a factorization of n into distinct prime powers. It is well known that

$$\#\{d|\ d|n\} = (s_1 + 1) \cdots (s_r + 1).$$

Hence

$$\begin{aligned}\#\{d|\ d|n\} \text{ is odd } &\Leftrightarrow \text{ all } (s_i + 1) \text{ are odd}\\ &\Leftrightarrow \text{ all } s_i \text{ are even}\\ &\Leftrightarrow n \text{ is a perfect square and } S_0(p_i^{s_i}) = 1 \text{ for all } i\\ &\Leftrightarrow n \text{ is a perfect square and } S_0(n) = 1.\end{aligned}$$

□

Corollary 2.3. *(a) If n is a perfect square, then $S_0(n) = 1$.*
(b) $\lambda(n) = 1$ if and only if $S_k(n) > 0$ for all $k \in \mathbb{N}$.

Proof. It is trivial by Lemma 2.2 □

Remark 2.2. Let p be a prime such that $p \equiv 1 \pmod 4$. If s is odd, then $S_k(p^s) \equiv 0 \pmod 4$ by (2).

Theorem 2.4. *For a given positive integer n, if $n \equiv 1 \pmod 4$ and $\lambda(n) = -1$ then $S_k(n) \equiv 0 \pmod 4$*

Proof. Let $n = {p_1}^{s_1} \cdots p_r^{s_r}$ with $l = s_1 + \cdots + s_r$. Since $n \equiv 1 \pmod 4$, $\sum_{p_i \equiv 3 \pmod 4} s_i$ is even. Here since $\lambda(n) = -1$, l is odd and there is an odd number s_j such that $p_j \equiv 1 \pmod 4$. By Remark 2.2, $S_k(p_j^{s_j}) \equiv 0 \pmod 4$ and so $S_k(n) \equiv 0 \pmod 4$. □

Lemma 2.5. *For a given positive integer n, we have*

$$\hat{S}_k(n) = \sum_{d|n} \lambda(d)\, \sigma_k(d).$$

Proof. Let $n = {p_1}^{s_1} \cdots p_r^{s_r}$ be a factorization of n into distinct prime powers. Then

$$\begin{aligned}\hat{S}_k(n) &= \sum_{d|n} \lambda(d)\, S_k(n/d)\\ &= \lambda(1)\, S_k(n) + \lambda(p_1)\, S_k(n/p_1) + \lambda(p_2)\, S_k(n/p_2) + \cdots + \lambda(n)\, S_k(1)\\ &= \lambda(1)\{\lambda(1)1^k + \lambda(p_1)p_1^k + \lambda(p_2)p_2^k + \cdots + \lambda(n)(n)^k\}\\ &\quad + \lambda(p_1)\{\lambda(1)1^k + \lambda(p_1)p_1^k + \cdots + \lambda(n/p_1)(n/p_1)^k\}\\ &\quad + \lambda(p_2)\{\lambda(1)1^k + \lambda(p_2)p_2^k + \cdots + \lambda(n/p_2)(n/p_2)^k\}\\ &\quad + \cdots + \lambda(n)\lambda(1)1^k\\ &= \lambda(1)1^k + \lambda(p_1)\{1^k + p_1^k\} + \lambda(p_2)\{1^k + p_2^k\} + \cdots\\ &\quad + \lambda(n)\{1^k + p_1^k + p_2^k + \cdots + n^k\}\end{aligned}$$

$$= \sum_{d|n} \lambda(d) \sum_{d_1|d} d_1^k$$
$$= \sum_{d|n} \lambda(d)\, \sigma_k(d).$$

This completes the proof of Lemma 2.5. □

3. Proofs of Theorem 1.1, Theorem1.2 and Corollary 1.3

Proof of Theorem 1.1. Using Lemma 2.5, we get Theorem 1.1. □

Lemma 3.1. *For a prime p,*

$$\hat{S}_k(p^s) = \begin{cases} \lambda(p^s)\,(1^k + p^{2k} + p^{4k} + \cdots + p^{sk}) & \text{if } s \text{ is even,} \\ \lambda(p^s)\,(p^k + p^{3k} + p^{5k} + \cdots + p^{sk}) & \text{if } s \text{ is odd.} \end{cases} \tag{2.4}$$

Proof. By Lemma 2.5, we have

$$\begin{aligned} \hat{S}_k(p^s) &= \sum_{d|p^s} \lambda(d)\, \sigma_k(d) \\ &= \lambda(1)\,\sigma_k(1) + \lambda(p)\,\sigma_k(p) + \lambda(p^2)\,\sigma_k(p^2) + \cdots + \lambda(p^s)\,\sigma_k(p^s) \\ &= 1^k - (1^k + p^k) + (1^k + p^k + p^{2k}) - (1^k + p^k + p^{2k} + p^{3k}) + \cdots \\ &\quad + (-1)^{s-1}(1^k + p^k + \cdots + p^{(s-1)k}) + (-1)^s(1^k + p^k + \cdots + p^{(s-1)k} + p^{sk}). \end{aligned}$$

Therefore, we get

$$\hat{S}_k(p^s) = \begin{cases} 1^k + p^{2k} + \cdots + p^{sk} & \text{if } s \equiv 0 \pmod 2, \\ -(p^k + p^{3k} + \cdots + p^{sk}) & \text{otherwise.} \end{cases} \tag{3}$$

Thus, this completes the proof of Lemma 3.1. □

Proof of Theorem 1.2. If s is even, then we obtain

$$\begin{aligned} \hat{S}_k(p^s) = 1^k + p^{2k} + \cdots + p^{sk} &= \lambda(p^s) \frac{(p^{2k(\frac{s}{2}+1)} - 1)}{(p^{2k} - 1)} \\ &= \lambda(p^s) p^{\epsilon(s)k} \frac{(p^{2k([\frac{s}{2}]+1)} - 1)}{(p^{2k} - 1)}. \end{aligned}$$

And if s is odd, then we obtain

$$\begin{aligned} \hat{S}_k(p^s) = -p^k(1^k + p^{2k} + \cdots + p^{(s-1)k}) &= \lambda(p^s) p^k \frac{(p^{2k(\frac{s-1}{2}+1)} - 1)}{(p^{2k} - 1)} \\ &= \lambda(p^s) p^{\epsilon(s)k} \frac{(p^{2k([\frac{s}{2}]+1)} - 1)}{(p^{2k} - 1)}. \end{aligned}$$

Since $\hat{S}_k$ is multiplicative. we obtain

$$\hat{S}_k(n) = \hat{S}_k({p_1}^{s_1}) \times \cdots \times \hat{S}_k({p_i}^{s_i}) \times \cdots \times \hat{S}_k({p_r}^{s_r})$$

$$= \prod \lambda({p_i}^{s_i}) \prod p_i^{\epsilon(s_i)k} \prod \frac{(p^{2k([\frac{s_i}{2}]+1)}-1)}{(p_i^{2k}-1)}$$
$$= \lambda(n) \prod p_i^{\epsilon(s_i)k} \prod \frac{(p^{2k([\frac{s_i}{2}]+1)}-1)}{(p_i^{2k}-1)} .$$

This completes the proof of the theorem. □

Proof of Corollary 1.3. By Lemma 2.5, we have

$$\sum_{d|n} \lambda(d)\,\sigma_0(d) = \sum_{d|n} \lambda(d)\,S_0(n/d) = \sum_{d|n} S_0(d)\,\lambda(n/d)$$
$$= \sum_{\substack{d|n \\ d\,:\,\text{perfect square}}} S_0(d)\,\lambda(n/d).$$

Here, if d is a perfect square, then $S_0(d) = 1$ and $\lambda(n/d) = \lambda(n)/\lambda(d) = \lambda(n)$. Hence

$$\sum_{d|n} \lambda(d)\,\sigma_0(d) = \sum_{\substack{d|n \\ d\,:\,\text{perfect square}}} S_0(d)\,\lambda(n/d)$$
$$= \lambda(n) \sum_{\substack{d|n \\ d\,:\,\text{perfect square}}} 1.$$

Now, d is a perfect square divisor of n if and only if d is a product of each numbers chosen in the sets $\{1, p_1^2, \ldots, {p_1}^{2[(s_1/2)]}\}, \ldots, \{1, p_r^2, \ldots, {p_r}^{2[(s_r/2)]}\}$. Hence, we obtain

$$\sum_{d|n} \lambda(d)\,\sigma_0(d) = \lambda(n) \left\{ \left(\left[\frac{s_1}{2}\right] + 1 \right) \cdots \left(\left[\frac{s_r}{2}\right] + 1 \right) \right\}.$$

This completes the proof of the corollary. □

Example 3.2. Some value of $\hat{S}_i(n)$ (i = 1, 2, 3) are give Table 3.

n	2	3	4	6	8	9	12	16	24	36
$\hat{S}_0(n)$	-1	-1	2	1	-2	2	-2	3	2	4
$\hat{S}_1(n)$	-2	-3	5	6	-10	10	-15	21	30	50
$\hat{S}_2(n)$	-4	-9	17	36	-68	82	-153	273	612	1394

Table 3. Some values of $\hat{S}_i(n)$

4. Applications of quasi-Stirling number of the first kind

Recall that we defined

$$S_k(n) := \sum_{\substack{n\in\mathbb{N} \\ d|n}} \lambda(d)\,d^k \quad \text{and} \quad \bar{S}_k(n) = \left[\sqrt[k]{|S_k(n)|} \right].$$

Lemma 4.1. *If n is a positive integer greater than 1 and $k \in \mathbb{N}$, then*
(a) $\bar{S}_k(n) < n$.
(b) For a large integer k, $\bar{S}_k(n) = n - 1$.

Proof. If p is a prime and s is a nonnegative integer, from (2) we have

$$p^{sk}\left(1 - \frac{1}{p^k}\right) < |S_k(p^s)| = \left|\frac{1 + (-1)^s p^{(s+1)k}}{p^k + 1}\right| < p^{sk}$$

for all $k > 0$. If $n = {p_1}^{s_1} \cdots p_r^{s_r}$, then

$$n^k\left(1 - \frac{1}{p_1^k}\right) \cdots \left(1 - \frac{1}{p_r^k}\right) < |S_k(n)| < n^k.$$

(a) Hence, we have $\bar{S}_k(n) = \left[\sqrt[k]{|S_k(n)|}\right] \leq \sqrt[k]{|S_k(n)|} < n$.

(b) Take a large integer k such that $p_i^k \geq {p_1}^{s_1} \cdots p_r^{s_r} = n$ for all i and $k \geq r$. Then

$$1 - \frac{1}{n} \leq 1 - \frac{1}{p_i^k} \quad \text{for each } i.$$

Hence,

$$\left(1 - \frac{1}{n}\right)^k \leq \left(1 - \frac{1}{n}\right)^r \leq \left(1 - \frac{1}{p_1^k}\right) \cdots \left(1 - \frac{1}{p_r^k}\right)$$

and so

$$n^k\left(1 - \frac{1}{n}\right)^k \leq n^k\left(1 - \frac{1}{p_1^k}\right) \cdots \left(1 - \frac{1}{p_r^k}\right) < |S_k(n)|.$$

It implies that $(n-1)^k < |S_k(n)| < n^k$ and $\bar{S}_k(n) = \left[\sqrt[k]{|S_k(n)|}\right] = n - 1$. □

Recall that we defined $\bar{S}_k^{(1)}(n) = \bar{S}_k(n)$ and $\bar{S}_k^{(l)}(n) = \bar{S}_k(\bar{S}_k^{(l-1)}(n))$ where $l \geq 2$. Then we get the following corollary.

Corollary 4.2. *Let $n \in \mathbb{N} - \{1\}$. There exists $l \in \mathbb{N}$ satisfying*

$$\bar{S}_k^{(l)}(n) = 1 \quad \textit{and} \quad \bar{S}_k{}^{(l-1)}(n) \neq 1.$$

Proof. If we let $\bar{S}_k(n) = m$, then by the above Lemma 4.1, we have

$$n > \bar{S}_k(n) = m > \bar{S}_k(m) = \bar{S}_k(\bar{S}_k^{(1)}(n)) = \bar{S}_k^{(2)}(n).$$

By continuing this process, we obtain the following decreasing sequence:

$$n > \bar{S}_k(n) > \bar{S}_k^{(2)}(n) > \cdots > \bar{S}_k{}^{(l-1)}(n) > \bar{S}_k^{(l)}(n) = 1,$$

from which the corollary follows. □

Example 4.3. The first eleven values of $\bar{S}_i^{(l)}(12)$ (i =1,2,3) are given Table 4.

l	1	2	3	4	5	6	7	8	9	10	11
$\bar{S}_1^{(l)}(12)$	6	2	1	1	1	1	1	1	1	1	1
$\bar{S}_2^{(l)}(12)$	10	8	7	6	4	3	2	1	1	1	1
$\bar{S}_3^{(l)}(12)$	11	10	9	8	7	6	5	4	3	2	1

Table 4. Some values of $\bar{S}_i^{(l)}(12)$

Proof of Theorem 1.4. Let $\bar{S}_k(n) = m$, then

$$\begin{aligned}
&(x+\bar{S}_k(n))(x+\bar{S}_k^{(2)}(n))\cdots(x+\bar{S}_k^{(l)}(n))\\
&=(x+m)(x+\bar{S}_k(m))\cdots(x+1)\\
&=\sum_{i=0}^{m}\left\{{m \atop i}\right\}_k x^i.
\end{aligned}$$

Multiply $(x+n)$ to both hand sides of the above equality. Then we obtain

$$\begin{aligned}
\sum_{i=0}^{n}\left\{{n \atop i}\right\}_k x^i &= (x+n)(x+\bar{S}_k(n))(x+\bar{S}_k^{(2)}(n))\cdots(x+\bar{S}_k^{(l)}(n))\\
&=(x+n)\sum_{i=0}^{m}\left\{{m \atop i}\right\}_k x^i\\
&=\sum_{i=0}^{n}\left(n\left\{{m \atop i}\right\}_k+\left\{{m \atop i-1}\right\}_k\right)x^i\\
&=\sum_{i=0}^{n}\left(n\left\{{\bar{S}_k(n) \atop i}\right\}_k+\left\{{\bar{S}_k(n) \atop i-1}\right\}_k\right)x^i.
\end{aligned}$$

Hence, we obtain our result. □

Example 4.4. Let $n = 12$. Then $\bar{S}(12) = 6,\ \bar{S}(6) = 2$ and $\bar{S}^{(2)}(6) = 1$. From

$$\begin{aligned}
(x+6)(x+\bar{S}(6))(x+\bar{S}^{(2)}(6)) &= (x+6)(x+2)(x+1)\\
&=\sum_{k=0}^{6}\left\{{6 \atop k}\right\}x^k,
\end{aligned}$$

we have $\left\{{6 \atop 0}\right\} = 12$, $\left\{{6 \atop 1}\right\} = 20$, $\left\{{6 \atop 2}\right\} = 9$, $\left\{{6 \atop 3}\right\} = 1$, $\left\{{6 \atop 4}\right\} = \left\{{6 \atop 5}\right\} = \left\{{6 \atop 6}\right\} = 0$. Hence, by Theorem 1.4, we have

$$\begin{aligned}
&\left\{{12 \atop 0}\right\} = 12^2 = 144,\ \left\{{12 \atop 1}\right\} = 12\cdot 20+12 = 252,\ \left\{{12 \atop 2}\right\} = 12\cdot 9+10 = 118,\\
&\left\{{12 \atop 3}\right\} = 12\cdot 1+9 = 21,\ \left\{{12 \atop 4}\right\} = 12\cdot 0+1 = 1,\ \left\{{12 \atop 5}\right\} = \cdots = \left\{{12 \atop 12}\right\} = 0.
\end{aligned}$$

And so,

$$(x+12)(x+\bar{S}(12))(x+\bar{S}^{(2)}(12))\cdots(x+\bar{S}^{(l)}(12))$$
$$=144+252x+118x^2+21x^3+x^4.$$

Corollary 4.5. *If $n \le 5$, then*

$$\left\{ n \atop i \right\} = \left[n+1 \atop i+1 \right].$$

Proof. Since $\bar{S}(5)=4,\ \bar{S}^{(2)}(5)=3,\ \bar{S}^{(3)}(5)=2,\ \bar{S}^{(4)}(5)=2,\ \bar{S}^{(5)}(5)=1$, we have

$$\begin{aligned}\sum_{i=0}^{5}\left\{ 5 \atop i \right\}x^i &= (x+5)(x+\bar{S}(5))(x+\bar{S}^{(2)}(5))(x+\bar{S}^{(3)}(5))(x+\bar{S}^{(4)}(5))(x+\bar{S}^{(5)}(5))\\ &= (x+5)(x+4)(x+3)(x+2)(x+1)\\ &= \sum_{i=1}^{6}\left[6 \atop i \right]x^{i-1} = \sum_{i'=0}^{5}\left[6 \atop i'+1 \right]x^{i'}\\ &= \sum_{i=0}^{5}\left[6 \atop i+1 \right]x^{k}.\end{aligned}$$

Hence, the proof is completed. □

Remark 4.1. By using Theorem 1.4 and Corollary 4.5, we can compute $\left\{ n \atop i \right\}$. The first ten values of $\left\{ n \atop i \right\}$ $(i=1,\ldots,7)$ are given Table 5.

	$\left\{ n \atop 0 \right\}$	$\left\{ n \atop 1 \right\}$	$\left\{ n \atop 2 \right\}$	$\left\{ n \atop 3 \right\}$	$\left\{ n \atop 4 \right\}$	$\left\{ n \atop 5 \right\}$	$\left\{ n \atop 6 \right\}$	$\left\{ n \atop 7 \right\}$	$\bar{S}(n)$
$n=1$	1	1	0	0	0	0	0	0	
$n=2$	2	3	1	0	0	0	0	0	1
$n=3$	6	11	6	1	0	0	0	0	2
$n=4$	24	50	35	10	1	0	0	0	3
$n=5$	120	274	225	85	15	1	0	0	4
$n=6$	12	20	9	1	0	0	0	0	2
$n=7$	84	152	83	16	1	0	0	0	6
$n=8$	960	2312	1850	905	205	23	1	0	5
$n=9$	756	1452	899	227	25	1	0	0	7
$n=10$	240	524	380	135	20	1	0	0	4

Table 5.

Corollary 4.6. *For each n, there exists a large integer k such that*

$$\left\{ n \atop i \right\}_k = \left[n+1 \atop i+1 \right].$$

Proof. By Lemma 4.1, there exists a large integer k such that $\bar{S}_k(n) = n-1$. Hence

$$\begin{aligned}\sum_{i=0}^{n} \left\{ {n \atop i} \right\}_k x^i &= (x+n)(x+\bar{S}_k(n))(x+\bar{S}_k^{(2)}(n))\cdots(x+\bar{S}_k^{(l)}(n)) \\ &= (x+n)(x+(n-1))(x+(n-2))\cdots(x+1) \\ &= \sum_{i=1}^{n} \left[{n+1 \atop i} \right] x^{i-1} \\ &= \sum_{i=0}^{n} \left[{n+1 \atop i+1} \right] x^{i}.\end{aligned}$$

It implies our result. □

References

1. J.M. Conway, R.K. Guy, *The Book of Numbers*, Springer-Verlag, New York, 1996: 258-259.
2. L. Comtet, Advanced combinatorics, D Reidel Publishing Company, Boston. 1974.
3. A. Alaca, S. Alaca and K.S. Williams, *The convolution sum* $\sum_{m<n/16} \sigma(m)\sigma(n-16m)$, Canad. Math. Bull. **51** (2008), 3-14.
4. A. Alaca, S. Alaca and K.S. Williams, *The convolution sums* $\sum_{l+24m=n} \sigma(l)\sigma(m)$ *and* $\sum_{3l+8m=n} \sigma(l)\sigma(m)$, M. J. Okayama Univ. **49** (2007), 93-111.
5. B.C. Berndt, *Ramanujan's Notebooks*, Part II. Springer-Verlag, New York, 1989.
6. B. Cho, D. Kim and J.-K. Koo, *Divisor functions arising from q-series*, Publ. Math. Debrecen **76** (2010), 495-508.
7. B. Cho, D. Kim and J.-K. Koo, *Modular forms arising from divisor functions*, J. Math. Anal. Appl. **356** (2009), 537-547.
8. J.W.L. Glaisher, *On the square of the series in which the coefficients are the sums of the divisors of the exponents*, Mess. Math. **14** (1884), 156-163.
9. J.W.L. Glaisher, *On certain sums of products of quantities depending upon the divisors of a number*, Mess. Math. **15** (1885), 1-20.
10. J.W.L. Glaisher, *Expressions for the five powers of the series in which the coefficients are the sums of the divisors of the exponents*, Mess. Math. **15** (1885), 33-36.
11. H. Hahn, *Convolution sums of some functions on divisors*, Rocky Mountain J. Math. **37** (2007), 1593-1622.
12. J.G. Huard, Z.M. Ou, B.K. Spearman, and K.S. Williams, *Elementary Evaluation of Certain Convolution Sums Involving Divisor Functions*, Number theory for the millennium, II, (2002), 229-274.
13. G. Melfi, *On some modular identities*, de Gruyter, Berlin, 1998, 371-382.
14. S. Ramanujan, *On certain arithmetical functions*, Trans. Cambridge Philos. Soc. **22** (1916), no9, 159-184.

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