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ON CHARACTERIZATIONS OF THE INVERSE WEIBULL DISTRIBUTION BASED ON RECORD VALUES

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Abstract. In this paper, we obtain characterizations of the inverse Weibull distribution based on ratios of lower record values by the property of independence. *Main Facts.*

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1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed random variables with cumulative distribution function(cdf) $F(x)$ and probability density function(pdf) $f(x)$. Let $Y_n = \min\{X_1, X_2, \ldots, X_n\}$ for $n \geq 1$. We say X_j is a lower record value of this sequence, if $Y_j < Y_{j-1}$ for $j > 1$. The indices at which the lower record values occur are given by the record times $\{L(n), n \geq 1\}$, where $L(n) = \min\{j \mid j > L(n-1), X_j < X_{L(n-1)}, n \geq 2\}$ with $L(1) = 1$. We assume that all lower record values $X_{L(i)}$ for $i \geq 1$ occur at a sequence $\{X_n, n \geq 1\}$ of i.i.d. random variables.

A continuous random variable X is called the inverse Weibull distribution with parameters $c > 0$, $\alpha > 0$ if the corresponding probability cdf $F(x)$ of X is given by

$$F(x) = \begin{cases} e^{-cx^{-\alpha}}, & x > 0,\ c > 0,\ \alpha > 0, \\ 0, & \text{otherwise.} \end{cases}$$

The current investigation was induced by the characterizations of Weibull distribution in [3]. They proved that $F(x)$ has a Weibull distribution if and only if $X_{U(m)}/X_{U(n)}$ and $X_{U(n)}$ or $X_{U(n)}/(X_{U(n)} \pm X_{U(m)})$ and $X_{U(n)}$ are independent for $1 \leq m < n$.

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In this paper we extend the result of Lee and Lim [3] and obtain characterizations of the inverse Weibull distribution by the independence property on the lower record values.

2. Results

Theorem 2.1. *Let* $\{X_n, n \geq 1\}$ *be a sequence of independent and identically distributed nonnegative random variables with cdf* $F(x)$ *which is absolutely continuous with pdf* $f(x)$ *and* $F(x) < 1$ *for all* $x > 0$*. Then* $F(x) = e^{-cx^{-\alpha}}$ *for all* $x > 0$ *and* $c > 0$, $\alpha > 0$, *if and only if* $X_{L(n)}/X_{L(m)}$ *and* $X_{L(n)}$ *are independent for* $1 \leq m < n$.

Theorem 2.2. *Let* $\{X_n, n \geq 1\}$ *be a sequence of independent and identically distributed nonnegative random variables with cdf* $F(x)$ *which is absolutely continuous with pdf* $f(x)$ *and* $F(x) < 1$ *for all* $x > 0$*. Then* $F(x) = e^{-cx^{-\alpha}}$ *for all* $x > 0$ *and* $c > 0$, $\alpha > 0$, *if and only if* $X_{L(n)}/(X_{L(n)} + X_{L(m)})$ *and* $X_{L(n)}$ *are independent for* $1 \leq m < n$.

Theorem 2.3. *Let* $\{X_n, n \geq 1\}$ *be a sequence of independent and identically distributed nonnegative random variables with cdf* $F(x)$ *which is absolutely continuous with pdf* $f(x)$ *and* $F(x) < 1$ *for all* $x > 0$*. Then* $F(x) = e^{-cx^{-\alpha}}$ *for all* $x > 0$ *and* $c > 0$, $\alpha > 0$, *if and only if* $X_{L(n)}/(X_{L(m)} - X_{L(n)})$ *and* $X_{L(n)}$ *are independent for* $1 \leq m < n$.

3. Proofs

Proof of Theorem 2.1. The joint pdf $f_{m,n}(x,y)$ of $X_{L(m)}$ and $X_{L(n)}$ is

$$f_{m,n}(x,y) = \frac{H(x)^{m-1}}{\Gamma(m)} h(x) \frac{\{H(y) - H(x)\}^{n-m-1}}{\Gamma(n-m)} f(y)$$

where $H(x) = -\ln F(x)$ and $h(x) = -\frac{d}{dx}H(x) = \frac{f(x)}{F(x)}$, for $1 \leq m < n$.

Consider the functions $U = X_{L(n)}/X_{L(m)}$ and $W = X_{L(n)}$. It follows that $x_{L(m)} = w/u$, $x_{L(n)} = w$ and $J = -w/u^2$. Thus we can write the joint pdf $f_{u,w}(u,w)$ of U and W as

$$f_{U,W}(u,w) = \frac{H(\frac{w}{u})^{m-1}}{\Gamma(m)} h\left(\frac{w}{u}\right) \frac{\{H(w) - H(\frac{w}{u})\}^{n-m-1}}{\Gamma(n-m)} f(w) \frac{w}{u^2}$$

for $0 < u < 1$, $w > 0$. If $F(x) = e^{-cx^{-\alpha}}$ for all $x > 0$, $\alpha > 0$, $c > 0$, then we get

$$f_{U,W}(u,w) = \frac{c^n \alpha^2}{\Gamma(m)\Gamma(n-m)} u^{\alpha m-1}(1-u^\alpha)^{n-m-1} w^{-\alpha n-1} e^{-cw^{-\alpha}} \qquad (1)$$

for all $0 < u < 1$, $w > 0$ and $\alpha > 0$. After integrating (1) with respect to W, we get

$$\begin{aligned} f_U(u) &= \int_0^\infty f_{U,W}(u,w)dw \\ &= \frac{c^n\alpha^2}{\Gamma(m)\Gamma(n-m)} u^{\alpha m-1}(1-u^\alpha)^{n-m-1}\int_0^\infty w^{-\alpha n-1}e^{-cw^{-\alpha}}dw \\ &= \frac{\Gamma(n)\alpha}{\Gamma(m)\Gamma(n-m)} u^{\alpha m-1}(1-u^\alpha)^{n-m-1} \end{aligned} \tag{2}$$

for all $0 < u < 1$, $\alpha > 0$, and $c > 0$. Also, the pdf $f_W(w)$ of W is given by

$$f_W(w) = \frac{H(w)^{n-1}}{\Gamma(n)} f(w) = \frac{c^n\alpha}{\Gamma(n)} w^{-\alpha n-1}e^{-cw^{-\alpha}}. \tag{3}$$

From (1), (2) and (3), we obtain $f_{U,W}(u,w) = f_U(u)f_W(w)$. Hence U and W are independent for $1 \le m < n$.

Now we will prove the sufficient condition. Let us use the transformation $U = X_{L(n)}/X_{L(m)}$ and $W = X_{L(n)}$. The Jacobian of the transformation is $J = -w/u^2$. Thus we can write the joint pdf $f_{U,W}(u,w)$ of U and W as

$$f_{U,W}(u,w) = \frac{H(\frac{w}{u})^{m-1}}{\Gamma(m)} h\left(\frac{w}{u}\right) \frac{\{H(w) - H(\frac{w}{u})\}^{n-m-1}}{\Gamma(n-m)} f(w)\frac{w}{u^2} \tag{4}$$

for all $0 < u < 1$ and $w > 0$. The pdf $f_W(w)$ of W is given by

$$f_W(w) = \frac{H(w)^{n-1}}{\Gamma(n)} f(w) \tag{5}$$

for all $w > 0$. Since U and W are independent, we get the pdf $f_U(u)$ of U from (4) and (5) as

$$\begin{aligned} f_U(u) &= \frac{\Gamma(n)}{\Gamma(m)\Gamma(n-m)} \frac{\{H(\frac{w}{u})\}^{m-1}h(\frac{w}{u})\{H(w) - H(\frac{w}{u})\}^{n-m-1}}{H(w)^{n-1}} \frac{w}{u^2} \\ &= \frac{\Gamma(n)}{\Gamma(m)\Gamma(n-m)} \left\{\frac{H(\frac{w}{u})}{H(w)}\right\}^{m-1} \left\{1 - \frac{H(\frac{w}{u})}{H(w)}\right\}^{n-m-1} \frac{\partial}{\partial u}\left(\frac{H(\frac{w}{u})}{H(w)}\right) \\ &= \frac{\Gamma(n)}{\Gamma(m)\Gamma(n-m)} \left\{\frac{\ln F(\frac{w}{u})}{\ln F(w)}\right\}^{m-1} \left\{1 - \frac{\ln F(\frac{w}{u})}{\ln F(w)}\right\}^{n-m-1} \frac{\partial}{\partial u}\left(\frac{\ln F(\frac{w}{u})}{\ln F(w)}\right) \end{aligned}$$

where $H(x) = -\ln F(x)$ and $h(x) = -\frac{d}{dx}H(x)$.

By [2, Lemma, p.48], the pdf $f_U(u)$ of U is a function of u only. Thus we have

$$\ln F(w/u) = G(u) \ln F(w) \tag{6}$$

where $G(u)$ is a function of u only.

By the theory of functional equation [1], the only continuous solution of (6) with the boundary conditions $\lim_{x\to 0^+} F(x) = 0$ and $F(\infty) = 1$ is $F(x) = e^{-cx^{-\alpha}}$ for all $x > 0$, $c > 0$, $\alpha > 0$. This completes the proof. □

Proof of Theorem 2.2. The necessary condition is easy to establish. Now we prove the sufficient condition.

Let us use the transformation $U = X_{L(n)}/(X_{L(n)} + X_{L(m)})$, $W = X_{L(n)}$. The Jacobian of the transformation is $J = -w/u^2$. Thus we can write the joint pdf $f_{U,W}(u,w)$ of U and W as

$$f_{U,W}(u,w) = \frac{H(\frac{w(1-u)}{u})^{m-1}}{\Gamma(m)} h\left(\frac{w(1-u)}{u}\right) \frac{[H(w) - H(\frac{w(1-u)}{u})^{n-m-1}]}{\Gamma(n-m)} f(w) \frac{w}{u^2} \tag{7}$$

for $0 < u < 1$ and $w > 0$. The pdf $f_W(w)$ of W is given by

$$f_W(w) = \frac{H(w)^{n-1}}{\Gamma(n)} f(w) \text{ for } w > 0 \tag{8}$$

Since U and W are independent, we get the pdf $f_U(u)$ of U form (7) and (8) as

$$\begin{aligned} f_U(u) &= \frac{\Gamma(n)}{\Gamma(m)\Gamma(n-m)} \left\{ \frac{H(\frac{w(1-u)}{u})}{H(w)} \right\}^{m-1} \left\{ 1 - \frac{H(\frac{w(1-u)}{u})}{H(w)} \right\}^{n-m-1} \\ &\quad \times \frac{\partial}{\partial u} \left(\frac{H(\frac{w}{u}(1-u))}{H(w)} \right) \\ &= \frac{\Gamma(n)}{\Gamma(m)\Gamma(n-m)} \left\{ \frac{\ln F(\frac{w(1-u)}{u})}{\ln F(w)} \right\}^{m-1} \left\{ 1 - \frac{\ln F(\frac{w(1-u)}{u})}{\ln F(w)} \right\}^{n-m-1} \\ &\quad \times \frac{\partial}{\partial u} \left(\frac{\ln F(\frac{w(1-u)}{u})}{\ln F(w)} \right) \end{aligned}$$

where $H(x) = -\ln F(x)$ and $h(x) = -\frac{d}{dx} H(x)$.

By [2, Lemma, p.48], the pdf $f_U(u)$ of U is a function of u only. Thus we have

$$\ln F(w(1-u)/u) = G(u) \ln F(w) \tag{9}$$

where $G(u)$ is a function of u only.

By the theory of functional equation [1], the only continuous solution of (9) with the boundary conditions $\lim_{x \to 0^+} F(x) = 0$ and $F(\infty) = 1$ is $F(x) = e^{-cx^{-\alpha}}$ for all $x > 0$, $c > 0$, $\alpha > 0$. This completes the proof. □

Proof of Theorem 2.3. The necessary condition is easy to establish. Now we prove the sufficient condition.

Let us use the transformation $V = X_{L(n)}/(X_{L(m)} - X_{L(n)})$, $W = X_{L(n)}$. The Jacobian of the transformation is $J = -w/v^2$. Thus we can write the joint pdf $f_{V,W}(v,w)$ of V and W as

$$f_{V,W}(v,w) = \frac{H(\frac{w(1+v)}{v})^{m-1}}{\Gamma(m)} h\left(\frac{w(1+v)}{v}\right) \frac{[H(w) - H(\frac{w(1+v)}{v})^{n-m-1}]}{\Gamma(n-m)} f(w) \frac{w}{v^2}$$

for $v > 0$, $w > 0$. The rest of the proof is similar to that of Theorem 2.2.. □

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