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# ON INTEGRAL OPERATORS INVOLVING THE PRODUCT OF GENERALIZED BESSEL FUNCTION AND JACOBI POLYNOMIAL

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Abstract. The aim of this research note is to evaluate two generalized integrals involving the product of generalized Bessel function and Jacobi polynomial by employing the result of Obhettinger [2]. Also, by mean of the main results, we have established an interesting relation in between Kampé de Fériet and Srivastava and Daoust functions. Some interesting special cases of our main results are also indicated.

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## 1. Introduction

In recent years, many integrals involving some well known special functions of mathematical physics have been evaluated by a number of authors (see, for example [1], [3], [4], [5], [6], [7], [8], [9], [10]). In particular, motivated by the work of Khan et al. [7], we further establish two interesting integrals involving the product of generalized Bessel function and Jacobi polynomial.

In order to propose our present study, we begin by recalling here the following definitions of some well known functions:

The generalized Bessel function of first kind $w^d_{\nu,c}(z)$ of order $\nu$ is defined by (see [7]):

$$w^d_{\nu,c}(z) = \sum_{m=0}^{\infty} \frac{(-1)^m c^m (z/2)^{\nu+2m}}{m!\Gamma(\nu+m+\frac{1+d}{2})}. \tag{1.1}$$

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Also, we have the following special cases of (1.1) (see [7]):

$$w^{d}_{1-\frac{d}{2},-c^2}(z) = \left(\frac{2}{z}\right)^{\frac{d}{2}} \frac{\sin cz}{\sqrt{\pi}} \tag{1.2}$$

and

$$w^{d}_{d/2,c^2}(z) = \left(\frac{2}{z}\right)^{\frac{d}{2}} \frac{\cos cz}{\sqrt{\pi}}. \tag{1.3}$$

The Jacobi polynomial $P_n^{(\alpha,\beta)}(z)$ is defined by (see [1], [3]):

$$P_n^{(\alpha,\beta)}(z) = \frac{(1+\alpha)_n}{n!} {}_2F_1 \left[ \begin{array}{ll} -n, 1+\alpha+\beta+n; & \frac{1-z}{2} \\ 1+\alpha; & \end{array} \right], \tag{1.4}$$

or, equivalently

$$P_n^{(\alpha,\beta)}(z) = \sum_{k=0}^{n} \frac{(1+\alpha)_n (1+\alpha+\beta)_{n+k}}{k!\ (n-k)!(1+\alpha)_k (1+\alpha+\beta)_n} \left(\frac{z-1}{2}\right)^k. \tag{1.5}$$

From equation (1.4) and (1.5), we have

$$P_n^{(\alpha,\beta)}(1) = \frac{(1+\alpha)_n}{n!}, \tag{1.6}$$

where $P_n^{(\alpha,\beta)}(z)$ is a polynomial of degree $n$.

For $\beta = \alpha$, the polynomial $P_n^{(\alpha,\alpha)}(z)$ is called the ultraspherical polynomial and further on setting $\alpha = \beta = \mu - \frac{1}{2}$, (1.4) reduces to

$$P_n^{(\mu-\frac{1}{2},\mu-\frac{1}{2})}(z) = \frac{(\mu+\frac{1}{2})_n}{(2\mu)_n} C_n^{\mu}(z), \tag{1.7}$$

where $C_n^{\mu}(z)$ is the Gegenbauer polynomial (see [1], [3]).
Again for $\alpha = \beta = -\frac{1}{2}$ in (1.4), we have

$$P_n^{(-\frac{1}{2},-\frac{1}{2})}(z) = \frac{\left(\frac{1}{2}\right)_n}{n!} T_n(z) \tag{1.8}$$

and for $\alpha = \beta = \frac{1}{2}$ in (1.5), we have

$$P_n^{(\frac{1}{2},\frac{1}{2})}(z) = \frac{\left(\frac{3}{2}\right)_n}{(n+1)!} U_n(z), \tag{1.9}$$

where $T_n(z)$ and $U_n(z)$ are Tchebicheff polynomials of first and second kind respectively (see [1], [3]).
For $\alpha = \beta = 0$ in equation (1.4) and (1.5), we have

$$P_n^{(0,0)}(z) = P_n(z), \tag{1.10}$$

where $P_n(z)$ is the Legendre polynomial (see[1], [3]).
Also, we recall here the following definition of Kampé de Fériet and Srivastava and Daoust functions respectively (see [7]):

The Kampé de Fériet function is defined by (see[3], [7]):

$$F^{p:q;k}_{l:m;:n}\left[\begin{array}{ll}(a_p):(b_q);(c_k); & \\ & x,y\\ (\alpha_l):(\beta_m):(\gamma_n); & \end{array}\right]=\sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r}$$

$$\times\frac{\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r y^s}{r!\ s!}, \tag{1.11}$$

where, for convergence,
$(i) p+q<l+m+1, p+k<l+n+1, |x|<\infty, |y|<\infty, or$
$(ii) p+q=l+m+1, p+k=l+n+1,$
and

$$\begin{cases} |x|^{\frac{1}{p-l}}+|y|^{\frac{1}{p-1}}<1; & if p>l\\ \max\{|x|,|y|\}<1; & if p\leq l.\end{cases}$$

The Srivastava and Daoust multivariable hypergeometric function is given as follows (see [3], [7]):

$$F^{p:q_1;\dots q_r}_{l:m_1;\dots m_r}\left[\begin{array}{ll}(a_j:\alpha_j^1,\dots\alpha_j^{(r)}))_{1,p}:(c_j^1,r_j^1)_{1,q_1};\dots;(c_j^{(r)},r_j^{(r)})_{1,q_r}; & \\ & x_1,x_2,x_3\dots x_r\\ (b_j:\beta_j^1,\dots\beta_j^{(r)})_{1,l}:(d_j^1,\delta_j^1)_{1,m_1};\dots;(d_j^{(r)},\delta_j^{(r)})_{1,m_r}; & \end{array}\right]$$

$$=\sum_{n_1,n_2\dots n_r=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{n_1\alpha_j^1+\dots n_r\alpha^{(r)}}\prod_{j=1}^{q_1}(c_j^1)_{n_1 r_j^1\dots}}{\prod_{j=1}^{m_r}(d_j^{(r)})_{n_r\delta_j^{(r)}}}$$

$$\times\frac{\prod_{j=1}^{q_r}(c_j^{(r)})_{n_r r_j^{(r)}}}{\prod_{j=1}^{l}(b_j)_{n_1\beta_j^1+\dots n_r\beta^{(r)}}\prod_{j=1}^{m_1}(d_j^1)_{n_1\delta_j^1\dots}}\frac{x_1^{n_1}}{n_1!}\dots\frac{x_r^{n_r}}{n_r!}, \tag{1.12}$$

where the multiple hypergeometric series converges absolutely under the parametric variable constrains and $(\lambda)_\nu$ denotes the well known Pochhammer symbol.

For our present investigation the following interesting result of Obhettinger [2] will be required:

$$\int_0^\infty x^{\mu-1}(x+a+\sqrt{(x^2+2ax)})^{-\lambda}=2\lambda a^{-\lambda}\left(\frac{a}{2}\right)^{\mu}\frac{\Gamma(2\mu)\Gamma(\lambda-\mu)}{\Gamma(1+\lambda+\mu)}, \tag{1.13}$$

provided $0<\Re(\mu)<\Re(\lambda)$.

## 2. Main results

In this section, we establish two interesting integrals involving the product of generalized Bessel function and Jacobi polynomial, which are given in terms of Kampé de Fériet and Srivastava and Daoust functions.

**Theorem 2.1.** *For $\Re(\nu) > -\frac{(1+d)}{2}$, $0 < \Re(\mu) < \Re(\lambda+\nu)$, and $x > 0$, the following integral formula (in terms of Kampé de Fériet function) holds true:*

$$\int_0^\infty x^{\mu-1}[x+a+\sqrt{(x^2+2ax)}]^{-\lambda} w_{\nu,c}^{d}\left[\frac{y}{x+a+\sqrt{(x^2+2ax)}}\right] \times P_n^{(\alpha,\beta)}\left[1-\frac{by}{x+a+\sqrt{(x^2+2ax)}}\right]dx$$
$$= y^{\nu}2^{1-\nu-\mu}a^{\mu-\nu-\lambda}\frac{(1+\alpha)_n\Gamma(2\mu)\Gamma(\lambda+\nu+1)\Gamma(\lambda+\nu-\mu)}{n!\ \Gamma\left(\nu+\frac{1+d}{2}\right)\Gamma(\lambda+\nu)\Gamma(1+\lambda+\nu+\mu)}$$
$$\times\left\{F_{4:1;3}^{4:0;4}\left[\begin{array}{l}\triangle(2;\lambda+\nu+1),\triangle(2;\lambda+\nu-\mu):-;\triangle(2;-n),\triangle(2;1+\alpha+\beta+n);\\ \triangle(2;\lambda+\nu),\triangle(2;1+\lambda+\nu+\mu):\nu+\frac{1+d}{2};\triangle(2;1+\alpha),\frac{1}{2};\end{array}\frac{-cy^2}{4a^2},\frac{b^2y^2}{4a^2}\right]\right.$$
$$+\frac{byn(1+\alpha+\beta+n)(\lambda+\nu+1)(\lambda+\nu-\mu)}{2a(\lambda+\nu)(1+\lambda+\nu+\mu)(1+\alpha)}$$
$$\left.\times F_{4:1;3}^{4:0;4}\left[\begin{array}{l}\triangle(2;\lambda+\nu+2),\triangle(2;\lambda+\nu-\mu+1):-;\triangle(2;-n+1),\triangle(2;2+\alpha+\beta+n);\\ \triangle(2;\lambda+\nu+1),\triangle(2;2+\lambda+\nu+\mu):\nu+\frac{1+d}{2};\triangle(2;2+\alpha),\frac{3}{2};\end{array}\frac{-cy^2}{4a^2},\frac{b^2y^2}{4a^2}\right]\right\}, \tag{2.1}$$

*where $\triangle(m;l)$ abbreviates the array of $m$ parameters as $\frac{l}{m},\frac{l+1}{m},\cdots,\frac{l+m-1}{m}$, $m\geq 1$ and $F_{l:m;n}^{p:q;r}$ is the Kampé de Fériet function, given in (1.11).*

*Proof.* Let us denote the left hand side of (2.1) by $I_1$, expanding $w_{\nu,c}^{d}(z)$ and $P_n^{(\alpha,\beta)}(z)$ with the help of (1.1) and (1.4) and then interchanging the order of integration and summation (which is valid under the given conditions) to get

$$I_1 = y^{\nu}2^{-\nu}\frac{(1+\alpha)_n}{n!}\sum_{m=0}^{\infty}\sum_{k=0}^{n}\frac{(-n)_k(1+\alpha+\beta+n)_k}{m!\ k!\ (1+\alpha)_k\Gamma\left(m+\nu+\frac{1+d}{2}\right)}\left(\frac{-cy^2}{4}\right)^m\left(\frac{by}{2}\right)^k$$
$$\times\int_0^\infty x^{\mu-1}[x+a+\sqrt{(x^2+2ax)}]^{-(\lambda+\nu+2m+k)}dx. \tag{2.2}$$

Using (1.13) in the above expression and after a little simplification, we get

$$I_1 = y^{\nu} 2^{1-\nu-\mu} a^{\mu-\lambda-\nu} \frac{(1+\alpha)_n \Gamma(2\mu)\Gamma(\lambda+\nu+1)\Gamma(\lambda+\nu-\mu)}{n!\ \Gamma\left(\nu+\frac{1+d}{2}\right)\Gamma(\lambda+\nu)\Gamma(1+\lambda+\nu+\mu)}$$

$$\times \sum_{m=0}^{\infty}\sum_{k=0}^{n} \frac{(\lambda+\nu+1)_{2m+k}(\lambda+\nu-\mu)_{2m+k}(-n)_k(1+\alpha+\beta+n)_k}{m!\ k!\ (\lambda+\nu)_{2m+k}\left(\nu+\frac{1+d}{2}\right)_m(1+\alpha)_k(1+\lambda+\nu+\mu)_{2m+k}}$$

$$\times \left(\frac{-cy^2}{4a^2}\right)^m \left(\frac{by}{2a}\right)^k. \tag{2.3}$$

On separating the k-series into its even and odd terms and then using the result $(A)_{m+n} = (A)_m(A+m)_n$, in the second term of the given expression, we get

$$I_1 = y^{\nu} 2^{1-\nu-\mu} a^{\mu-\lambda-\nu} \frac{(1+\alpha)_n \Gamma(2\mu)\Gamma(\lambda+\nu+1)\Gamma(\lambda+\nu-\mu)}{n!\ \Gamma\left(\nu+\frac{1+d}{2}\right)\Gamma(\lambda+\nu)\Gamma(1+\lambda+\nu+\mu)}$$

$$\times \left\{ \sum_{m=0}^{\infty}\sum_{k=0}^{n} \frac{(\lambda+\nu+1)_{2(m+k)}(\lambda+\nu-\mu)_{2(m+k)}(-n)_{2k}}{m!\ k!\ 2^{2k}\left(\frac{1}{2}\right)_k(\lambda+\nu)_{2(m+k)}\left(\nu+\frac{1+d}{2}\right)_m(1+\alpha)_k} \right.$$

$$\times \frac{(1+\alpha+\beta+n)_{2k}}{(1+\lambda+\nu+\mu)_{2(m+k)}} \left(\frac{-cy^2}{4a^2}\right)^m \left(\frac{by}{2a}\right)^{2k}$$

$$+ \frac{by(1+\alpha+\beta+n)(\lambda+\nu+1)(\lambda+\nu-\mu)}{2a(\lambda+\nu)(1+\lambda+\nu+\mu)(1+\alpha)}$$

$$\times \sum_{m=0}^{\infty}\sum_{k=0}^{n} \frac{(\lambda+\nu+2)_{2(m+k)}(\lambda+\nu-\mu+1)_{2(m+k)}(-n+1)_{2k}}{m!\ k!\ 2^{2k}\left(\frac{3}{2}\right)_k(\lambda+\nu+1)_{2(m+k)}\left(\nu+\frac{1+d}{2}\right)_m}$$

$$\left. \times \frac{(2+\alpha+\beta+n)_{2k}}{(2+\alpha)_k(2+\lambda+\nu+\mu)_{2(m+k)}} \left(\frac{-cy^2}{4a^2}\right)^m \left(\frac{by}{2a}\right)^{2k} \right\}. \tag{2.4}$$

Finally, the use of (1.11), yields the desired result.

□

**Theorem 2.2.** *For $\Re(\nu) > -\frac{(1+d)}{2}$, $0 < \Re(\mu) < \Re(\lambda+\nu)$ and $x > 0$, the following integral formula (in terms of Srivastava and Daoust function) holds true:*

$$\int_0^{\infty} x^{\mu-1}[x+a+\sqrt{(x^2+2ax)}]^{-\lambda} w_{\nu,c}^{d}\left[\frac{y}{x+a+\sqrt{(x^2+2ax)}}\right]$$

$$\times P_n^{(\alpha,\beta)}\left[1-\frac{by}{x+a+\sqrt{(x^2+2ax)}}\right] dx$$

$$= y^{\nu} 2^{1-\nu-\mu} a^{\mu-\nu-\lambda} \frac{\Gamma(2\mu)\Gamma(\lambda+\nu+1)\Gamma(\lambda+\nu-\mu)}{\Gamma\left(\nu+\frac{1+d}{2}\right)\Gamma(\lambda+\nu)\Gamma(1+\lambda+\nu+\mu)}$$

$$\times F^{4:0;0}_{5:0;1}\left[\begin{array}{l}(\lambda+\nu+1:2,3),(\lambda+\nu-\mu:2,3),(1+\alpha+\beta:1,2),(1+\alpha:1,1):\\(\lambda+\nu:2,3),(1+\lambda+\nu+\mu:2,3),(\nu+\frac{1+b}{2}:1,1),(1+\alpha+\beta:1,1),\end{array}\right.$$

$$\left.\begin{array}{l}-;-;\\(1:1,1):-;(1+\alpha,1);\end{array}\ \frac{-cy^2}{4a^2},\frac{bcy^3}{8a^3}\right]. \tag{2.5}$$

*Proof.* Let us denote the left-hand side of (2.5) by $I_2$, expanding $w^d_{\nu,c}(z)$ and $P^{(\alpha,\beta)}_n(z)$ in their series form with the help of (1.1) and (1.5), respectively, and then by using the lemma (see [1]):

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}B(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}B(k,n+k),$$

we get

$$I_2=y^\nu 2^{-\nu}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(1+\alpha)_{n+k}(1+\alpha+\beta)_{n+2k}(-b)_k}{(n+k)!\ n!\ k!\ \Gamma\left(n+\nu+\frac{1+d}{2}+k\right)(1+\alpha+\beta)_{n+k}(1+\alpha)_k}$$

$$\times\left(\frac{y}{2}\right)^{2n+3k}(-1)^{n+k}c^{n+k}\int_0^\infty x^{\mu-1}[x+a+\sqrt{(x^2+2ax)}]^{-\lambda-\nu-2n-3k}dx. \tag{2.6}$$

On using (1.13) and after a little simplification, we arrive at

$$I_2=y^\nu 2^{1-\nu-\mu}a^{\mu-\lambda-\nu}\frac{\Gamma(2\mu)\Gamma(\lambda+\nu-\mu)\Gamma(\lambda+\nu+1)}{\Gamma\left(\nu+\frac{1+d}{2}\right)\Gamma(\lambda+\nu)\Gamma(1+\lambda+\nu+\mu)}$$

$$\times\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(1+\alpha)_{n+k}(1+\alpha+\beta)_{n+2k}(\lambda+\nu-\mu)_{2n+3k}(\lambda+\nu+1)_{2n+3k}}{(1)_{n+k}n!\ k!\ \left(\nu+\frac{1+d}{2}\right)_{n+k}(1+\alpha+\beta)_{n+k}(\lambda+\nu)_{2n+3k}(1+\alpha)_k}$$

$$\times\frac{1}{(1+\lambda+\nu+\mu)_{2n+3k}}\left(\frac{-y^2c}{4a^2}\right)^n\left(\frac{cby^3}{8a^3}\right)^k. \tag{2.7}$$

Finally, summing up the above series with the help of (1.12), we arrive at the right hand side of (2.5).

□

**Remark 2.1.** On setting $c=d=1$ in Theorem 2.1 and Theorem 2.2, respectively, we easily get the equations (2.1) and (2.5) of Khan et al. [7], which further on setting $\alpha=\beta=b=0$ reduces to the well known result of Choi et al. [5].

## 3. Special Cases

In this section, we have given following (presumably) new integrals as a special cases of our main results:

**Corollary 3.1.** *For $\beta = \alpha$ in (2.1), the following integral formula (which is valid under given conditions) holds true:*

$$\int_0^\infty x^{\mu-1}[x+a+\sqrt{(x^2+2ax)}]^{-\lambda} w_{\nu,c}^d \left[\frac{y}{x+a+\sqrt{(x^2+2ax)}}\right] \times P_n^{(\alpha,\alpha)}\left[1-\frac{by}{x+a+\sqrt{(x^2+2ax)}}\right] dx$$

$$= y^\nu 2^{1-\nu-\mu} a^{\mu-\nu-\lambda} \frac{(1+\alpha)_n \Gamma(2\mu)\Gamma(\lambda+\nu+1)\Gamma(\lambda+\nu-\mu)}{n!\ \Gamma\left(\nu+\frac{1+d}{2}\right)\Gamma(\lambda+\nu)\Gamma(1+\lambda+\nu+\mu)}$$

$$\times \left\{ F_{4:1;3}^{4:0;4}\left[\begin{array}{l} \triangle\,(2;\lambda+\nu+1), \triangle\,(2;\lambda+\nu-\mu): -; \triangle\,(2;-n), \triangle\,(2;1+2\alpha+n); \\ \triangle\,(2;\lambda+\nu), \triangle\,(2;1+\lambda+\nu+\mu): \nu+\frac{1+d}{2}; \triangle\,(2;1+\alpha), \frac{1}{2}; \end{array} \frac{-cy^2}{4a^2}, \frac{b^2y^2}{4a^2}\right] + \frac{byn(1+2\alpha+n)(\lambda+\nu+1)(\lambda+\nu-\mu)}{2a(\lambda+\nu)(1+\lambda+\nu+\mu)(1+\alpha)} \right.$$

$$\left. \times F_{4:1;3}^{4:0;4}\left[\begin{array}{l} \triangle\,(2;\lambda+\nu+2), \triangle\,(2;\lambda+\nu-\mu+1): -; \triangle\,(2;-n+1), \triangle\,(2;2\alpha+n); \\ \triangle\,(2;\lambda+\nu+1), \triangle\,(2;2+\lambda+\nu+\mu): \nu+\frac{1+d}{2}; \triangle\,(2;2+\alpha), \frac{3}{2}; \end{array} \frac{-cy^2}{4a^2} \frac{b^2y^2}{4a^2}\right]\right\}. \tag{3.1}$$

*where $P_n^{(\alpha,\alpha)}(z)$ is known as the ultraspherical polynomial (see [3], [7]).*

**Corollary 3.2.** *On taking $\beta = \alpha = l - \frac{1}{2}$ in (2.1) and then by using (1.7), the following integral formula (which is valid under the given conditions) holds true:*

$$\int_0^\infty x^{\mu-1}[x+a+\sqrt{(x^2+2ax)}]^{-\lambda} w_{\nu,c}^d \left[\frac{y}{x+a+\sqrt{(x^2+2ax)}}\right] \times C_n^l\left[1-\frac{by}{x+a+\sqrt{(x^2+2ax)}}\right] dx$$

$$= y^\nu 2^{1-\nu-\mu} a^{\mu-\nu-\lambda} \frac{(2l)_n \Gamma(2\mu)\Gamma(\lambda+\nu+1)\Gamma(\lambda+\nu-\mu)}{n!\ \Gamma\left(\nu+\frac{1+d}{2}\right)\Gamma(\lambda+\nu)\Gamma(1+\lambda+\nu+\mu)}$$

$$\times \left\{ F_{4:1;3}^{4:0;4}\left[\begin{array}{l} \triangle\,(2;\lambda+\nu+1), \triangle\,(2;\lambda+\nu-\mu): -; \triangle\,(2;-n), \triangle\,(2;2l+n); \\ \triangle\,(2;\lambda+\nu), \triangle\,(2;1+\lambda+\nu+\mu): \nu+\frac{1+d}{2}; \triangle\,(2;l+1/2), \frac{1}{2}; \end{array}\right.\right.$$

$$\left.\frac{-cy^2}{4a^2},\frac{b^2y^2}{4a^2}\right]+\frac{byn(2l+n)(\lambda+\nu+1)(\lambda+\nu-\mu)}{2a(\lambda+\nu)(1+\lambda+\nu+\mu)(l+1/2)}$$

$$\times F^{4:0;4}_{4:1;3}\left[\begin{array}{l}\triangle\,(2;\lambda+\nu+2),\triangle\,(2;\lambda+\nu-\mu+1):-;\triangle\,(2;-n+1),\\ \\ \triangle\,(2;\lambda+\nu+1),\triangle\,(2;2+\lambda+\nu+\mu):\nu+\frac{1+d}{2};\triangle\,(2;l+3/2),\end{array}\right.$$

$$\left.\left.\begin{array}{c}\triangle\,(2;2l+n+1);\\ \frac{3}{2};\end{array}\quad\frac{-cy^2}{4a^2}\frac{b^2y^2}{4a^2}\right]\right\}. \tag{3.2}$$

*where $C_n^l(z)$ is known as the Gegenbauer polynomial (see [3], [7]).*

**Corollary 3.3.** *Assuming $\beta=\alpha=-\frac{1}{2}$ in (2.1) and then by using (1.8), the following integral formula (which is valid under the given conditions) holds true:*

$$\int_0^\infty x^{\mu-1}[x+a+\sqrt{(x^2+2ax)}]^{-\lambda}w^d_{\nu,c}\left[\frac{y}{x+a+\sqrt{(x^2+2ax)}}\right]$$

$$\times T_n\left[1-\frac{by}{x+a+\sqrt{(x^2+2ax)}}\right]dx$$

$$=y^\nu 2^{1-\nu-\mu}a^{\mu-\nu-\lambda}\frac{\Gamma(2\mu)\Gamma(\lambda+\nu+1)\Gamma(\lambda+\nu-\mu)}{\Gamma\left(\nu+\frac{1+d}{2}\right)\Gamma(\lambda+\nu)\Gamma(1+\lambda+\nu+\mu)}$$

$$\times\left\{F^{4:0;4}_{4:1;3}\left[\begin{array}{l}\triangle\,(2;\lambda+\nu+1),\triangle\,(2;\lambda+\nu-\mu):-;\triangle\,(2;-n),\triangle\,(2;n);\\ \\ \triangle\,(2;\lambda+\nu),\triangle\,(2;1+\lambda+\nu+\mu):\nu+\frac{1+d}{2};\triangle\,(2;\frac{1}{2}),\frac{1}{2};\end{array}\right.\right.$$

$$\left.\frac{-cy^2}{4a^2},\frac{b^2y^2}{4a^2}\right]+\frac{byn^2(\lambda+\nu+1)(\lambda+\nu-\mu)}{a(\lambda+\nu)(1+\lambda+\nu+\mu)}$$

$$\times F^{4:0;4}_{4:1;3}\left[\begin{array}{l}\triangle\,(2;\lambda+\nu+2),\triangle\,(2;\lambda+\nu-\mu+1):-;\triangle\,(2;-n+1),\\ \\ \triangle\,(2;\lambda+\nu+1),\triangle\,(2;2+\lambda+\nu+\mu):\nu+\frac{1+d}{2};\triangle\,(2;\frac{3}{2}),\end{array}\right.$$

$$\left.\left.\begin{array}{c}\triangle\,(2;1+n);\\ \frac{3}{2};\end{array}\ \frac{-cy^2}{4a^2},\frac{b^2y^2}{4a^2}\right]\right\}, \tag{3.3}$$

*where $T_n(z)$ is the Tchebicheff polynomial of first kind (see [3], [7]).*

**Corollary 3.4.** *On taking $\beta=\alpha=\frac{1}{2}$ in (2.1) and then by using (1.9), the following integral formula (which is valid under the given conditions) holds true:*

$$\int_0^\infty x^{\mu-1}[x+a+\sqrt{(x^2+2ax)}]^{-\lambda}w^d_{\nu,c}\left[\frac{y}{x+a+\sqrt{(x^2+2ax)}}\right]$$

$$\times U_n\left[1-\frac{by}{x+a+\sqrt{(x^2+2ax)}}\right]dx$$

$$= y^{\nu}2^{1-\nu-\mu}a^{\mu-\nu-\lambda}\frac{(n+1)\Gamma(2\mu)\Gamma(\lambda+\nu+1)\Gamma(\lambda+\nu-\mu)}{\Gamma\left(\nu+\frac{1+d}{2}\right)\Gamma(\lambda+\nu)\Gamma(1+\lambda+\nu+\mu)}$$

$$\times\left\{F^{4:0;4}_{4:1;3}\left[\begin{array}{l}\triangle\,(2;\lambda+\nu+1),\triangle\,(2;\lambda+\nu-\mu):-;\triangle\,(2;-n),\triangle\,(2;2+n);\\ \triangle\,(2;\lambda+\nu),\triangle\,(2;1+\lambda+\nu+\mu):\nu+\frac{1+d}{2};\triangle\,(2;\frac{3}{2}),\frac{1}{2};\end{array}\right.\right.$$

$$\left.\frac{-cy^2}{4a^2},\frac{b^2y^2}{4a^2}\right]+\frac{byn(2+n)(\lambda+\nu+1)(\lambda+\nu-\mu)}{3a(\lambda+\nu)(1+\lambda+\nu+\mu)}$$

$$\times F^{4:0;4}_{4:1;3}\left[\begin{array}{l}\triangle\,(2;\lambda+\nu+2),\triangle\,(2;\lambda+\nu-\mu+1):-;\triangle\,(2;-n+1),\\ \triangle\,(2;\lambda+\nu+1),\triangle\,(2;2+\lambda+\nu+\mu):\nu+\frac{1+d}{2};\triangle\,(2;\frac{5}{2}),\end{array}\right.$$

$$\left.\left.\begin{array}{l}\triangle\,(2;3+n);\\ \frac{3}{2};\end{array}\frac{-cy^2}{4a^2},\frac{b^2y^2}{4a^2}\right]\right\}, \tag{3.4}$$

*where $U_n(z)$ is the Tchebicheff polynomial of seconr kind (see [3], [7]).*

**Corollary 3.5.** *Setting $\beta=\alpha=0$ in (2.1) and then by using (1.10), the following integral formula (which is valid under the given conditions) holds true:*

$$\int_0^{\infty}x^{\mu-1}[x+a+\sqrt{(x^2+2ax)}]^{-\lambda}w^{d}_{\nu,c}\left[\frac{y}{x+a+\sqrt{(x^2+2ax)}}\right]$$

$$\times P_n\left[1-\frac{by}{x+a+\sqrt{(x^2+2ax)}}\right]dx$$

$$= y^{\nu}2^{1-\nu-\mu}a^{\mu-\nu-\lambda}\frac{\Gamma(2\mu)\Gamma(\lambda+\nu+1)\Gamma(\lambda+\nu-\mu)}{\Gamma\left(\nu+\frac{1+d}{2}\right)\Gamma(\lambda+\nu)\Gamma(1+\lambda+\nu+\mu)}$$

$$\times\left\{F^{4:0;4}_{4:1;3}\left[\begin{array}{l}\triangle\,(2;\lambda+\nu+1),\triangle\,(2;\lambda+\nu-\mu):-;\triangle\,(2;-n),\triangle\,(2;1+n);\\ \triangle\,(2;\lambda+\nu),\triangle\,(2;1+\lambda+\nu+\mu):\nu+\frac{1+d}{2};\triangle\,(2;1),\frac{1}{2};\end{array}\right.\right.$$

$$\left.\frac{-cy^2}{4a^2},\frac{b^2y^2}{4a^2}\right]+\frac{byn(1+n)(\lambda+\nu+1)(\lambda+\nu-\mu)}{2a(\lambda+\nu)(1+\lambda+\nu+\mu)}$$

$$\times F^{4:0;4}_{4:1;3}\left[\begin{array}{l}\triangle\,(2;\lambda+\nu+2),\triangle\,(2;\lambda+\nu-\mu+1):-;\triangle\,(2;-n+1),\\ \triangle\,(2;\lambda+\nu+1),\triangle\,(2;2+\lambda+\nu+\mu):\nu+\frac{1+d}{2};\triangle\,(2;2),\end{array}\right.$$

$$\left.\left.\begin{array}{l}\triangle\,(2;2+n);\\ \frac{3}{2};\end{array}\frac{-cy^2}{4a^2},\frac{b^2y^2}{4a^2}\right]\right\}, \tag{3.5}$$

*where $P_n(z)$ is the Legendre polynomial (see [3], [7]).*

**Corollary 3.6.** *By taking $\nu = 1 - \frac{d}{2}$ and replacing c by $c^2$ in (2.1) and then by using (1.2), the following integral formula (which is valid under the given conditions) holds true:*

$$\int_0^\infty x^{\mu-1}[x+a+\sqrt{(x^2+2ax)}]^{-(\lambda-1/2)} \sin\left(\frac{yc}{(x+a+\sqrt{(x^2+2ax)})}\right) \times P_n^{(\alpha,\beta)}\left[1-\frac{by}{x+a+\sqrt{(x^2+2ax)}}\right]dx$$

$$= y^{1-\frac{d}{2}} 2^{\frac{d}{2}-\mu} a^{\mu-1+\frac{d}{2}-\lambda} \frac{(1+\alpha)_n \Gamma(2\mu)\Gamma(\lambda-\frac{d}{2}+2)\Gamma(\lambda+1-\frac{d}{2}-\mu)}{n!\,\Gamma\left(\frac{3}{2}\right)\Gamma(\lambda+1-\frac{d}{2})\Gamma(2+\lambda-\frac{d}{2}+\mu)}$$

$$\times\left\{F^{4:0;4}_{4:1;3}\left[\begin{array}{l}\triangle(2;\lambda-\frac{d}{2}+2), \triangle(2;\lambda+1-\frac{d}{2}-\mu) : -; \triangle(2;-n), \\ \triangle(2;\lambda+1-\frac{d}{2}), \triangle(2;1+\lambda+1-\frac{d}{2}+\mu) : \frac{3}{2}; \triangle(2;1+\alpha),\end{array}\right.\right.$$

$$\left.\begin{array}{l}\triangle(2;1+\alpha+\beta+n); \\ \frac{1}{2};\end{array} \frac{-c^2y^2}{4a^2}, \frac{b^2y^2}{4a^2}\right]$$

$$+\frac{byn(1+\alpha+\beta+n)(\lambda-\frac{d}{2}+2)(\lambda+1-\frac{d}{2}-\mu)}{2a(\lambda+1-\frac{d}{2})(\lambda-\frac{d}{2}+\mu+2)(1+\alpha)}$$

$$\times F^{4:0;4}_{4:1;3}\left[\begin{array}{l}\triangle(2;\lambda-\frac{d}{2}+3), \triangle(2;\lambda-\frac{d}{2}-\mu+2) : -; \triangle(2;-n+1), \\ \triangle(2;\lambda-\frac{d}{2}+2), \triangle(2;3+\lambda-\frac{d}{2}+\mu) : \frac{3}{2}; \triangle(2;2+\alpha),\end{array}\right.$$

$$\left.\left.\begin{array}{l}\triangle(2;2+\alpha+\beta+n); \\ \frac{3}{2};\end{array} \frac{-c^2y^2}{4a^2}, \frac{b^2y^2}{4a^2}\right]\right\}. \tag{3.6}$$

**Corollary 3.7.** *On setting $\nu = -\frac{d}{2}$ and replacing c by $c^2$ in (2.1) and then by using (1.3), the following integral formula (which is valid under the given conditions) holds true:*

$$\int_0^\infty x^{\mu-1}[x+a+\sqrt{(x^2+2ax)}]^{-(\lambda-1/2)} \cos\left(\frac{yc}{(x+a+\sqrt{(x^2+2ax)})}\right) \times P_n^{(\alpha,\beta)}\left[1-\frac{by}{x+a+\sqrt{(x^2+2ax)}}\right]dx$$

$$= y^{-\frac{d}{2}} 2^{1+\frac{d}{2}-\mu} a^{\mu+\frac{d}{2}-\lambda} \frac{(1+\alpha)_n \Gamma(2\mu)\Gamma(\lambda-\frac{d}{2}+1)\Gamma(\lambda-\frac{d}{2}-\mu)}{n!\,\Gamma\left(\frac{1}{2}\right)\Gamma(\lambda-\frac{d}{2})\Gamma(1+\lambda-\frac{d}{2}+\mu)}$$

$$\times\left\{F^{4:0;4}_{4:1;3}\left[\begin{array}{l}\triangle(2;\lambda-\frac{d}{2}+1), \triangle(2;\lambda-\frac{d}{2}-\mu) : -; \triangle(2;-n), \\ \triangle(2;\lambda+\nu), \triangle(2;1+\lambda+\nu+\mu) : \frac{1}{2}; \triangle(2;1+\alpha),\end{array}\right.\right.$$

$$\left.\begin{matrix} \triangle\,(2;1+\alpha+\beta+n); \\ \frac{1}{2}; \end{matrix} \frac{-c^2y^2}{4a^2},\frac{b^2y^2}{4a^2}\right]$$

$$+\frac{byn(1+\alpha+\beta+n)(\lambda-\frac{d}{2}+1)(\lambda-\frac{d}{2}-\mu)}{2a(\lambda-\frac{d}{2})(1+\lambda-\frac{d}{2}+\mu)(1+\alpha)}$$

$$\times F^{4:0;4}_{4:1;3}\left[\begin{matrix} \triangle\,(2;\lambda-\frac{d}{2}+2),\triangle\,(2;\lambda-\frac{d}{2}-\mu+1):-;\triangle\,(2;-n+1), \\ \triangle\,(2;\lambda+\nu+1),\triangle\,(2;2+\lambda+\nu+\mu):\frac{1}{2};\triangle\,(2;2+\alpha), \end{matrix}\right.$$

$$\left.\left.\begin{matrix} \triangle\,(2;2+\alpha+\beta+n); \\ \frac{3}{2}; \end{matrix} \frac{-c^2y^2}{4a^2},\frac{b^2y^2}{4a^2}\right]\right\}. \tag{3.7}$$

## 4. Relation between Kampé de Fériet and Srivastava-Daoust functions

Here, we establish the following interesting relation in between Kampé de Fériet and Srivastava and Daoust functions:

$$F^{4:0;4}_{4:1;3}\left[\begin{matrix} \triangle\,(2;\lambda+\nu+1),\triangle\,(2;\lambda+\nu-\mu):-;\triangle\,(2;-n),\triangle\,(2;1+\alpha+\beta+n); \\ \triangle\,(2;\lambda+\nu),\triangle\,(2;1+\lambda+\nu+\mu):\nu+\frac{1+d}{2};\triangle\,(2;1+\alpha),\frac{1}{2}; \end{matrix}\right.$$

$$\left.\frac{-cy^2}{4a^2},\frac{b^2y^2}{4a^2}\right]+\frac{byn(1+\alpha+\beta+n)(\lambda+\nu+1)(\lambda+\nu-\mu)}{2a(\lambda+\nu)(1+\lambda+\nu+\mu)(1+\alpha)}$$

$$\times F^{4:0;4}_{4:1;3}\left[\begin{matrix} \triangle\,(2;\lambda+\nu+2),\triangle\,(2;\lambda+\nu-\mu+1):-;\triangle\,(2;-n+1), \\ \triangle\,(2;\lambda+\nu+1),\triangle\,(2;2+\lambda+\nu+\mu):\nu+\frac{1+d}{2};\triangle\,(2;2+\alpha), \end{matrix}\right.$$

$$\left.\begin{matrix} \triangle\,(2;2+\alpha+\beta+n); \\ \frac{3}{2}; \end{matrix} \frac{-cy^2}{4a^2},\frac{b^2y^2}{4a^2}\right]$$

$$=\frac{n!}{(1+\alpha)_n}$$

$$\times F^{4:0;0}_{5:0;1}\left[\begin{matrix} (\lambda+\nu+1:2,3),(\lambda+\nu-\mu:2,3),(1+\alpha+\beta:1,2),(1+\alpha:1,1): \\ (\lambda+\nu:2,3),(1+\lambda+\nu+\mu:2,3),(\nu+\frac{1+d}{2}:1,1),(1+\alpha+\beta:1,1), \end{matrix}\right.$$

$$\left.\begin{matrix} -;-; \\ (1:1,1):-;(1+\alpha,1); \end{matrix} \frac{-cy^2}{4a^2},\frac{bcy^3}{8a^3}\right]. \tag{4.1}$$

**Concluding Remarks:** In this paper, we have evaluated two interesting integrals associated with the product of generalized Bessel function and Jacobi polynomial in the form of Theorem 2.1 and Theorem 2.2, whose explicit representations are given in terms of Kampé de Fériet function and Srivastava-Daoust function, respectively. By mean of Theorem 2.1, we have derived some (presumably) new integrals as its special cases. It is noticed that, in a similar manner we can obtain various other useful integrals with the help of Theorem 2.2.

Also, it is remarked that if we replace the integral operator (1.13) (which is used to establish the main results) by any other operator, then we get a number of new interesting results.

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