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FINITE DIFFERENCE SCHEME FOR SINGULARLY PERTURBED SYSTEM OF DELAY DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS

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ABSTRACT. In this paper we consider a class of singularly perturbed system of delay differential equations of convection diffusion type with integral boundary conditions. A finite difference scheme on an appropriate piecewise Shishkin type mesh is suggested to solve the problem. We prove that the method is of almost first order convergent. An error estimate is derived in the discrete maximum norm. Numerical experiments support our theoretical results.

1. INTRODUCTION

In natural or technological control problems, a controller monitors the state of the system, and makes adjustments to the system based on its observations. Since these adjustments can never be made instantaneously, a delay arises between the observation and the control action. This kind of systems are governed by differential equations with delay arguments. A subclass of these equations consists of singularly perturbed differential equations with delay are typically characterized by the presence of small positive parameter ε multiplying some or all of the highest derivatives present in the differential equation. These equations arise in mathematical models of biological science and engineering. Differential equations with integral boundary conditions have plenty of applications. A Parabolic equation with nonlocal boundary conditions arising from electro chemistry is well studied by Choi and Chan [7]. In [8], Day have discussed Parabolic equations and thermodynamics. Cannon [6] have worked for the solution of the heat equation subject to the specification of energy, etc. The authors of [4, 9, 14] have proved that the problem of differential equations with integral boundary conditions is well posed. The authors of [1, 5, 16, 20, 21] have developed various numerical schemes on uniform meshes for singularly perturbed first and second order differential equations with integral boundary conditions. The standard numerical methods used for solving singularly perturbed differential equation are some time ill posed and fail to give analytical solution when the perturbation parameter ε is small. Therefore, it is necessary to improve suitable numerical

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methods which are uniformly convergent to solve this type of differential equations. Many authors have worked on singularly perturbed differential equations with small and large delay using uniformly convergent numerical methods. In [13], Lange and Miura have discussed singularly perturbed linear second order differential-difference equations with small delay. In [10, 11, 12, 17, 18, 19] finite difference and finite element methods are proposed to solve these kind of equations with large and small shifts.

In the present paper, motivated by the works of [1, 2, 3], we analyze a fitted finite difference scheme on a piecewise uniform mesh for the numerical solution of a class of second order singularly perturbed system of delay differential equations with integral boundary conditions. The present paper is arranged as follows. Statement of the problem is given in section 2. In section 3, maximum principle, stability result and appropriate bounds for the derivatives of the solution of the problem are presented. Section 4 describes the numerical method. Error analysis for approximate solution is given section 5. Numerical results are given in section 6. Section 7 includes the conclusion part. Throughout our analysis we use the following notations: $\bar{\Omega} = [0,2]$, $\Omega = (0,2)$, $\Omega_1 = (0,1)$, $\Omega_2 = (1,2)$, $\Omega^* = \Omega_1 \cup \Omega_2$. $\bar{\Omega}^{2N} = \{0,1,2,...,2N\}$, $\Omega_1^{2N} = \{1,2,...,N-1\}$, $\Omega_2^{2N} = \{N+1,...,2N-1\}$. C, C_1 are generic positive constants that are independent of parameter ε and $2N$ mesh points. We assume that $\varepsilon \leq CN^{-1}$. The supremum norm used for studying the convergence of the numerical solution to the exact solution of a singular perturbation problem is $\|u\|_\Omega = \sup_{x\in\Omega} |u(x)|$.

2. STATEMENT OF THE PROBLEM

Find $\bar{u} = (u_1,\ u_2)^T,\ u_1,\ u_2 \in Y = C^0(\bar{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega_1 \cup \Omega_2)$ such that

$$\begin{cases} -\varepsilon u_1''(x) + a_1(x)u_1'(x) + b_{11}(x)u_1(x) + b_{12}(x)u_2(x) + c_{11}(x)u_1(x-1) \\ \qquad +c_{12}(x)u_2(x-1) = f_1(x),\ x \in \Omega_1 \cup \Omega_2, \\ -\varepsilon u_2''(x) + a_2(x)u_2'(x) + b_{21}(x)u_1(x) + b_{22}(x)u_2(x) + c_{21}(x)u_1(x-1) \\ \qquad +c_{22}(x)u_2(x-1) = f_2(x),\ x \in \Omega_1 \cup \Omega_2, \\ u_1(x) = \phi_1(x),\ x \in [-1,0],\ \mathcal{K}_1 u_1(2) = u_1(2) - \varepsilon \int_0^2 g_1(x)u_1(x)dx = l_1, \\ u_2(x) = \phi_2(x),\ x \in [-1,0],\ \mathcal{K}_2 u_2(2) = u_2(2) - \varepsilon \int_0^2 g_2(x)u_2(x)dx = l_2. \end{cases} \tag{2.1}$$

where $0 < \varepsilon << 1$ is a small positive parameter, the functions $a_1(x), a_2(x), b_{11}(x), b_{12}(x), b_{21}(x), b_{22}(x), c_{11}(x), c_{12}(x), c_{21}(x), c_{22}(x), f_1(x), f_2(x)$ are sufficiently smooth on $\bar{\Omega} = [0,2]$ and satisfy the following assumptions.

$$\begin{cases} a_1(x) \geq \alpha_1 > 0,\ a_2(x) \geq \alpha_2 > 0,\ 0 < \alpha < \min\{\alpha_1, \alpha_2\},\ x \in \bar{\Omega} \\ b_{11}(x) \geq 0,\ b_{12}(x) \leq 0,\ b_{21}(x) \leq 0,\ b_{22}(x) \geq 0, \\ b_{11}(x) + b_{12}(x) \geq \beta_1 \geq 0,\ b_{21}(x) + b_{22}(x) \geq \beta_2 \geq 0, \\ c_{11}(x) \leq 0,\ c_{12}(x) \leq 0,\ c_{21}(x) \leq 0,\ c_{22}(x) \geq 0, \\ c_{11}(x) + c_{12}(x) \geq \gamma_1 \geq 0,\ c_{21}(x) + c_{22}(x) \geq \gamma_2 \geq 0, \\ g_i \text{ are nonnegative and } 1 - \int_0^2 g_i(x)dx > 0,\ i = 1,2. \end{cases}$$

The problem (2.1) can be rewritten as,

$$\mathcal{L}_1\bar{u}(x):=\begin{cases}-\varepsilon u_1''(x)+a_1(x)u_1'(x)+b_{11}(x)u_1(x)+b_{12}(x)u_2(x)\\ \qquad\qquad =f_1(x)-c_{11}(x)\phi_1(x-1)-c_{12}(x)\phi_2(x-1),\ x\in\Omega_1,\\ -\varepsilon u_1''(x)+a_1(x)u_1'(x)+b_{11}(x)u_1(x)+b_{12}(x)u_2(x)+c_{11}(x)u_1(x-1)\\ \qquad\qquad +c_{12}(x)u_2(x-1)=f_1(x), \qquad\qquad x\in\Omega_2,\end{cases} \tag{2.2}$$

$$\mathcal{L}_2\bar{u}(x):=\begin{cases}-\varepsilon u_2''(x)+a_2(x)u_2'(x)+b_{21}(x)u_1(x)+b_{22}(x)u_2(x)\\ \qquad\qquad =f_2(x)-c_{21}(x)\phi_1(x-1)-c_{22}(x)\phi_2(x-1),\ x\in\Omega_1,\\ -\varepsilon u_2''(x)+a_2(x)u_2'(x)+b_{21}(x)u_1(x)+b_{22}(x)u_2(x)+c_{21}(x)u_1(x-1)\\ \qquad\qquad +c_{22}(x)u_2(x-1)=f_2(x), \qquad\qquad x\in\Omega_2,\end{cases} \tag{2.3}$$

$$\begin{cases}u_1(0)=\phi_1(0),\ u_1(1-)=u_1(1+),\ u_1'(1-)=u_1'(1+),\\ \mathcal{K}_1u_1(2)=u_1(2)-\varepsilon\int_0^2 g_1(x)u_1(x)dx=l_1,\\ u_2(0)=\phi_2(0),\ u_2(1-)=u_2(1+),\ u_2'(1-)=u_2'(1+),\\ \mathcal{K}_2u_2(2)=u_2(2)-\varepsilon\int_0^2 g_2(x)u_2(x)dx=l_2,\end{cases} \tag{2.4}$$

3. THE CONTINUOUS PROBLEM

Theorem 3.1. *(Maximum Principle) Let* $\bar{w}=(w_1,\ w_2)^T,\ w_1,\ w_2\in C^0(\bar{\Omega})\cap C^2(\Omega_1\cup\Omega_2)$ *be any function satisfying* $w_i(0)\geq 0,\ i=1,2,\ \mathcal{K}_iw_i(2)\geq 0,\ i=1,2,\ \mathcal{L}_i\bar{w}(x)\geq 0,\ \forall x\in\Omega_1\cup\Omega_2,\ i=1,2$ *and* $w_i'(1+)-w_i'(1-)=[w_i'](1)\leq 0,\ i=1,2.$ *Then* $w_i(x)\geq 0,\ \forall x\in\bar{\Omega},\ i=1,2.$

Proof. Let $\bar{s}=(s_1,\ s_2)^T$ be a function defined by

$$s_i(x)=\begin{cases}\frac{1}{8}+\frac{x}{2}, & x\in[0,1]\\ \frac{3}{8}+\frac{x}{4}, & x\in[1,2].\end{cases}\quad i=1,2. \tag{3.1}$$

It is easy to see that, $s_i(x)>0,\forall x\in\bar{\Omega},\ i=1,2,\ \mathcal{L}_i\bar{s}_i(x)>0,\forall x\in\Omega_1\cup\Omega_2,\ i=1,2$ and $[s_i'](1)<0,\ i=1,2$. Let

$$\mu=\max\Big\{\max_{\bar{\Omega}}\{\frac{-w_1(x)}{s_1(x)}\},\ \max_{\bar{\Omega}}\{\frac{-w_2(x)}{s_2(x)}\}\Big\}.$$

Then there exists at least one point $x_0\in\bar{\Omega}$ such that $w_1(x_0)+\mu s_1(x_0)=0$ or $w_2(x_0)+\mu s_2(x_0)=0$ or both and $w_i(x)+\mu s_i(x)\geq 0,\ \forall x\in\bar{\Omega},\ i=1,2$. With out loss of generality we assume that $w_1(x_0)+\mu s_1(x_0)=0$. Therefore the function $(w_1+\mu s_1)$ attains its minimum at $x=x_0$. Suppose the theorem does not hold true, then $\mu>0$.
Case(i): $x_0\in\Omega_1$

$$\begin{aligned}0<\mathcal{L}_1(\bar{w}+\mu\bar{s})(x_0)=&-\varepsilon(w_1+\mu s_1)''(x_0)+a_1(x_0)(w_1+\mu s_1)'(x_0)\\ &+b_{11}(x_0)(w_1+\mu s_1)(x_0)+b_{12}(x_0)(w_2+\mu s_2)(x_0)\leq 0.\end{aligned}$$

Case (ii): $x_0 = 1$

$$0 \leq [(\bar{w} + \mu\bar{s})'](1) = [\bar{w}'](1) + \mu[\bar{s}'](1) < 0$$

Case(iii): $x_0 \in \Omega_2$

$$\begin{aligned} 0 < \mathcal{L}_1(\bar{w} + \mu\bar{s})(x_0) &= -\varepsilon(w_1 + \mu s_1)''(x_0) + a_1(x_0)(w_1 + \mu s_1)'(x_0) \\ &+ b_{11}(x_0)(w_1 + \mu s_1)(x_0) + b_{12}(x_0)(w_2 + \mu s_2)(x_0) \\ &+ c_{11}(x_0)(w_1 + \mu s_1)(x_0 - 1) + c_{12}(x_0)(w_2 + \mu s_2)(x_0 - 1) \leq 0. \end{aligned}$$

Case (iv): $x_0 = 2$

$$0 < \mathcal{K}_1(\bar{w} + \mu\bar{s})(2) = (w_1 + \mu s_1)(2) - \varepsilon \int_0^2 g_1(x)(w_1 + \mu s_1)(x)dx \leq 0.$$

Observe that in all the four cases we arrived a contradiction. Therefore $\mu > 0$ is not possible. This shows that $w_1(x) \geq 0$. Similarly $w_2(x) \geq 0$. □

Corollary 3.2. *(Stability Result) Let $\bar{u} = (u_1,\ u_2)^T, u_1,\ u_2 \in Y$ be any function. Then,*

$$| u_i(x) | \leq C\ \max\left\{ \max_{j=1,2}\{| u_j(0) |\},\ \max_{j=1,2}\{| \mathcal{K}_j u_j(2) |\},\ \max_{j=1,2}\{ \sup_{x \in \Omega_1 \cup \Omega_2} | \mathcal{L}_j \bar{u}(x) |\} \right\},$$
$$\forall\, x \in \bar{\Omega},\ i = 1, 2.$$

Proof. Define $\bar{\psi}^{\pm}(x) = (\psi_1^{\pm}(x),\ \psi_2^{\pm}(x))^T,\ x \in \bar{\Omega}$, where

$$\psi_i^{\pm}(x) = CMs_i(x) \pm u_i(x),\ x \in \bar{\Omega},\ i = 1, 2,$$

$$M = \max\{\max_{j=1,2}\{| u_j(0) |\},\ \max_{j=1,2}\{| \mathcal{K}_j u_j(2) |\},\ \max_{j=1,2}\{ \sup_{x \in \Omega_1 \cup \Omega_2} | \mathcal{L}_j \bar{u}(x) |\}\}$$

and $\bar{s}$ is defined by (3.1). Using the above barrier functions $\psi^{\pm}(x)$ and Theorem 3.1, this corollary can be proved easily. □

Bounds for the derivatives of $\bar{u}(x)$ are given in the following lemma.

Lemma 3.3. *Let $\bar{u}(x)$ be the solution of* (1). *Then, for* $1 \leq k \leq 3$,

$$|u_j^{(k)}(x)| \leq C\varepsilon^{-k},\ j = 1, 2.$$

Proof. Using Corollary 3.2 and applying arguments as given in [15] this lemma gets proved. □

The uniform error estimates can be derived using the sharper bounds on the derivatives of the solution $\bar{u}(x)$. To get sharper bounds we write the analytical solution $\bar{u}(x)$ in the form $\bar{u}(x) = \bar{v}(x) + \bar{w}(x)$, where $\bar{v}(x) = (v_1(x), v_2(x))^T$ and $\bar{w}(x) = (w_1(x), w_2(x))^T$. The regular component $\bar{v}(x)$ can be written as $\bar{v}(x) = \bar{v}_0(x) + \varepsilon\bar{v}_1(x) + \varepsilon^2\bar{v}_2(x)$, where $\bar{v}_0 = (v_{01}, v_{02})^T$, $\bar{v}_1 = (v_{11}, v_{12})^T$ and $\bar{v}_2 = (v_{21}, v_{22})^T$ satisfy the following equations:

$$\begin{cases} a_1(x)v'_{01}(x) + b_{11}(x)v_{01}(x) + b_{12}(x)v_{02}(x) + c_{11}(x)v_{01}(x-1) \\ \qquad\qquad +c_{12}(x)v_{02}(x-1) = f_1(x), \\ a_2(x)v'_{02}(x) + b_{21}(x)v_{01}(x) + b_{22}(x)v_{02}(x) + c_{21}(x)v_{01}(x-1) \\ \qquad\qquad +c_{22}(x)v_{02}(x-1) = f_2(x), \\ v_{01}(x) = u_1(x),\ v_{02}(x) = u_2(x),\ x \in [-1, 0] \end{cases} \tag{3.2}$$

$$\begin{cases} a_1(x)v'_{11}(x) + b_{11}(x)v_{11}(x) + b_{12}(x)v_{12}(x) + c_{11}(x)v_{11}(x-1) \\ \qquad\qquad +c_{12}(x)v_{12}(x-1) = f_1(x), \\ a_2(x)v'_{12}(x) + b_{21}(x)v_{11}(x) + b_{22}(x)v_{12}(x) + c_{21}(x)v_{11}(x-1) \\ \qquad\qquad +c_{22}(x)v_{12}(x-1) = f_2(x), \\ v_{11}(x) = u_1(x),\ v_{12}(x) = u_2(x),\ x \in [-1, 0] \end{cases} \tag{3.3}$$

$$\begin{cases} \mathcal{L}_1 v_2(x) = v''_{11}(x),\ v_{21}(x) = 0,\ x \in [-1, 0],\ \mathcal{K}_1 v_{21}(2) = 0, \\ \mathcal{L}_2 v_2(x) = v''_{12}(x),\ v_{22}(x) = 0,\ x \in [-1, 0],\ \mathcal{K}_2 v_{22}(2) = 0. \end{cases} \tag{3.4}$$

Thus the regular component $\bar{v}(x)$ is the solution of

$$\begin{cases} \mathcal{L}_1 v(x) = f_1(x),\ v_1(x) = u_1(x),\ x \in [-1, 0], \\ \mathcal{K}_1 v_1(2) = \mathcal{K}_1 v_{01}(2) + \varepsilon\mathcal{K}_1 v_{11}(2) + \varepsilon^2\mathcal{K}_1 v_{21}(2). \\ \mathcal{L}_2 v(x) = f_2(x),\ v_2(x) = u_2(x),\ x \in [-1, 0], \\ \mathcal{K}_2 v_2(2) = \mathcal{K}_2 v_{02}(2) + \varepsilon\mathcal{K}_2 v_{12}(2) + \varepsilon^2\mathcal{K}_2 v_{22}(2). \\ \bar{v}(1) = \bar{v}_0(1) + \varepsilon\bar{v}_1(1) + \varepsilon^2\bar{v}_2(1). \end{cases} \tag{3.5}$$

and $\bar{w}(x)$ is the solution of

$$\begin{cases} \mathcal{L}_1 w_1(x) = 0,\ w_1(x) = 0,\ x \in [-1, 0],\ \mathcal{K}_1 w_1(2) = \mathcal{K}_1 u_1(2) - \mathcal{K}_1 v_1(2). \\ \mathcal{L}_2 w_2(x) = 0,\ w_2(x) = 0,\ x \in [-1, 0],\ \mathcal{K}_2 w_2(2) = \mathcal{K}_2 u_2(2) - \mathcal{K}_2 v_2(2). \\ [\bar{w}'](1) = -[\bar{v}'](1) \end{cases} \tag{3.6}$$

We further decompose $\bar{w}(x)$ as $\bar{w}(x) = \bar{w}_B(x) + \bar{w}_I(x)$,where the function $\bar{w}_B(x)$ is boundary layer component and $\bar{w}_I(x)$ is interior layer component, which are the solution of the following problems respectively:
Find $\bar{w}_B(x) \in X$ such that

$$\begin{cases} \mathcal{L}_1 w_{B1}(x) = 0,\ w_{B1}(x) = 0,\ x \in [-1, 0],\ \mathcal{K}_1 w_{B1}(2) = \mathcal{K}_1 u_1(2) - \mathcal{K}_1 v_1(2). \\ \mathcal{L}_2 w_{B2}(x) = 0,\ w_{B2}(x) = 0,\ x \in [-1, 0],\ \mathcal{K}_2 w_{B2}(2) = \mathcal{K}_2 u_2(2) - \mathcal{K}_2 v_2(2). \end{cases} \tag{3.7}$$

Find $w_I(x) \in C^0(\overline{\Omega}) \cap C^2(\Omega^*)$ such that

$$\begin{cases} \mathcal{L}_1 w_{I1}(x) = 0,\ w_{1I}(x) = 0,\ x \in [-1, 0],\ [w'_{I1}](1) = -[v'](1),\ \mathcal{K} w_{I1}(2) = 0. \\ \mathcal{L}_2 w_{I2}(x) = 0,\ w_{I2}(x) = 0,\ x \in [-1, 0],\ [w'_{I2}](1) = -[v'](1),\ \mathcal{K} w_{I2}(2) = 0. \end{cases} \tag{3.8}$$

Theorem 3.4. *Let $\bar{u}(x)$ be the solution of the problem* (1) *and $\bar{v}_0(x)$ be its reduced problem solution defined in* (6)*. Then*

$$|u_j(x) - v_{0j}(x)| \le C(\varepsilon + e^{-\alpha(2-x)/\varepsilon}),\ x \in \bar{\Omega},\ j = 1, 2.$$

Proof. Consider the barrier functions $\bar{\psi}^{\pm}(x) = (\psi_1^{\pm}(x), \psi_2^{\pm}(x))^T$, where

$$\psi_j^{\pm}(x) = C(\varepsilon s_j(x) + e^{-\alpha(2-x)/\varepsilon}) \pm (u_j(x) - v_{0j}(x)),\ x \in \bar{\Omega}\ j = 1, 2.$$

Note that $\psi_j^{\pm}(x) \in C^0(\bar{\Omega}) \cap C^2(\Omega)$. It is easy to see that, $\psi_1^{\pm}(0) \ge 0$ for a suitable choice of $C > 0$. Further

$$\begin{aligned} \mathcal{K}_1 \psi_1^{\pm}(2) &= \psi_1^{\pm}(2) - \varepsilon \int_0^2 g_1(x) \psi_1^{\pm}(x) dx \\ &\ge C(2\varepsilon + 1) - 2C\varepsilon \int_0^2 g_1(x) dx - C\varepsilon \int_0^2 g_1(x) dx \pm \mathcal{K}_1(u_1 - v_{01})(2) \ge 0 \end{aligned}$$

for a suitable choice of $C > 0$.

Let $x \in (0, 1)$. Then

$$\begin{aligned} \mathcal{L}_1 \bar{\psi}^{\pm}(x) &= C\varepsilon[a_1(x) s'_1(x) + b_{11}(x) s_1(x) + b_{12}(x) s_2(x)] \\ &\quad + C[\frac{\alpha}{\varepsilon}(a_1(x) - \alpha) + b_{11}(x) + b_{12}(x)] e^{-\alpha(1-x)/\varepsilon} \\ &\quad \pm \mathcal{L}_1(\bar{u} - \bar{v}_{01})(x) \ge 0, \end{aligned}$$

by a proper choice of $C > 0$. Let $x \in \Omega_2$. Then

$$\begin{aligned} \mathcal{L}_1 \bar{\psi}^{\pm}(x) = C\Bigg[&(\frac{\alpha}{\varepsilon}(a_1(x) - \alpha) + b_{11}(x) s_1(x) + b_{12}(x) s_2(x) + (c_{11}(x) s_1(x) + c_{12}(x) s_2(x)) \\ &\exp(-\frac{\alpha}{\varepsilon})) \exp(\frac{-\alpha(2-x)}{\varepsilon}) + \varepsilon(a_1(x) + b_{11}(x) s_1(x) + b_{12}(x) s_2(x) \\ &+ (c_{11}(x) s_1(x-1) + c_{12}(x) s_2(x-1))\Bigg] \pm \varepsilon v''_0(x), \\ \ge 0. & \end{aligned}$$

for a suitable choice of $C > 0$.

Similarly one can prove that $\mathcal{L}_2 \bar{\psi}^{\pm}(x) \ge 0$ and $\mathcal{K}_2 \psi_2^{\pm}(2) \ge 0$. Then by maximum principle we have $\psi_i^{\pm}(x) \ge 0,\ x \in \bar{\Omega},\ i = 1, 2.$ □

Lemma 3.5. *The regular component $\bar{v}(x)$ and the singular component $\bar{w}(x)$ of the solution $\bar{u}(x)$ satisfy the following bounds.*

$$\|v_j^k(x)\|_{\Omega^*} \le C(1 + \varepsilon^{2-k}),\ \ for\ k = 0, 1, 2, 3 \tag{3.9}$$

$$|w_{Bj}^k(x)| \leq C\varepsilon^{-k}\exp(\frac{-\alpha(2-x)}{\varepsilon}),\ x \in \Omega^*,\ k = 0,1,2,3 \tag{3.10}$$

$$|w_{Ij}^k(x)| \leq C\begin{cases}\varepsilon^{1-k}\exp(\frac{-\alpha(1-x)}{\varepsilon}), & x \in \Omega_1,\\ \varepsilon^{1-k}, & x \in \Omega_2,\end{cases}\ k = 0,1,2,3, \tag{3.11}$$

where $j = 1, 2$

Proof. Integrating (3.2) and (3.4) and using the stability result, the inequalities (3.9) can be proved easily. To prove the inequalities (3.11), consider the barrier functions

$$\Phi_j^{\pm}(x) = C_1(\exp(\frac{-\alpha(2-x)}{\varepsilon})) \pm w_{Bj}(x),\ x \in \bar{\Omega}.\ j = 1,2$$

It is easy to see that $\Phi_1^{\pm}(0) \geq 0$.

Further,

$$\begin{aligned}\mathcal{K}_1\Phi_1^{\pm}(2) &= \Phi_1^{\pm}(2) - \varepsilon\int_0^2 g_1(x)\Phi_1^{\pm}(x)dx\\ &= C[1-\varepsilon\int_0^2 g_1(x)\exp(\frac{-\alpha(2-x)}{\varepsilon})dx] \pm \mathcal{K}_1 w_{B1}(2)\\ &\geq 0\end{aligned}$$

Also

$$\begin{aligned}\mathcal{L}\Phi^{\pm}(x) &= C_1\left[\frac{\alpha}{\varepsilon}(a_1(x)-\alpha) + b_{11}(x) + b_{12}(x) + (c_{11}(x)+c_{12}(x))\exp(-\frac{\alpha}{\varepsilon})\right]\\ &\quad \exp(\frac{-\alpha(2-x)}{\varepsilon}) \pm\ \mathcal{L}w_{B1}(x)\\ &\geq C_1\left[\frac{\alpha}{\varepsilon}(\alpha_1-\alpha) + \beta + \gamma\exp(-\frac{\alpha}{\varepsilon})\right]\exp(\frac{-\alpha(2-x)}{\varepsilon}) \pm\ 0\\ &\geq 0.\end{aligned}$$

By the Theorem 3.1,

$$|w_{B1}(x)| \leq C\exp(-\alpha(2-x)/\varepsilon).$$

Integration of (3.7) yields the estimates of $|w'_{B1}(x)|$. From the differential equations (3.6), one can derive the rest of the derivative estimates (3.11). Similarly it is easy to prove w_{B2} is bounded. To prove the inequalities (3.11), consider the barrier functions

$$\Phi_j^{\pm}(x) = C_1\varepsilon(\exp(-\alpha(1-x)/\varepsilon)) \pm w_{Ij}(x),\ x \in [0,1],\ j = 1,2.$$

Clearly, $\Phi^{\pm}(0) \geq 0$ and also $\mathcal{L}_j\Phi(x_i) \geq 0$, $j = 1, 2$ easily proves the first inequality.

Similarly, consider the following barrier functions

$$\Phi_j^{\pm}(x) = C_1 x\varepsilon \pm w_{Ij}(x),\ x \in [1,2].$$

Note that

$$\mathcal{K}_j\Phi^{\pm}(2) = \Phi_j^{\pm}(2) - \varepsilon\int_0^2 g_j(x)\Phi_j^{\pm}(x)dx$$

$$\begin{aligned} &= C\varepsilon[2-\varepsilon\int_0^2 xg_j(x)dx] \pm \mathcal{K}_j w_{Ij}(2) \\ &= 2C\varepsilon[1-\varepsilon\int_0^2 g_j(x)dx] \pm\ 0 \\ &\geq 0 \end{aligned}$$

$$\mathcal{L}_j\Phi_j^{\pm}(x) = -\varepsilon(\Phi_j^{\pm})''(x) + a(x)(\Phi_j^{\pm})' + b(x)\Phi_j^{\pm}(x) + c(x)\Phi_j^{\pm}(x-1) \geq 0$$

Hence the proof. □

Note: From the above lemma, it is not difficult to prove

$$|u_j(x) - v_j(x)| \leq C \begin{cases} \varepsilon\ \exp(\frac{-\alpha(1-x)}{\varepsilon}) + \exp(\frac{-\alpha(2-x)}{\varepsilon}), & x \in \Omega_1 \\ \varepsilon + \exp(\frac{-\alpha(2-x)}{\varepsilon}), & x \in \Omega_2. \end{cases} \text{where} j = 1, 2. \quad (3.12)$$

4. THE DISCRETE PROBLEM

The BVP (2.1) exhibits strong boundary layer at $x = 2$ and interior layer at $x = 1$.

The interval $[0, 1]$ is partitioned into $[0, 1-\sigma]$ and $[1-\sigma, 1]$ and the interval $[1, 2]$ is partitioned as $[1, 2-\sigma]$ and $[2-\sigma, 2]$, where σ is transition parameter for this mesh defined by

$$\sigma = \min\{\frac{1}{2}, 2\frac{\varepsilon}{\alpha}\ln N\}.$$

The mesh $\bar{\Omega}^{2N} = \{x_0, x_1, \cdots\cdots, x_{2N}\}$ is defined by

$x_0 = 0$, $x_i = x_0 + iH$, $i = 1$ to $\frac{N}{2}$, $x_{i+\frac{N}{2}} = x_{\frac{N}{2}} + ih$, $i = 1$ to $\frac{N}{2}, x_{i+N} = x_N + iH$, $i = 1$ to $\frac{N}{2}$, $x_{i+\frac{3N}{2}} = x_{\frac{3N}{2}} + ih$, $i = 1$ to $\frac{N}{2}$ where $h = \frac{2\sigma}{N}$, $H = \frac{2(1-\sigma)}{N}$.

The discrete problem corresponding to (2.2)-(2.4) is: Find $\bar{U}(x_i) = (U_1(x_1), U_2(x_2))^T$ such that

$$\begin{cases} \mathcal{L}_1^N\bar{U}(x_i) = -\varepsilon\delta^2 U_1(x_i) + a_1(x_i)D^- U_1(x_i) + b_{11}(x_i)U_1(x_i) \\ +b_{12}(x_i)U_2(x_i) + c_{11}(x_i)U_1(x_{i-N}) + c_{12}(x_i)U_2(x_{i-N}) = f_1(x_i),\ \forall x_i \in \Omega^{2N} \\ \mathcal{L}_2^N\bar{U}(x_i) = -\varepsilon\delta^2 U_2(x_i) + a_2(x_i)D^- U_2(x_i) + b_{21}(x_i)U_1(x_i) \\ +b_{22}(x_i)U_2(x_i) + c_{21}(x_i)U_1(x_{i-N}) + c_{22}(x_i)U_2(x_{i-N}) = f_2(x_i),\ \forall x_i \in \Omega^{2N}. \end{cases} \quad (4.1)$$

$$\begin{cases} U_j(x_i) = \phi_j(x_i),\ i = -N, -N+1, ..., 0, \\ \mathcal{K}_j^N U_j(x_N) = U_j(x_N) - \varepsilon\sum_{i=1}^{2N} \frac{g_j(x_{i-1})U_j(x_{i-1}) + g_j(x_i)U_j(x_i)}{2} h_i = l_j, \forall x_i \in \Omega^{2N} \\ D^- U_j(x_N) = D^+ U_j(x_N), \end{cases} \quad (4.2)$$

where

$$\begin{aligned} \delta^2 U_j(x_i) &= \frac{2}{h_{i+1} + h_i}\left(\frac{U_j(x_{i+1}) - U_j(x_i)}{h_{i+1}} - \frac{U_j(x_i) - U_j(x_{i-1})}{h_i}\right), \\ D^- U_j(x_i) &= \frac{U_j(x_i) - U_j(x_{i-1})}{h_{i-1}},\ j = 1, 2 \end{aligned}$$

Theorem 4.1. ***(Discrete Maximum Principle)*** *Let* $\bar{\Psi}(x_i) = (\Psi_1(x_i), \Psi_2(x_i))^T$ *be the mesh function satisfying* $\Psi_1(x_0) \geq 0, \Psi_2(x_0) \geq 0, \mathcal{K}_1^N \Psi_1(x_{2N}) \geq 0, \mathcal{K}_2^N \Psi_2(x_{2N}) \geq 0, \mathcal{L}_1^N \bar{\Psi}(x_i) \geq 0, \mathcal{L}_2^N \bar{\Psi}(x_i) \geq 0$ *and* $[D]U_j(x_N)) \leq 0,\ j = 1, 2$. *Then* $\bar{\Psi}(x_i) \geq 0$, $x_i \in \bar{\Omega}^{2N}$.

Proof. Define $\bar{S}_j(x_i) = (S_1(x_i), S_2(x_i))^T$,

$$S_j(x_i) = \begin{cases} \frac{1}{8} + \frac{x_i}{2}, & x_i \in [0,1] \cap \bar{\Omega}^{2N}, \\ \frac{3}{8} + \frac{x_i}{4}, & x_i \in [1,2] \cap \bar{\Omega}^{2N}, \end{cases} \quad j = 1, 2.$$

Note that $\bar{S}_j(x_i) > 0, \forall x_i \in \bar{\Omega}^{2N}$, $\mathcal{K}_1^N S_1(x_{2N}) > 0, \mathcal{K}_2^N S_2(x_{2N}) > 0, \mathcal{L}_1^N \bar{S}_1(x_i) > 0$ and $\mathcal{L}^N \bar{S}_2(x_i) > 0, \forall x_i \in \Omega^{2N}$. Let

$$\mu = \max\left\{ \max_{x_i \in \bar{\Omega}^{2N}} \left(\frac{-\Psi_1(x_i)}{S_1(x_i)} \right), \max_{x_i \in \bar{\Omega}^{2N}} \left(\frac{-\Psi_2(x_i)}{S_2(x_i)} \right) \right\}.$$

Then there exists one $x_k \in \bar{\Omega}^{2N}$ such that $\Psi_1(x_k) + \mu S_1(x_k) = 0$ or $\Psi_2(x_k) + \mu S_2(x_k) = 0$ or both. We have $\Psi_j(x_i) + \mu S_j(x_i) \geq 0, x_i \in \bar{\Omega}^{2N}, j = 1, 2$. Therefore either $(\Psi_1 + \mu S_1)$ or $(\Psi_2 + \mu S_2)$ attains minimum at $x_i = x_k$. Suppose the theorem does not hold true, then $\mu > 0$.
Case (i): $x_k = x_0$

$$0 < (\Psi_j + \mu S_j)(x_0) = 0$$

It is a contradiction.
Case (ii): $x_k \in \Omega_1^{2N}$

$$0 < \mathcal{L}_j^N (\Psi_j + \mu S_j)(x_k) \leq 0,\ j = 1, 2.$$

It is a contradiction.
Case (iii): $x_k = x_N$

$$0 \leq [D_j(\Psi_j + \mu S_j)](x_N) < 0,\ j = 1, 2.$$

It is a contradiction.
Case (iv): $x_k \in \Omega_2^{2N}$

$$0 < \mathcal{L}_j^N (\Psi_j + \mu S_j)(x_k) \leq 0,\ j = 1, 2.$$

It is a contradiction.
Case (v): $x_k = x_{2N}$

$$0 < \mathcal{K}_j^N (\Psi_j + \mu S_j) x_{2N} \leq 0,\ j = 1, 2$$

It is a contradiction. Hence the proof of the theorem. □

Lemma 4.2. ***(Discrete Stability Result)*** *Let* $\bar{U}(x_i) = (U_1(x_i), U_2(x_i))^T$ *be any mesh function. Then*

$$\mid U_k(x) \mid \leq C \ \max\Big\{ \max_{j=1,2} \{\mid U_j(0) \mid\},\ \max_{j=1,2} \{\mid \mathcal{K}_j U_j(2) \mid\},\ \max_{j=1,2} \{ \sup_{x_i \in \Omega_1 \cup \Omega_2} \mid \mathcal{L}_j \bar{U}(x_i) \mid \} \Big\},$$
$$\forall\, x_i \in \bar{\Omega}^{2N},\ k = 1, 2.$$

Proof. By choosing suitable barrier functions and using Theorem 4.1, one can establish the above inequality. □

Analogous to the continuous case, the discrete solution $\bar{U}(x_i)$ can be decomposed as

$$\bar{U}(x_i) = \bar{V}(x_i) + \bar{W}(x_i),$$

where $\bar{V}(x_i)$ and $\bar{W}(x_i)$ are respectively, the solutions of the problems:

$$\begin{cases} \mathcal{L}_1^N V_1(x_i) = f_1(x_i),\ x_i \in \Omega^{2N},\ V_1(x_0) = v_1(0), \\ \qquad [D]V_1(x_N) = [v_1'](1),\ \mathcal{K}_1^N V_1(x_{2N}) = \mathcal{K}_1 v_1(2) \\ \mathcal{L}_2^N V_2(x_i) = f_2(x_i),\ x_i \in \Omega^{2N},\ V_2(x_0) = v_2(0), \\ \qquad [D]V_2(x_N) = [v_2'](1),\ \mathcal{K}_2^N V_2(x_{2N}) = \mathcal{K}_2 v_2(2). \\ [D]\bar{V}(x_N) = [\bar{v}'](1) \end{cases} \tag{4.3}$$

$$\begin{cases} \mathcal{L}_1^N W_1 = 0,\ x_i \in \Omega_\epsilon^{2N},\ W_1(x_0) = w_1(0), \\ \qquad [D]W_1(x_N) = -[D]V_1(x_N),\ \mathcal{K}_1^N W_1(x_{2N}) = \mathcal{K}_1 w_1(2) \\ \mathcal{L}_2^N W_2 = 0,\ x_i \in \Omega_\epsilon^{2N},\ W_2(x_0) = w_2(0), \\ \qquad [D]W_2(x_N) = -[D]V_2(x_N),\ \mathcal{K}_2^N W_2(x_{2N}) = \mathcal{K}_1 w_1(2). \\ [D]\bar{W}(x_N) = -[D]\bar{V}(x_N) \end{cases} \tag{4.4}$$

The following theorem gives an estimate for the difference of the solutions of $(4.1) - (4.2)$ and (4.3).

Theorem 4.3. *Let $\bar{U}(x_i)$ be a numerical solution of* $(2.2) - (2.4)$ *defined by* $(4.1) - (4.2)$ *and $\bar{V}(x_i)$ be a numerical solution of* (3.5) *defined by* (4.3). *Then*

$$|\bar{U}_j(x_i) - \bar{V}_j(x_i)| \le C \begin{cases} N^{-1},\ i = 0, 1, \cdots, \frac{3N}{2} \\ N^{-1} + |l_j - \mathcal{K}_j^N V_j(X_{2N})|\ i = \frac{3N}{2} + 1, \cdots, 2N. \end{cases} \quad j = 1, 2.$$

Proof. Consider a mesh function $\bar{\Psi}^{\pm}(x_i) = (\Psi_1^{\pm}(x_i), \Psi_2^{\pm}(x_i))^T$, where

$$\Psi_j^{\pm}(x_i) = CN^{-1}S_j(x_i) + Cx_i\varphi(x_i) \pm (U_j(x_i) - V_j(x_i)),\ x_i \in \bar{\Omega}^{2N},$$

$$\varphi(x_i) = \begin{cases} 0,\ i = 0, 1, \cdots, \frac{3N}{2} \\ |l_j - \mathcal{K}_j^N V_j(X_{2N})|\ i = \frac{3N}{2} + 1, \cdots, 2N \end{cases} \quad j = 1, 2.$$

It is clear that $\Psi^{\pm}(x_0) \ge 0$ and $\mathcal{K}\Psi^{\pm}(x_{2N}) \ge 0$.

If $\forall\ x_i \in \Omega_1^{2N}$

$$\mathcal{L}_j^N \Phi^{\pm}(x_i) \ge 0,\ j = 1, 2$$

If $\forall\ x_i \in \Omega_2^{2N}$

$$\mathcal{L}_j^N \Psi^{\pm}(x_i) \ge 0,\ j = 1, 2 \text{ and}$$

$[D]^+\Psi_j^{\pm}(x_N) < 0$, j=1,2, for a suitable choice of $C_1 > 0$. By Theorem 4.1, this theorem gets proved. □

5. Error estimates for the solution

We obtain separate error estimates for each component of the numerical solution.

Theorem 5.1. *Let $\bar{V}(x_i)$ be a numerical solution of* (3.5) *defined by* (4.3). *Then*

$$|v_j(x_i) - V_j(x_i)| \le CN^{-1},\ x_i \in \bar{\Omega}^{2N},\ \textit{where } j = 1,2.$$

Proof. If $x_i \in \Omega_1^{2N}$ and $x_i \in \Omega_2^{2N}$ then by [15], we have

$$|\mathcal{L}^N(v_j(x_i) - V_j(x_i))| \le CN^{-1}, i \in \Omega_1^{2N} \cup \Omega_2^{2N}.$$

By the Lemma 4.2, we have

$$|v_j(x_i) - V_j(x_i)| \le CN^{-1},\ i \in \Omega_1^{2N} \cup \Omega_2^{2N}.$$

At the point $x_i = x_{2N}$,

$$\begin{aligned}
\mathcal{K}_j^N(V_j - v_j)(x_{2N}) &= \mathcal{K}_j^N V_j(x_{2N}) - \mathcal{K}_j^N v_j(x_{2N}) \\
&= l - \mathcal{K}_j^N v_j(x_{2N}) \\
&= \mathcal{K}_j v_j(x_{2N}) - \mathcal{K}_j^N v_j(x_{2N}) \\
&= v_j(x_{2N}) - \int_{x_0}^{x_{2N}} g_j(x)v(x)dx - v_j(x_{2N}) + \sum_{i=1}^{2N} \frac{g_{i-1}v_{i-1} + g_i v_i}{2} h_i \\
|\mathcal{K}_j^N(V_j - v_j)(x_{2N})| &\le C\varepsilon((h_1^3 v''(\chi_1) + \cdots + h_{2N}^3 v''(\chi_{2N})) \\
&\le C\varepsilon(h_1^3 + \cdots + h_{2N}^3) \\
&\le CN^{-2} \\
&\le CN^{-1},\ \text{where } x_{i-1} \le \chi_i \le x_i,\ j = 1,2,\ 1 \le i \le 2N.
\end{aligned}$$

Applying Lemma 4.2, we have $|(V_j - v_j)(x_{2N})| \le CN^{-1}$.

Hence $|v_j(x_i) - V_j(x_i)| \le CN^{-1},\ i \in \bar{\Omega}^{2N},$ where $j = 1,2$. □

Theorem 5.2. *Let $\bar{W}(x_i)$ be a numerical solution of* (3.6) *defined by* (4.4). *Then*

$$|w_j(x_i) - W_j(x_i)| \le CN^{-1} ln^2 N,\ x_i \in \bar{\Omega}^{2N},\ \textit{where } j = 1,2.$$

Proof. Note that

$$|w_j(x_i) - W_j(x_i)| \le |u_j(x_i) - U_j(x_i)| + |v_j(x_i) - V_j(x_i)|$$

Then by (3.12), Theorem 3.4 and Theorem 4.3, we have

$$|u_j(x_i) - U_j(x_i)| \le |U_j(x_i) - V_j(x_i)| + |v_j(x_i) - V_j(x_i)| + |u_j(x_i) - v_j(x_i)|.$$

Now,

$$\begin{aligned}
|w_j(x_i) - W_j(x_i)| &\le |U_j(x_i) - V_j(x_i)| + 2|v_j(x_i) - V_j(x_i)| + |u_j(x_i) - v_j(x_i)|, \\
&\le C_1 N^{-1} + C_1 \exp(\frac{-\alpha(2 - x)}{\varepsilon}) + \varepsilon
\end{aligned}$$

$$\leq \quad C_1 \exp(\frac{-\alpha\sigma}{\varepsilon}) + C_1 N^{-1} \leq CN^{-1}, i = 0 \text{ to } \frac{3N}{2} \qquad (5.1)$$

Consider mesh functions

$$\phi_j^{\pm}(x_i) = C_1 N^{-1}\bar{s}(x_i) + C_1 N^{-1}\frac{\sigma}{\varepsilon^2}(x_i - (2-\sigma)) \pm (w_j(x_i) - W_j(x_i))\; x_i \in [2-\sigma, 2] \cap \bar{\Omega}^{2N}.$$

From (5.1), it is easy to check $\phi_j^{\pm}(x_{\frac{3N}{2}}) \geq 0$ and $\mathcal{K}_j\phi_j^{\pm}(x_{2N}) \geq 0$, for a suitable choice of $C_1 > 0$.

$$\begin{aligned} \mathcal{L}_j^N \phi_j^{\pm}(x_i) \quad &\geq \quad C_1 N^{-1}[\beta_j + \gamma_j] + C_1 N^{-1}\frac{\sigma}{\varepsilon^2}[\alpha + \beta_j(x_i + \sigma - 2) + \gamma_j(x_{i+\frac{N}{2}} + \sigma - 2)] \\ &\quad \pm CN^{-1}\varepsilon^{-2} \\ &\geq \quad 0 \end{aligned}$$

Then by the Theorem 5.1, we have $\phi_j^{\pm}(x_i) \geq 0,\ x_i \in \bar{\Omega}^{2N}$. Therefore

$$|w_j(x_i) - W_j(x_i)| \leq CN^{-1}ln^2 N,\ x_i \in \bar{\Omega}^{2N},\ \text{where } j = 1, 2.$$

Hence the proof. □

Theorem 5.3. *Let $\bar{U}(x_i)$ be the solution of $(2.2) - (2.4)$ defined in $(4.1) - (4.2)$. Then*

$$|u_j(x_i) - U_j(x_i)|_{\bar{\Omega}^{2N}} \leq CN^{-1}(\ln N)^2,\ \textit{where } j = 1, 2.$$

Proof. Combining Theorem 5.1 and Theorem 5.2, the proof gets completed. □

6. Numerical Results

Example 6.1.

$$\begin{cases} -\varepsilon u_1''(x) + 11u'(x) + 10u_1(x) - 2u_2(x) - x^2 u_1(x-1) - xu_2(x-1) = e^x,\ x \in \Omega^* \\ -\varepsilon u_1''(x) + 16u'(x) - 2u_1(x) + 10u_2(x) - xu_1(x-1) - xu_2(x-1) = e^{x^2},\ x \in \Omega^* \end{cases}$$

with the boundary conditions

$$\begin{cases} u_1(0) = 1,\ u_1(2) - \varepsilon \int_0^2 \frac{x}{3} u_1(x)dx = 2,\ x \in \bar{\Omega} \\ u_2(0) = 1,\ u_2(2) - \varepsilon \int_0^2 \frac{x}{3} u_2(x)dx = 2,\ x \in \bar{\Omega}. \end{cases}$$

Example 6.2.

$$\begin{cases} -\varepsilon u_1''(x) + 11u'(x) + 6u_1(x) - 2u_2(x) - u_1(x) = 0,\ x \in \Omega^* \\ -\varepsilon u_1''(x) + 16u'(x) - 2u_1(x) + 5u_2(x) - u_2(x) = 0,\ x \in \Omega^* \end{cases}$$

with the boundary conditions

$$\begin{cases} u_1(0) = 1,\ u_1(2) - \varepsilon \int_0^2 \frac{x}{3} u_1(x)dx = 2,\ x \in \bar{\Omega} \\ u_2(0) = 1,\ u_2(2) - \varepsilon \int_0^2 \frac{x}{3} u_2(x)dx = 2,\ x \in \bar{\Omega}. \end{cases}$$

TABLE 1. Maximum pointwise errors and order of convergence for Example 6.1

	Number of mesh points $2N$					
	32	64	128	256	512	1024
D_1^N	6.0996e-03	3.0853e-03	1.5512e-03	7.7754e-04	3.8912e-04	1.9459e-04
P_1^N	9.8329e-01	9.9200e-01	9.9644e-01	9.9869e-01	9.9981e-01	-
D_2^N	4.5859e-03	2.3027e-03	1.1532e-03	5.7681e-04	2.8833e-04	1.4408e-04
P_2^N	9.9391e-01	9.9764e-01	9.9949e-01	1.0004e+00	1.0009e+00	-

TABLE 2. Maximum pointwise errors and order of convergence for Example 6.2

	Number of mesh points $2N$					
	32	64	128	256	512	1024
D_1^N	4.3386e-03	2.2283e-03	1.1292e-03	5.6836e-04	2.8512e-04	1.4283e-04
P_1^N	9.6127e-01	9.8063e-01	9.9047e-01	9.9523e-01	9.9723e-01	-
D_2^N	4.4291e-03	2.2517e-03	1.1349e-03	5.6958e-04	2.8527e-04	1.4276e-04
P_2^N	9.7596e-01	9.8846e-01	9.9462e-01	9.9756e-01	9.9878e-01	-

The analytical solution of the above example are not available. Therefore, we estimate the error using double mesh principle which is defined as $D_\varepsilon^N = \max_{x_i \in \bar{\Omega}_\varepsilon^{2N}} |U^N(x_i) - U^{2N}(x_i)|$ and $D^N = \max_{\varepsilon} D_\varepsilon^N$ where $U^N(x_i)$ and $U^{2N}(x_i)$ denote the numerical solution computed using N and $2N$ mesh points. From these quantities the order of convergence is defined as $P^N = \log_2(\frac{D^N}{D^{2N}})$. In Tables 1 and 2, D_1^N and D_2^N denote the maximum pointwise errors of U_1 and U_2 respectively, P_1^N and P_2^N denote the order of convergence with respect to U_1 and U_2 respectively.

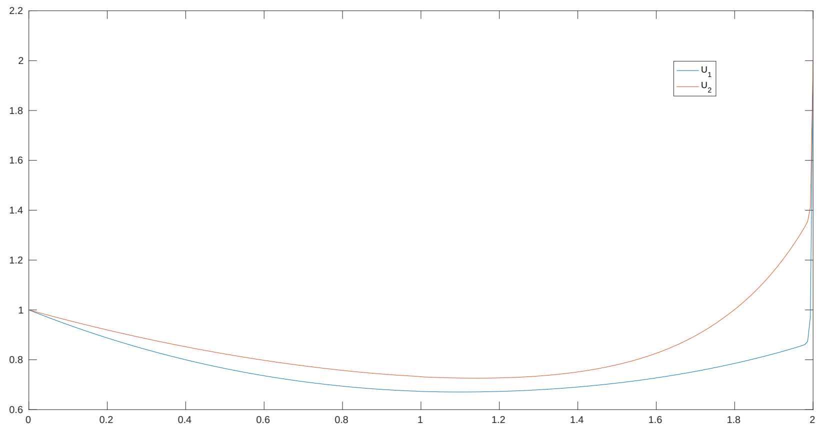

FIGURE 1. Graph of the numerical solution of Example 6.1 for $n = 128$ and $\varepsilon = 2^{-8}$.

The numerical solution of Example (6.2) is plotted in Figure 2.

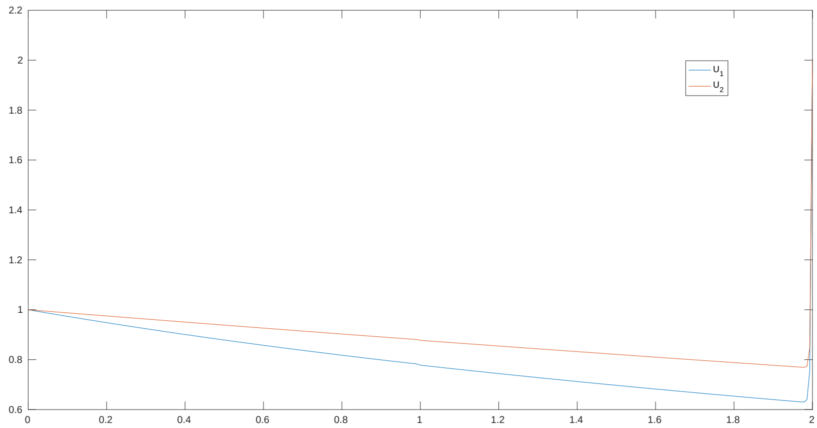

FIGURE 2. Graph of the numerical solution of Example 6.2 for $n = 128$ and $\varepsilon = 2^{-12}$.

7. CONCLUSION

We have solved a class of system of singularly perturbed boundary value problem (2.1) with integral boundary conditions, using a finite difference method on Shishkin mesh. The method is shown to be of order $O(N^{-1}\ln^2 N)$, that is, the method has almost first order convergence with respect to ε. Two examples are given to illustrate the numerical method. Our numerical results reflect the theoretical estimates. Maximum pointwise errors and order of convergence of the Examples (6.1) and (6.2) are given in Table 1 and 2 respectively. The numerical solution of Example (6.1) is plotted in Figure 1.

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