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A STUDY ON CONDENSATION IN ZERO RANGE PROCESSES

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Abstract. We investigate the condensation transition of a zero range process with jump rate g given by

$$g(k) = \begin{cases} \dfrac{M}{k^\alpha}, & \text{if } k \le an \\ \\ \dfrac{1}{k^\alpha}, & \text{if } k > an, \end{cases} \tag{0.1}$$

where $\alpha > 0$ and $a(0 < a < 1/2)$ is a rational number. We show that for any $\epsilon > 0$, there exists $M^* > 0$ such that, for any $0 < M \le M^*$, the maximum cluster size is between $(a-\epsilon)n$ and $(a+\epsilon)n$ for large n.

1. Introduction

The zero range process introduced by Spitzer in 1970, is not only an important example of interacting particle systems in probability theory, but also a model for a wide range of real phenomena such as metastability, granular clustering, wealth condensation, hub formation in complex networks, or even jamming in traffic flow [1, 3, 6, 13, 15, 16].

The zero range process describes the dynamics in which m non-distinguishable particles are distributed over n sites and particles move around the site according to rules based on the transition matrix and positive function g defined on natural numbers. Each particle at one site jumps to other site with a probability given by transition matrix, after waiting exponentially distributed amount of time with parameter $g(k)$ where k is the size of the cluster which contains the particle.

For given m and n, this process form a finite state irreducible Markov Chain and the system converges to a unique invariant measure. Let $Z \doteq (Z_1, Z_2, \cdots, Z_n)$ be the random vector

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corresponding to the invariant measure, and let

$$Z_n^* = \max_{1 \le i \le n} Z_i. \tag{1.1}$$

be the size of the largest cluster.

One of the most interesting phenomena about the zero range process is that it accounts for condensation. Condensation is a phenomenon in which very small particles gather to form large clusters as the water vapor becomes a water droplet. A Bose-Einstein condensate which is a state of dilute gaseous bosons cooled to a temperature very close to 0 K, in particular, explains the important phenomenon in physics that a finite fraction of the bosons take on the lowest quantum state to form a large cluster.

In the zero range process, condensation, then, can be determined by how large the biggest cluster size of the invariant measure is, compared to the total number of particles m of which we generally assume $m = n$ so that the total density $m/n = 1$. Using the above notation, we may say that, if $\lim_{n\to\infty} \frac{Z_n^*}{n} \to a$ where $0 < a < 1$, condensation occurs. It has a great attention after the discovery of existence of condensation transition by Jeon et al. [13] and Evans [5] independently in zero range processes. Indeed, they found that $\frac{Z_n^*}{n}$ converges to a, with $0 < a < 1$ for the jump rates given by

$$g(k) = 1 + \frac{\beta}{k}, \qquad \beta > 0. \tag{1.2}$$

Since then, there have been many studies involving condensation [2, 3, 4, 7, 8, 9, 10, 12].

As the rigorous mathematical proof involves a lot of elaboration, Jeon considered a different type of rate function to find a case where the model is more intuitive and easier to apply [11]. In his model g is given by

$$g(k) = \begin{cases} k^{-\alpha} & \text{if} \quad a < k/n \\ Mk^{-\alpha} & \text{if} \quad k/n \le a \end{cases} \tag{1.3}$$

with $\alpha > 0, M > 0$ and $1/2 < a < 1$, and he was able to show with much simpler way that the largest cluster size is close to an, i.e., $\frac{Z_n^*}{n}$ converges to a. More precisely, for any ϵ there exists $M^* > 0$ such that, for any $0 < M \le M^*$ the maximum cluster size is between $(a - \epsilon)n$ and $(a + \epsilon)n$ for large n. Though providing a case where condensation occurs, he was not able to find out the size of the largest cluster for the case that $0 < a \le 1/2$, and it was remained as an open problem.

In this paper, we prove that the largest cluster size for the case that $0 < a < 1/2$ and a is a rational number is about an. (See Theorem 3.9 and Theorem 3.10.) This result suggests, at least theoretically, a very simple and intuitive way to create a cluster of size an and/or to maintain the rates in order to limit the cluster size to an or less. This method can be used to control large masses produced naturally in the systems, such as traffic congestion and distribution of wealth, etc. by not exceeding a certain scale.

This study is organized as follows: Section 2 briefly introduces the zero range processes and invariant measures; and Section 3 presents proofs of the main theorems.

2. ZERO-RANGE PROCESS

In this section, we introduce the zero range process rigorously. consider the subset of natural numbers $N_n = \{1, 2, \cdots, n\}$, the configuration space $\Omega_n^* = \{0, 1, 2, \dots\}^{N_n}$ and a symmetric irreducible stochastic matrix $\{P_{ij}\}_{1 \le i,j \le n}$. Note that $P_{ij} = P_{ji}$ and $\sum_{j=1}^n P_{ij} = 1$ for all i. Let g be a positive rate function which is defined on $N = \{1, 2, \cdots\}$. Now we define a stochastic process on Ω_n^* as follows.

Suppose the process is in state η at certain time, which means that at site i there is an $\eta(i)$-cluster. Then at any site, say i, the $\eta(i)$-cluster waits for exponentially distributed amount of time with parameter $g(\eta(i))$, and picks site j with probability P_{ij} and gives one particle to the cluster at site j. As a result, $\eta(i)$ decreases to $\eta(i) - 1$, while $\eta(j)$ increases to $\eta(j) + 1$. Let $\eta_t \doteq (\eta_t(1), \eta_t(2), \cdots, \eta_t(n)), 0 \le t < \infty$, be the Markov process which represents such a dynamics. Since η_t preserves the total number of particles, i.e., $\sum_{i=1}^n \eta_t(i) = \sum_{i=1}^n \eta_0(i)$ for all t, and since P_{ij} is irreducible, if we let

$$\Omega_n^m = \{\eta \in \Omega_n^* : \sum_{i=1}^n \eta(i) = m\}, 1 \le m < \infty, \tag{2.1}$$

then there is a unique invariant measure on Ω_n^m, say ν_n^m, which gives the steady state of the process. Let $Z \doteq (Z_1, Z_2, \cdots, Z_n)$ be the random vector corresponding to the invariant measure. The following proposition gives the explicit invariant measure on Ω_n^m.

Lemma 2.1. *For any rate function $g(l)$, and for any $\eta \in \Omega_n^m$, let*

$$\mu_n^m(\eta) = \prod_{i=1}^n \{g!(\eta(i))\}^{-1}, \tag{2.2}$$

where $g!(l) = g(l)g(l-1)g(l-2)\cdots g(1)g(0)$, with convention $g!(0) = g(0) = 1$. Let

$$\nu_n^m(\eta) = \frac{1}{\Gamma}\mu_n^m(\eta), \text{where } \Gamma = \mu_n^m(\Omega_n^m) = \sum_{\eta \in \Omega_n^m} \mu_n^m(\eta). \tag{2.3}$$

Then ν_n^m is the equilibrium measure corresponding to $g(l)$ [16].

Let $|\Omega_n^m|$ be the number of elements in Ω_n^m. Since Ω_n^m is the set of nonnegative integers of the equation, we have

$$x_1 + x_2 + \cdots + x_n = m,$$

and elementary combinatorics gives $|\Omega_n^m| = \binom{n+m-1}{n-1}$

3. THE MAIN THEOREMS AND PROOFS

Let $a = p/q \in (0, 1/2) \bigcap \mathbb{Q}$ be fixed, where p and q are relatively prime. Then we can express a by

$$a = \frac{p}{p\, m + \lambda_p}, \tag{3.1}$$

where $m \geq 2,\ m \in \mathbb{N}$ and $\lambda_p \in \{1, 2, \cdots, p-1\}$. Choose large an such that $\lfloor n/\lfloor an \rfloor \rfloor$ is sufficiently smaller than $\lfloor an \rfloor$, where $\lfloor \cdot \rfloor$ is a floor function. Since $an - 1 < \lfloor an \rfloor \leq an$, we get

$$\frac{1}{a} + \frac{1}{a(an-1)} > \frac{n}{\lfloor an \rfloor} \geq \frac{1}{a}. \tag{3.2}$$

Therefore, for sufficiently large n we have

$$\left\lfloor \frac{n}{\lfloor an \rfloor} \right\rfloor = \left\lfloor \frac{1}{a} \right\rfloor. \tag{3.3}$$

Since $\lfloor an \rfloor \leq an < \lfloor an \rfloor + 1$, it is enough to consider the case that

$$an = \lfloor an \rfloor. \tag{3.4}$$

From (3.1) and (3.4), we have

$$\begin{aligned} b &= n \left(1 - \left\lfloor \frac{1}{a} \right\rfloor a \right) \qquad (3.5) \\ &= n \left(1 - \left\lfloor \frac{p\, m + \lambda_p}{p} \right\rfloor \frac{p}{p\, m + \lambda_p} \right) \\ &= n \left(1 - m\, \frac{p}{p\, m + \lambda_p} \right) \\ &= \frac{\lambda_p\, n}{q}\ . \end{aligned}$$

Consider the rate function g denoted by

$$g(k) = \begin{cases} \dfrac{M}{k^{\alpha}}, & \text{if } k \leq an \\[2ex] \dfrac{1}{k^{\alpha}}, & \text{if } k > an, \end{cases} \tag{3.6}$$

where $0 < a < 1/2$. Let A_k be the set of configurations of which the maximum cluster size is less than or equal to k denoted by

$$A_k := \{\eta \in \Omega_n : \max_{1 \leq j \leq n} \eta(j) \leq k\}.$$

Let $B_k := A_k - A_{k-1}$ be the set of configurations of which the maximum cluster size is exactly k. Define $\gamma := |\{j : \eta(j) \geq 1\}|$, γ indicates the total number of occupied sites. Let $C_{k,l}$ be the set of configurations in B_k with exactly l occupied sites denoted by

$$C_{k,l} = \{\eta \in B_k : \gamma(\eta) = l\}.$$

Note that $|C_{k,l}|$ is bounded by $P(n)\binom{n}{l}\binom{n}{l}$, where P is some polynomial.

Lemma 3.1. *Let $m = lk + r,\ 0 \leq r < k$, and let $\eta_* = (k, k, \cdots, k, r, 0, \cdots, 0) \in A_k$, where the k's are repeated l times. Let g be a rate function with corresponding invariant measure ν_n on Ω_n. Then for any $\eta \in A_k$:*

(a) $\nu_n(\eta_*) \geq \nu_n(\eta)$*, if* g *is decreasing.*
(b) $\nu_n(\eta_*) \leq \nu_n(\eta)$*, if* g *is increasing.*

Proof. See Lemma 1.3 in [13]. □

Proposition 3.2. *Suppose rate function* g *is given by (3.6). Let* $a = p/q < 1/2$*, where* p *and* q *are relatively prime. Then*

$$\nu_n(A_{n/2q}) \to 0 \text{ as } n \to \infty. \tag{3.7}$$

Proof. Choose configurations denoted by

$$\eta_0 = (an, \cdots, an,\ b,\ 0, \cdots, 0) \in \Omega_n,$$
$$\eta_1 = (n/2q, \cdots, n/2q,\ d, 0, \cdots, 0) \in \Omega_n,$$

where $n = \lfloor \frac{1}{a} \rfloor an + b$ and $b = \frac{\lambda_p\, n}{q}$, for some $\lambda_p \in \{0, 1, 2, \cdots, p-1\}$ and $n = t \cdot (n/2q) + d$ with $0 \leq d < n/2q$. From Lemma 3.1, $\mu_n(\eta_0) \geq \mu_n(\eta)$, for any $\eta \in A_{an}$ and $\mu_n(\eta_1) \geq \mu_n(\eta)$ for any $\eta \in A_{\frac{n}{2q}}$. Moreover, we note

$$n = \left\lfloor \frac{q}{p} \right\rfloor \frac{pn}{q} + \frac{\lambda_p n}{q} = \frac{n}{2q} \left(\left\lfloor \frac{q}{p} \right\rfloor 2p + 2\lambda_p \right) .$$

From Stiring's formula

$$\begin{aligned}
\nu_n(A_{\frac{n}{2q}}) &= \frac{\mu_n(A_{\frac{n}{2q}})}{\mu_n(\Omega_n)} \\
&\leq \frac{\mu_n(\eta_1)\binom{2n}{n}}{\mu_n(\eta_0)} \\
&\leq P(n) \left(\frac{((n/2q)!)^{\lfloor q/p \rfloor 2p + 2\lambda_p}}{((pn/q)!)^{\lfloor q/p \rfloor} ((pn/q)!)} \right)^{\alpha} \binom{2n}{n} \\
&\leq Q(n) \left(\frac{4}{2^{\alpha}} \right)^n \left(\frac{1}{p^{(p/q)\lfloor q/p \rfloor}} \right)^{n\alpha} \left(\frac{1}{\lambda_p^{\lambda_p\, \alpha/q}} \right)^n ,
\end{aligned}$$

where $\alpha > 2$ and degrees of polynomials P and Q are independent of n. Therefore the right hand side goes to zero as $n \to \infty$. □

Lemma 3.3.

$$\frac{\mathrm{d}}{\mathrm{d}x} \log \Gamma(x+1) = -\gamma + \sum_{p=1}^{\infty} \left(\frac{1}{p} - \frac{1}{x+p} \right), \tag{3.8}$$

where $\Gamma(\cdot)$ *is Gamma function and* γ *Euler gamma constant.*

Lemma 3.4. *If* $n \in \mathbb{N}$*, then*

$$\frac{\mathrm{d}}{\mathrm{d}x} \log \Gamma(x+1)|_{x=n} = -\gamma + \sum_{p=1}^{n} \frac{1}{p}. \tag{3.9}$$

Proposition 3.5. *Suppose rate function g is given by (3.6). Let $a = p/q < 1/2$, where p and q are relatively prime. For any small $\varepsilon > 0$,*

$$\nu_n \left(A_{(a-\varepsilon)n} - A_{n/2q}\right) \to 0 \text{ as } n \to \infty. \tag{3.10}$$

Proof. Without loss of generality, we may assume $n/2q = \lfloor n/2q \rfloor$.
First,

$$\begin{aligned} &n = 2q \ (n/2q) \\ &n = (2q-1)(n/2q+1) + n/2q - (2q-1) \\ &n = (2q-1)(n/2q+2) + n/2q - 2(2q-1) \\ &\quad \cdots \\ &n = (2q-1)(n/2q+m_1) + n/2q - m_1(2q-1), \end{aligned} \tag{3.11}$$

where m_1 is the largest integer with

$$n/2q - m_1(2q-1) > 0.$$

Second,

$$\begin{aligned} &n = (2q-2)(n/2q+m_1+1) + n/2q \cdot 2 - (m_1+1)(2q-2) \\ &n = (2q-2)(n/2q+m_1+2) + n/2q \cdot 2 - (m_1+2)(2q-2) \\ &\quad \cdots \\ &n = (2q-2)(n/2q+m_1+m_2) + n/2q \cdot 2 - (m_1+m_2)(2q-2), \end{aligned} \tag{3.12}$$

where m_2 is the largest integer with

$$n/2q \cdot 2 - (m_1+m_2)(2q-2) > 0.$$

Inductively,

$$n = (2q-k)(n/2q+m_1+\cdots+m_k) + n/2q \cdot k - (m_1+\cdots+m_k)(2q-k), \tag{3.13}$$

where m_k is the largest integer with

$$n/2q \cdot k - (m_1+\cdots+m_k)(2q-k) > 0.$$

And we choose $l = l(\varepsilon) \in \mathbb{N}$ such that

$$(a-\varepsilon)n = n/2q + m_1 + \cdots + m_{l-1} + \hat{m}_l, \tag{3.14}$$

where $1 \le \hat{m}_l \le m_l$.

Set

$$E = \{x \in \mathbb{N} : n/2q + 1 \le x \le (a-\varepsilon)n\}.$$

Consider the disjoint collection $\{E_i\}_{i=1}^{l}$ of E denoted by

$$\begin{aligned} E_1 &= \{x \in \mathbb{N} : n/2q + 1 \le x \le n/2q + m_1\} \\ E_2 &= \{x \in \mathbb{N} : n/2q + m_1 + 1 \le x \le n/2q + m_1 + m_2\} \\ &\dots \\ E_{l-1} &= \{x \in \mathbb{N} : n/2q + m_1 + \cdots + m_{l-2} + 1 \le x \le n/2q + m_1 + \cdots + m_{l-2} + m_{l-1}\} \\ E_l &= \{x \in \mathbb{N} : n/2q + m_1 + \cdots + m_{l-1} + 1 \le x \le n/2q + m_1 + \cdots + m_{l-1} + \hat{m}_l\}. \end{aligned} \tag{3.15}$$

For $j = 1, 2, \cdots l$, set G_j be the real-valued extension super set of E_j denoted by

$$\begin{aligned} G_1 &= \{x \in \mathbb{R} : n/2q + 1 \le x \le n/2q + m_1\} \\ &\dots \\ G_{l-1} &= \{x \in \mathbb{R} : n/2q + m_1 + \cdots + m_{l-2} + 1 \le x \le n/2q + m_1 + \cdots + m_{l-2} + m_{l-1}\} \\ G_l &= \{x \in \mathbb{R} : n/2q + m_1 + \cdots + m_{l-1} + 1 \le x \le n/2q + m_1 + \cdots + m_{l-1} + \hat{m}_l\}. \end{aligned} \tag{3.16}$$

For any $k \in \{1, 2, \cdots, l\}$, let $x \in E_k$, by division algorithm,

$$n = (2q - k)\, x + d_k(x), \tag{3.17}$$

where $0 \le d_k(x) < x$. From (3.17), if x is increasing on E_k, $d_k(x)$ is decreasing on E_k, but $2q - k$ is constant on E_k. In this paper, we use the symbol ! as Gamma function notation which means that $x \in \mathbb{R},\ x! = \Gamma(x + 1)$.

Lemma 3.6. *For $k \in \{1, 2, \cdots, l\}$, $x \in E_k$, let $\eta_x^k = (\underbrace{x, \cdots, x}_{2q-k \text{ times}}, d_k(x), 0, \cdots, 0)$ with $n = (2q - k)\ x + d_k(x),\ 0 \le d_k(x) < x$. Suppose rate function g is given by (3.6). Then the unnormalized measure*

$$\mu_n(\eta_x^k) := \Big((x!)^{2q-k}\ d_k(x)!\Big)^{\alpha}$$

is increasing on E_k, where $\alpha > 0$.

Proof. For $x \in G_k$, set φ_k be denoted by

$$\varphi_k(x) = (x!)^{2q-k} d_k(x)!. \tag{3.18}$$

From (3.17), $d_k(x) = n - (2q - k)\ x$, if we take derivation by x except for end points of G_k, then

$$\frac{\mathrm{d}}{\mathrm{d}x} d_k(x) = -(2q - k). \tag{3.19}$$

We take a logarithm for φ_k , then

$$\log \varphi_k(x) = (2q - k) \log x! + \log d_k(x)!\ . \tag{3.20}$$

By Lemma 3.3 and from (3.19), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\log\varphi_k(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left((2q-k)\ \log x!\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(\log d_k(x)!\right) \\ &= (2q-k)\left(\sum_{p=1}^{\infty}\left(\frac{1}{p}-\frac{1}{x+p}\right) - \sum_{p=1}^{\infty}\left(\frac{1}{p}-\frac{1}{d_k(x)+p}\right)\right).\end{aligned} \tag{3.21}$$

If $x \in E_k$, then by Lemma 3.4 and from (3.21), we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\log\varphi_k(x) = (2q-k)\left(\sum_{p=1}^{x}\frac{1}{p} - \sum_{p=1}^{d_k(x)}\frac{1}{p}\right) > 0, \tag{3.22}$$

since $0 \le d_k(x) < x$. Thus $\mu_n(\eta_x^k)$ is increasing on E_k. □

Set x_L^k be the last element of E_k and $x_F^{k+1} = x_L^k + 1$ the first element of E_{k+1}. Then we note that

$$\begin{aligned} n &= (2q-k)\, x_L^k + d_k\left(x_L^k\right), \\ n &= (2q-k-1)\, x_F^{k+1} + d_{k+1}\left(x_F^{k+1}\right), \\ &= (2q-k-1)\, (x_L^k+1) + d_k\left(x_L^k\right) + x_L^k - (2q-k-1). \end{aligned} \tag{3.23}$$

Lemma 3.7.

$$\frac{\left(x_L^k!\right)^{2q-k}\ d_k\left(x_L^k\right)!}{\left(x_F^{k+1}!\right)^{2q-k-1}\ d_{k+1}\left(x_F^{k+1}\right)!} \ \le\ 1. \tag{3.24}$$

Proof. Set $x_L^k = y$. From (3.23), we have

$$\begin{aligned} & \frac{(y!)^{2q-k} d_k(y)!}{((y+1)!)^{2q-k-1}\ (d_k(y)+y-(2q-k-1))!} \\ = \ & \frac{y!}{(y+1)^{2q-k-1}\,(d_k(y)+y-(2q-k-1))\cdots(d_k(y)+1)} \\ = \ & \frac{\overbrace{y\cdots(y-(2q-k-2))}^{2q-k-1 \text{ times}}\ \overbrace{(y-(2q-k-1))\cdots\ \cdots 1}^{y-(2q-k-1) \text{ times}}}{\underbrace{(y+1)\cdots(y+1)}_{2q-k-1 \text{ times}}\ \underbrace{(d_k(y)+y-(2q-k-1))\cdots(d_k(y)+1)}_{y-(2q-k-1) \text{ times}}} \ \le\ 1. \end{aligned}$$

□

For $n/2q \le x \le (a-\varepsilon)n$, consider configurations $\eta_x,\ \eta_1 \in \Omega_n$, denoted by

$$\eta_x = (\underbrace{x, \cdots, x}_{s \text{ times}}, d_x, 0, \cdots, 0), \tag{3.25}$$

$$\eta_1 = (x, an-x, \underbrace{an, \cdots, an}_{\lfloor 1/a \rfloor - 1 \text{ times}}, b, 0, \cdots, 0),$$

where $n = s\ x + d_x,\ 0 \le d_x < x$ and $n = \lfloor 1/a \rfloor an + b,\ 0 \le b < an.$ Then we have the following Lemma.

Lemma 3.8. *For $1 \le x \le an$, choose configurations $\eta_x,\ \eta_1$ denoted by (3.25). Suppose rate function g is given by (3.6). Then*

$$\mu_n(\eta_x) \le \mu_n(\eta_1).$$

Proof. By definition of the un-normalized measure, it is enough to show that

$$(x!)^s\ d_x! \le x!\ (an-x)!\ ((an)!)^{\lfloor 1/a \rfloor - 1}\ b!\ . \tag{3.26}$$

To prove this lemma, we use induction. For $x = an$, (3.26) is satisfied. i.e.

$$((an)!)^{\lfloor 1/a \rfloor} b!\ \le ((an)!)\ 0!\ ((an)!)^{\lfloor 1/a \rfloor - 1} b!\ . \tag{3.27}$$

From division algorithm, we assume that

$$n = l_k \cdot k + d_k,\ 0 \le d_k < k, \tag{3.28}$$

$$n = l_{k+1} \cdot (k+1) + d_{k+1},\ 0 \le d_{k+1} < k+1, \tag{3.29}$$

where $1 \le k \le an-1$. From Lemma 3.6 and Lemma 3.7, l_{k+1} is classified into two cases:

- case1: $l_{k+1} = l_k$, set $l_k = l$.

$$\begin{aligned} n &= l_{k+1} \cdot (k+1) + d_{k+1}, \\ &= l \cdot (k+1) + d_k - l. \end{aligned} \tag{3.30}$$

- case2: $l_{k+1} = l_k - 1$, set $l_k = l$.

$$\begin{aligned} n &= l_{k+1} \cdot (k+1) + d_{k+1}, \\ &= (l-1) \cdot (k+1) + d_k + k - l + 1. \end{aligned} \tag{3.31}$$

Set $L(x) = (x!)^s\ d_x!$ and $R(x) = x!\ (an-x)!\ ((an)!)^{\lfloor 1/a \rfloor} b!$. Consider the ratio between $L(k+1)$ and $L(k)$. From (3.30) and (3.31), $L(k+1)/L(k)$ is classified into two cases:

- case1:

$$\begin{aligned} \frac{L(k+1)}{L(k)} &= \frac{((k+1)!)^l\ (d_k - l)!}{(k!)^l\ d_k!} \\ &= \frac{(k+1)^l}{d_k\ (d_k-1) \cdots (d_k - l + 1)}\ >\ 1. \end{aligned} \tag{3.32}$$

- case2:

$$\begin{aligned} & \frac{L(k+1)}{L(k)} \\ = & \frac{((k+1)!)^{l-1}\,(d_k+k-l+1)!}{(k!)^l\,d_k!} \\ = & \frac{(k+1)^{l-1}\,(d_k+k-l+1)\,(d_k+k-l)\cdots(d_k+2)\,(d_k+1)}{\underbrace{k\,(k-1)\cdots(k-l+2)}_{l-1 \text{ times}}\,\underbrace{(k-l+1)\,(k-l)\cdots 2\cdot 1}_{k-l \text{ times}}} \\ > & \ 1\,. \end{aligned} \tag{3.33}$$

Consider the ratio between $R(k+1)$ and $R(k)$.

$$\begin{aligned} \frac{R(k+1)}{R(k)} &= \frac{(k+1)!\,(an-k-1)!\,((an)!)^{\lfloor 1/a \rfloor -1}\,b!}{k!\,(an-k)!\,((an)!)^{\lfloor 1/a \rfloor -1}\,b!} \\ &= \frac{k+1}{an-k}\,. \end{aligned} \tag{3.34}$$

If $1 \le k < an/2$, then we have

$$\frac{R(k+1)}{R(k)} = \frac{k+1}{an-k} \le 1.$$

For $1 \le k < an/2$, we get

$$L(k) < L(k+1) \le R(k+1) \le R(k).$$

Hence (3.26) is true for the case that $1 \le k < an/2$.

If $an/2 \le k \le an-1$, then we get

$$\frac{R(k+1)}{R(k)} = \frac{k+1}{an-k} > 1.$$

Since $L(k+1) > L(k)$ and $R(k+1) > R(k)$, to prove (3.26), we must show that

$$\frac{L(k+1)}{L(k)} \ge \frac{R(k+1)}{R(k)}, \tag{3.35}$$

for $an/2 \le k \le an-1$. From (3.30) and (3.31), (3.35) is classified as follows:

- case1 :

$$\frac{(k+1)^l}{d_k\,(d_k-1)\cdots(d_k-l+1)} \ \ge\ \frac{k+1}{an-k}\,. \tag{3.36}$$

- case2 :

$$\frac{(k+1)^{l-1}\,(d_k+k-l+1)\,(d_k+k-l)\cdots(d_k+2)\,(d_k+1)}{\underbrace{k\,(k-1)\cdots(k-l+2)}_{l-1 \text{ times}}\,\underbrace{(k-l+1)\,(k-l)\cdots 2\cdot 1}_{k-l \text{ times}}} \ \ge\ \frac{k+1}{an-k}\,. \tag{3.37}$$

Case1 (3.36) becomes

$$(k+1)^{l-1}\ (an-k)\ \geq\ d_k\ (d_k-1)\cdots(d_k-l+1). \tag{3.38}$$

Set $\hat{L}(k)=(k+1)^{l_k-1}\ (an-k)$ and $\hat{R}(k)=d_k\ (d_k-1)\cdots(d_k-l_k+1)$ with $n=l_k\cdot k+d_k$. To prove (3.38), we use an induction. For $k=an/2$, (3.38) is satisfied. i.e.

$$\left(\frac{an}{2}+1\right)^{\lfloor 2/a\rfloor-1}\ \left(\frac{an}{2}\right)\geq d\ (d-1)\ \cdots(d-\lfloor 2/a\rfloor+1), \tag{3.39}$$

with $n=\lfloor 2/a\rfloor\ (an/2)+d$ and $0\leq d<an/2$. From (3.30), say case1, and $l_k=l_{k+1}=l>2$, we easily get

$$\frac{\hat{L}(k+1)}{\hat{L}(k)}=\frac{(k+2)^{l-1}\ (an-k-1)}{(k+1)^{l-1}\ (an-k)}\ >\ 1. \tag{3.40}$$

And

$$\begin{aligned}\frac{\hat{R}(k+1)}{\hat{R}(k)}&=\frac{d_{k+1}\ (d_{k+1}-1)\cdots(d_{k+1}-l+1)}{d_k\ (d_k-1)\cdots(d_k-l+1)}\\&=\frac{(d_k-l)\ (d_k-l-1)\cdots(d_k-2l+1)}{d_k\ (d_k-1)\cdots(d_k-l+1)}\ <\ 1.\end{aligned} \tag{3.41}$$

From (3.39), (3.40) and (3.41), we note

$$\hat{L}(k+1)>\hat{L}(k)\geq\hat{R}(k)>\hat{R}(k+1). \tag{3.42}$$

Thus (3.38) is satisfied.

Let's consider another case now. Then case2 (3.37) becomes

$$(k+1)^{l-2}\ (an-k)\ (d_k+k-l+1)\ \cdots(d_k+2)(d_k+1)\ \geq\ k!\,. \tag{3.43}$$

Set $\tilde{L}(k)=(k+1)^{l_k-2}\ (an-k)\ (d_k+k-l_k+1)\cdots(d_k+2)\ (d_k+1)$ and $\tilde{R}(k)=k!$ with $n=l_k\cdot k+d_k$. Similarly, we use an induction. For $k=an/2$, (3.43) is satisfied. i.e.

$$\begin{aligned}&\left(\frac{an}{2}+1\right)^{\lfloor 2/a\rfloor-2}\left(\frac{an}{2}\right)\left(d_k+\frac{an}{2}-\lfloor 2/a\rfloor+1\right)\\&\left(d_k+\frac{an}{2}-\lfloor 2/a\rfloor\right)\cdots(d_k+2)\ (d_k+1)\ \geq\ \left(\frac{an}{2}\right)\ \left(\frac{an}{2}-1\right)\ \cdots\ 2\cdot 1\,,\end{aligned} \tag{3.44}$$

with $n=\lfloor 2/a\rfloor\ (an/2)+d$ and $0\leq d<an/2$. From (3.31), say case2, and $l_{k+1}=l_k-1=l-1$, we easily get

$$\frac{\tilde{L}(k+1)}{\tilde{L}(k)}=\frac{(k+2)^{l-3}\ (an-k-1)\ \overbrace{(d_k+2k-2l+4)\cdots(d_k+k-l+2)}^{k-l+1\text{ times}}}{(k+1)^{l-2}\ (an-k)\ (d_k+k-l+1)\cdots(d_k+1)}\,. \tag{3.45}$$

Since the size of k is sufficiently larger that that of l, we get

$$\frac{\tilde{L}(k+1)}{\tilde{L}(k)}>1\,. \tag{3.46}$$

Clearly,

$$\frac{\tilde{R}(k+1)}{\tilde{R}(k)} = \frac{(k+1)!}{k!} = k+1 > 1\,. \tag{3.47}$$

Consider the ratio denoted by

$$\frac{\tilde{L}(k+1)}{\tilde{L}(k)} \Big/ \frac{\tilde{R}(k+1)}{\tilde{R}(k)}. \tag{3.48}$$

Then (3.48) becomes

$$\frac{(k+2)^{l-3}\,(an-k-1)\,\overbrace{(d_k+2k-2l+4)\cdots(d_k+k-l+2)}^{k-l+1 \text{ times}}}{(k+1)^{l-3}\,(an-k)\,(d_k+k-l+1)\cdots(d_k+1)}\,. \tag{3.49}$$

Clearly, this ratio is greater than 1. Thus

$$\frac{\tilde{L}(k+1)}{\tilde{L}(k)} > \frac{\tilde{R}(k+1)}{\tilde{R}(k)}. \tag{3.50}$$

Therefore from (3.44) and (3.50), (3.43) is satisfied. In all cases, this completes the proof. □

Consider configurations

$$\begin{aligned}\eta_0 &= (\underbrace{an,\cdots,an}_{\lfloor 1/a\rfloor \text{ times}}, b, 0, \cdots, 0) \in \Omega_n,\\ \eta_x &= (\underbrace{x,\cdots,x}_{s \text{ times}}, d_x, 0, \cdots, 0) \in \Omega_n,\end{aligned}$$

where $n = s\cdot x + d_x,\ 0 \le d_x < x$ and $n/2q < x \le (a-\varepsilon)n$.

(1) For $x \le b,\ 0 \le b < an$, choose $\eta_1 = (x, an-x, \underbrace{an,\cdots,an}_{\lfloor 1/a\rfloor -1 \text{ times}}, b, 0, \cdots, 0) \in \Omega_n$. From Lemma 3.1, $\mu_n(\eta_x) \ge \mu_n(\eta)$ for any $\eta \in C_{x,l}$. From Lemma 3.8, we get $\mu_n(\eta_1) \ge \mu_n(\eta_x)$. From Stirling's formula,

$$\begin{aligned}\frac{\mu_n(\eta_1)}{\mu_n(\eta_0)} &= \left(\frac{x!\,(an-x)!(an!)^{\lfloor 1/a\rfloor -1}b!}{(an!)^{\lfloor 1/a\rfloor}b!}\right)^{\alpha} \qquad (3.51)\\ &= \left(\frac{x!\,(an-x)!}{(an)!}\right)^{\alpha}\\ &\le P(n)\left(\frac{x^x\,(an-x)^{an-x}}{(an)^{an}}\right)^{\alpha}\\ &= P(n)\left(\frac{(x/n)^{x/n}\,(a-x/n)^{a-x/n}}{(a)^a}\right)^{\alpha n}\\ &\le P(n)\,\gamma_1^n, \qquad (3.52)\end{aligned}$$

for some $0 < \gamma_1 < 1$. The degree of the polynomial $P(n)$ which may vary in each expression is independent of n.

(2) For $b < x \le b + \varepsilon_0 n$, where $0 < \varepsilon_0 < \varepsilon$. Choose

$$\tilde{\eta_0} = (an - \varepsilon_0 n,\ b + \varepsilon_0 n,\ \underbrace{an, \cdots, an}_{\lfloor 1/a \rfloor - 1 \text{ times}},\ 0, \cdots, 0),$$

$$\eta_1 = (x,\ an - x - \varepsilon_0 n,\ b + \varepsilon_0 n,\ \underbrace{an, \cdots, an}_{\lfloor 1/a \rfloor - 1 \text{ times}},\ 0, \cdots, 0),$$

From Lemma 3.6 and Lemma 3.7, we note that

$$(((a-\varepsilon_0)n)!)^{\lfloor 1/(a-\varepsilon_0) \rfloor - 1}\ d! \ \le\ (an!)^{\lfloor 1/a \rfloor - 1}\ (b + \varepsilon_0 n)!, \tag{3.53}$$

where $n - (a-\varepsilon_0)n = (\lfloor 1/(a-\varepsilon_0) \rfloor - 1)\ (a-\varepsilon_0)n + d$, $0 \le d < (a-\varepsilon_0)n$. And we should choose small $\varepsilon_0 > 0$ with $0 \le b + \varepsilon_0 n < an$. From (3.53), we get

$$(((a-\varepsilon_0)n)!)^{\lfloor 1/(a-\varepsilon_0) \rfloor}\ d!\ \le (a-\varepsilon_0)n!\ (b+\varepsilon_0 n)!\ (an!)^{\lfloor 1/a \rfloor - 1}, \tag{3.54}$$

where $n = \lfloor 1/(a-\varepsilon_0) \rfloor\ (a-\varepsilon_0)n + d$, $0 \le d < (a-\varepsilon_0)n$. This fact corresponds to that of (3.27) in Lemma 3.8. Hence, by the similar inductive method of Lemma 3.8, we have

$$(x!)^s\ d_x!\ \le\ x!((a-\varepsilon_0)n - x)!\ (b+\varepsilon_0 n)!\ (an!)^{\lfloor 1/a \rfloor - 1} b! \tag{3.55}$$

Thus

$$\mu_n(\eta_1) \ge \mu_n(\eta_x)\ . \tag{3.56}$$

From Lemma 3.1, $\mu_n(\eta_0) \ge \mu_n(\tilde{\eta_0})$ and $\mu_n(\eta_x) \ge \mu_n(\eta)$ for any $\eta \in C_{x,l}$. In all, we have

$$\mu_n(\eta_1) \ge \mu_n(\eta) \ \text{ for any }\ \eta \in C_{x,l}.$$

Hence

$$\begin{aligned} \frac{\mu_n(\eta_1)}{\mu_n(\eta_0)} \le \frac{\mu_n(\eta_1)}{\mu_n(\tilde{\eta_0})} &= \left(\frac{x!(an - x - \varepsilon_0 n)!}{(an - \varepsilon_0 n)!} \right)^{\alpha} \\ &\le P(n) \left(\frac{(x/n)^{x/n}\ (a - \varepsilon_0 - x/n)^{a-\varepsilon_0 - x/n}}{(a-\varepsilon_0)^{a-\varepsilon_0}} \right)^{\alpha n} \\ &\le P(n)\ \gamma_2^n, \end{aligned} \tag{3.57}$$

for some $0 < \gamma_2 < 1$. The degree of polynomial $P(n)$ which may vary in each expression is independent of n.

(3) For $x > b + \varepsilon_0 n$, choose

$$\eta_1 = (x,\ an - x,\ b,\ \underbrace{an, \cdots, an}_{\lfloor 1/a \rfloor - 1 \text{ times}},\ 0, \cdots, 0) \in \Omega_n,$$

$$\eta_2 = (x,\ an - x + b,\ \underbrace{an, \cdots, an}_{\lfloor 1/a \rfloor - 1 \text{ times}},\ 0, \cdots, 0) \in \Omega_n.$$

Since $(an-x+b)! \geq (an-b)!\, b!$, we get

$$\mu_n(\eta_2) \geq \mu_n(\eta_1)\ . \tag{3.58}$$

From Lemma 3.1 and Lemma 3.8, we have

$$\mu_n(\eta_1) \geq \mu_n(\eta_x) \geq \mu_n(\eta) \text{ for any } \eta \in C_{x,l}\ .$$

In all,

$$\mu_n(\eta_2) \geq \mu_n(\eta) \text{ for any } \eta \in C_{x,l}\ . \tag{3.59}$$

Hence from Stiring's formula,

$$\begin{aligned}
\frac{\mu_n(\eta_2)}{\mu_n(\eta_0)} &= \left(\frac{x!(an-x+b)!}{(an)!b!}\right)^{\alpha} \qquad (3.60)\\
&\leq P(n)\ \left(\frac{x^x\ (an+b-x)^{an+b-x}}{(an)^{an}\ b^b}\right)^{\alpha}\\
&\leq P(n)\ \left(\frac{\left(\frac{x}{an+b}\right)^{\frac{x}{an+b}}\ \left(1-\frac{x}{an+b}\right)^{1-\frac{x}{an+b}}}{\left(\frac{an}{an+b}\right)^{\frac{an}{an+b}}\ \left(1-\frac{an}{an+b}\right)^{1-\frac{an}{an+b}}}\right)^{\alpha}\\
&\leq P(n)\ \gamma_3^n,
\end{aligned}$$

for some $0<\gamma_3<1$. The degree of polynomial $P(n)$ which may vary in each expression is independent of n. Since $x > b+\varepsilon_0 n$, we note

$$1-\frac{x}{an+b} \neq \frac{an}{an+b}. \tag{3.61}$$

Moreover, since $\dfrac{an}{an+b} > \frac{1}{2},\ x < an$ and from (3.61), the above last inequality of (3.60) is verified.

In all, we choose $0<\gamma<1$ with $\gamma > \max\{\gamma_1,\gamma_2,\gamma_3\}$. Therefore,

$$\begin{aligned}
\nu_n\left(\bigcup_{x=n/2q}^{(a-\varepsilon)n}\bigcup_{l} C_{x,l}\right) &\leq \frac{\sum_x\sum_l \mu_n(C_{x,l})}{\mu_n(\eta_0)} \qquad (3.62)\\
&\leq \sum_l P(n)\binom{n}{l}\binom{n}{l}\gamma^n, \qquad (3.63)
\end{aligned}$$

for some polynomial $P(n)$. Let $\varepsilon_1>0$ be sufficiently small.

(a) For $l<\varepsilon_1 n$, we have

$$\binom{n}{l} \leq \binom{n}{\varepsilon_1 n}.$$

Then

$$\nu_n\left(\bigcup_{x=n/2q}^{(a-\varepsilon)n}\bigcup_{l<\varepsilon_1 n} C_{x,l}\right) \le n^2 P(n)\binom{n}{\varepsilon_1 n}\binom{n}{\varepsilon_1 n}\gamma^n \tag{3.64}$$

$$\le Q(n)\left(\frac{\gamma}{(\varepsilon_1^{\varepsilon_1}(1-\varepsilon_1)^{1-\varepsilon_1})^2}\right)^n, \tag{3.65}$$

where ε_1 can be chosen $\left(\varepsilon_1^{\varepsilon_1}(1-\varepsilon_1)^{1-\varepsilon_1}\right)^2 > \gamma$. Degrees of $P(n),\ Q(n)$ are independent of n. The right hand side goes to 0 as $n\to\infty$.

(b) For $l \ge \varepsilon_1 n$, assume $n = s\cdot x + d,\ 0\le d < s$.

- (i) If $d < l - s$, then we choose

$$\tilde{\eta_1} = (\underbrace{x,\cdots,x}_{s-1\text{ times}},\ n-x(s-1)-l+s,\ \underbrace{1,\cdots,1}_{l-s\text{ times}},\ 0,\cdots,0)\in\Omega_n. \tag{3.66}$$

- (ii) If $d \ge l-s$, then we choose

$$\tilde{\eta_1} = (\underbrace{x,\cdots,x}_{s\text{ times}},\ d-l+s,\ \underbrace{1,\cdots,1}_{l-s-1\text{ times}},\ 0,\cdots,0)\in\Omega_n. \tag{3.67}$$

Clearly, $\mu_n(\tilde{\eta_1}) \ge \mu_n(\eta)$ for any $\eta\in C_{x,l}$.

Then

- (i) $d < l-s : n - x(s-1) + s + 1 \ = \ d + x + s + 1 \ge \varepsilon n.$

$$\begin{aligned}\frac{\mu_n(\tilde{\eta_1})}{\mu_n(\eta_0)} &= \left(\frac{(x!)^{s-1}\,(n-x(s-1)-l+s)!}{(an)!^{\lfloor 1/a\rfloor}b!}\right)^{\alpha} \\ &\le \left(\frac{(x!)^{s-1}\,(n-x(s-1))!}{(an)!^{\lfloor 1/a\rfloor}b!\big(n-x(s-1)-1\big)\cdots\big(n-x(s-1)-l+s+1\big)}\right)^{\alpha} \\ &\le \left(\frac{n!\ \cdot\ 1}{(an)!^{\lfloor 1/a\rfloor}b!\big(n-x(s-1)-1\big)\cdots\big(n-x(s-1)-\varepsilon_1 n+s+1\big)}\right)^{\alpha} \\ &\le \left(\frac{C_1}{((\varepsilon-\varepsilon_1)n)^{\alpha\varepsilon_1}}\right)^{n},\end{aligned} \tag{3.68}$$

where $C_1 > 0$ is some constant.

- (ii) $d\ge l-s$: note that $d+s\ge l\ge\varepsilon_1 n$ and s is sufficiently smaller than l. Thus we have

$$d! \ge (\varepsilon_1 n - s)! \sim (\tau n)^{\tau n}\, e^{-\tau n},\ \text{for some } \tau > 0.$$

$$\begin{aligned}\frac{\mu_n(\tilde{\eta_1})}{\mu_n(\eta_0)} &= \left(\frac{(x!)^s\,(d-l+s)!}{(an)!^{\lfloor 1/a \rfloor} b!}\right)^\alpha \\ &\le \left(\frac{(x!)^s\,d!}{(an)!^{\lfloor 1/a \rfloor} b!\;d(d-1)\cdots(d-l+s+1)}\right)^\alpha \\ &\le \left(\frac{(x!)^s\,d!}{n!}\;\frac{n!}{(an)!^{\lfloor 1/a \rfloor} b!}\;\frac{1}{d(d-1)\cdots(d-\varepsilon_1 n+s+1)}\right)^\alpha \\ &\le \left(\frac{(x!)^s\,d!}{n!}\;\frac{n!}{(an)!^{\lfloor 1/a \rfloor} b!}\;\frac{1}{(\varepsilon_1 n-s)(\varepsilon_1 n-s-1)\cdots 1}\right)^\alpha \\ &\le \left(\frac{C_2}{(\tau n)^{\tau\alpha}}\right)^n,\end{aligned} \tag{3.69}$$

where C_2, τ are some positive constants.

Therefore, from (3.68) and (3.69), we note that

$$\frac{\mu_n(\tilde{\eta_1})}{\mu_n(\eta_0)} \le \left(\frac{C_3}{n^\xi}\right)^n, \tag{3.70}$$

where $C_3,\ \xi$ are some positive constants. Thus

$$\begin{aligned}\nu_n\left(\bigcup_{x=n/2q}^{(a-\varepsilon)n}\ \bigcup_{l\ge\varepsilon_1 n} C_{x,\,l}\right) &\le \frac{\mu_n\left(\bigcup_x\bigcup_l C_{x,\,l}\right)}{\mu_n(\eta_0)} \\ &\le \frac{\sum_x\sum_l\ \mu_n\left(C_{x,\,l}\right)}{\mu_n(\eta_0)} \\ &\le P(n)\binom{n}{l}\binom{n}{l}\left(\frac{C_3}{n^\xi}\right)^n \\ &\le \left(\frac{C}{n^\delta}\right)^n,\end{aligned} \tag{3.71}$$

where $C,\ \delta$ are some positive constants and the degree of $P(n)$ is independent of n. In all cases,

$$\nu_n\left(A_{(a-\varepsilon)n} - A_{n/2q}\right) \to 0 \text{ as } n \to \infty.$$

□

Theorem 3.9. *Suppose rate function g is given by (3.6). Let $a = p/q < 1/2$, where p and q are relatively prime. For any small $\varepsilon > 0$, there exists M such that*

$$Z_n^* \ge (a-\varepsilon)n, \text{ for large } n.$$

Theorem 3.10. *Suppose rate function g is given by (3.6). Let $a = p/q < 1/2$, where p and q are relatively prime. For any small $\varepsilon > 0$, there exists $M,\ 0 < M < 1$ such that*

$$Z_n^* \le (a+\varepsilon)n, \text{ for large } n.$$

Proof. For any $l,\ 1 \le l \le n$. Let $\eta_0 = (an, \cdots, an, b, 0, \cdots, 0) \in \Omega_n$, where $n = \lfloor \frac{1}{a} \rfloor an + b,\ 0 \le b < an$. Consider $\eta_1,\ \eta_2 \in \Omega_n$ where

$$\eta_1 = (\underbrace{l, \cdots l}_{t}, d, 0, \cdots, 0),\ \eta_2 = (l, n-l, 0, \cdots, 0) \in \Omega_n, \tag{3.72}$$

with $n = t \cdot l + d,\ 0 \le d < l$. By Lemma 3.1, $\mu_n(\eta_1) \ge \mu_n(\eta)$ for any $\eta \in B_l$ and

$$\frac{\mu_n(\eta_2)}{\mu_n(\eta_1)} = \left(\frac{l!\,(n-l)!}{(l!)^t\, d!} \right)^\alpha = \left(\frac{(n-l)!}{(l!)^{t-1}\, d!} \right)^\alpha \ge 1, \tag{3.73}$$

since $n - l = (t-1)l + d$. Define a function $\phi : \Omega_n \to \{0, 1, 2, \cdots\}$ which counts the number of Ms contained in $\mu_n(\eta)$ as $\phi(\eta) = m$. Clearly $\phi(\eta_0) = n,\ \phi(\eta_2) < n$, since $l \ge (a+\varepsilon)n$. Thus there exists $\omega > 0$ such that

$$\phi(\eta_0) - \phi(\eta_2) = \omega n.$$

For $l \ge (a+\varepsilon)n$,

$$\begin{aligned} P\{Z_n \in B_l\} &= \nu_n(B_l) \\ &\le \frac{\mu_n(B_l)}{\mu_n(\eta_0)} \\ &\le \frac{n \binom{2n}{n} \mu_n(\eta_2)}{\mu_n(\eta_0)} \\ &\le p(n)\, 2^n\, M^{\phi(\eta_0) - \phi(\eta_2)} \left(\frac{l!\,(n-l)!}{(an)!^{\lfloor 1/a \rfloor} b!} \right)^\alpha \\ &\le p(n)\, 2^n\, M^{\omega n} \left(\frac{l!\,(n-l)!}{n!} \frac{n!}{(an)!^{\lfloor 1/a \rfloor} b!} \right)^\alpha \\ &\le p(n)\, M^{\omega n}\, \lambda^n, \end{aligned}$$

where $\lambda > 1$ is some constant and the degree of polynomial p is independent of n. Hence

$$P\{Z_n \ge (a+\varepsilon)n\} \ \le\ n\, p(n)\, M^{\omega n}\, \lambda^n. \tag{3.74}$$

If $M < (1/\lambda)^{1/\omega}$, the right hand side goes to zero as $n \to \infty$. This completes the proof. $\square$

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