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On Regularity and *drs*-invariant of a Homogeneous Cohen-Macauly Ring

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Abstract

We show that the (Castelnuovo-Mumford) regularity of a homogeneous Cohen-Macaulay ring agrees with some of the invariants defined the papers, and we study a ring of small index.

Keywords: Regularity, Generalized Loewy Length, Index, Invariants, Cohen-Macaulay Rings

1. Introduction

Koh and Lee show[1] that there are certain restrictions on the entries of the maps in the minimal free resolutions of finitely generated modules of infinite projective dimension over Noetherian local rings A.

From this fact, some of new invariants were studied[2]: They are $col(A)$ [resp. $row(A)$] for a number associated with the columns [resp. rows] of the maps, and $crs(A)$ and $drs(A)$, which are associated with the cyclic modules determined by regular sequences and their Matlis duals.

The purpose of this paper is to relate the (Castelnuovo-Mumford) regularity of a homogeneous ring to some of these invariants, $crs(A)$ and $drs(A)$.

Let A be a homogeneous ring, i.e., $A=\oplus_{n\in\mathbb{N}}A_n$ is a Noetherian N-graded ring such that $A_0=K$ is a field and $A=K[A_1]$. We show that all invariants considered in the paper[2] are bounded by $1+reg(A)$, where $reg(A)$ denotes the (Castelnuovo-Mumford) regularity which is, in general, much smaller than the multiplicity which we used to bound these invariants defined in the paper[2] when K is infinite. For example, a result in the paper[3] shows that $reg(A)\approx \deg(X)/codim(X)$ for arithmetically Buchsbaum varieties X in P^r. We also show that $drs(A)=\ell\ell(A)=reg(A)+1$ for homogeneous Cohen-Macaulay rings over an infinite field K. It follows that $reg(A)=red(A)$ in this case, where $red(A)$ denotes the reduction number of A. At the end of this article, we provide a ring of small index.

Although all rings we consider in this paper are commutative, Noetherian with identity, and all modules are unital, we emphasize the Noetherian property in our statements. We use the usual notation $E(A/\mathfrak{m})$ for the injective hull of $A/\mathfrak{m}$ and $M^\vee$ for Matlis dual, $Hom_A(-,E(A/\mathfrak{m}))$.

2. Regularity on Homogeneous Rings

We first recall the invariants defined in the paper[2], the generalized Loewy length $\ell\ell(A)$, and the reduction number $red(A)$ for a ring A. We also state the basic properties of these invariants.

Definition 2.1. Let $(A,\mathfrak{m})$ be a Noetherian local ring, and M a finite A-module. We use the usual notation $Soc(M):=Hom_A(A/\mathfrak{m},M)$ to denote the socle of M, and $M^\vee$ to denote the Matlis dual of M.

i) $crs(A)=\inf\{\, t\geq 1 : Soc(A/\boldsymbol{x})\not\subseteq \mathfrak{m}^t(A/\boldsymbol{x})$ for some maximal regular sequence $\boldsymbol{x}=x_1,\cdots,x_d\,\}$.

When A is Cohen-Macaulay,

ii) $drs(A) = \inf\{t\geq 1$: $Soc((A/(\boldsymbol{x}))^\vee)\not\subseteq \mathfrak{m}^t((A/(\boldsymbol{x}))^\vee)$ for some system of parameters $\boldsymbol{x}=x_1,\cdots,x_d\,\}$.

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We have the following propositions:

Proposition 2.2.([2,Proposition1.4]) Let $(A,\mathfrak{m})$ be a Cohen-Macaulay ring of dimension n. Then

(1) $drs(A) = inf\{t \geq 1 : \mathrm{m}^t(A/(x)) = 0$ for some system of parameters $\boldsymbol{x} = x_1, \cdots, x_n\}$.

(2) $crs(A) \leq drs(A)$.

Proposition 2.3.([2,Proposition1.8]) Let $(A,\mathfrak{m})$ be a Noetherian local ring. Let x be a non zero-divisor of A. Then $crs(A) \leq crs(A/xA)$. If A is Cohen-Macaulay, then $drs(A) \leq drs(A/xA)$.

We recall the definition of the generalized Loewy length of A[4,5]:

$\ell\ell(A) = \inf\ \{t \geq 1$: $\mathfrak{m}^t \subseteq (\boldsymbol{x})$ for some system of parameters $\boldsymbol{x}\}$. By Proposition 2.2, it is easy to obtain \$$drs(A) = \ell\ell(A)$. Also, we immediately know by definitions that $\ell\ell(A) \leq 1 + red(A)$, where $red(A) = inf\{r : (x)\mathrm{m}^r = \mathrm{m}^{r+1}$ for a system of parameters $\boldsymbol{x}\}$ denotes the reduction number of A.

From now on, we consider only homogeneous rings. Let $A = \oplus_{n \in \mathbb{N}} A_n$ be a homogeneous ring, which is generated over A_0 by elements of degree 1, i.e., $A_0 = K$ is a field and $A = K[A_1]$. We show that all six invariants considered in the paper[2] are bounded by $reg(A)$, the regularity of A. We also show that $drs(A) = \ell\ell(A) = 1 + reg(A)$ if A is Cohen-Macaulay and K is infinite. We first recall the definition of regularity of a finitely generated graded module M over a polynomial ring $S = K[x_1, \cdots, x_r]$ with the natural grading (see the paper[6] for detail).

For simplicity of notation, we assume that M is generated by elements of degree ≥ 0. Let ℓ be a projective dimension of M over S, and express a minimal graded resolution $F_\bullet$ of M as

$$F_\bullet : 0 \to \oplus_{0 \leq j \leq q} S(-\ell - j)^{e_{\ell j}} \to \cdots$$
$$\to \oplus_{0 \leq j \leq q} S(-i-j)^{e_{ij}} \to \cdots$$
$$\to \oplus_{0 \leq j \leq q} S(-j)^{e_{0j}} \to 0$$

For each $0 \leq i \leq \ell$, we define $\rho_i(M) = \max\{j : e_{ij} \neq 0\}$. The regularity of M is defined by

$$reg(M) = \max\{\rho_i(M) : 0 \leq i \leq \ell\}.$$

We point out that since $e_{ij} \neq 0$ if and only if $Tor_i^S(K,M)_{i+j} \neq 0$, our description is equivalent to the usual definition

$reg(M) = inf\{t \geq 1 : \mathrm{Tor}_i^S(K,M)_{i+j} = 0$ for all $0 \leq i \leq \ell$ and $j > t\}$.

We note that

Lemma 2.4. With notation as above, the following properties are true:

(1) $\rho_\ell(M) = -\min\{\deg(v) :\ v$ is a part of a minimal generators of $Ext^{\ell}{}_S(M, S(-\ell))\}$.

(2) $reg(M) = reg(M/xM)$ if x is a non zero-divisor of M and $\deg(x) = 1$.

Proof. We have $Hom_S(\oplus_{0 \leq j \leq q} S(-\ell-j)^{e_{\ell j}}, S(-\ell)) \cong \oplus_{0 \leq j \leq q} S(j)^{e_{\ell j}}$, and (1) follows from this isomorphism. The part (2) follows from the fact that $F_\bullet \otimes_S S/xS$ is a minimal graded resolution of M/xM. ■

We recall the Graded Local Duality over \$S\$ (see the paper[7]):

Graded Local Duality. Let $S = K[x_1, \cdots, x_r]$ be a polynomial ring with a natural grading. Let M be a finitely generated graded S-module. Then for all $0 \leq i \leq r$, $Hom_K(Ext_S^i(M, S(-r)), K) \cong H^{r-i}{}_{\mathfrak{m}}(M)$.

Before we prove our main theorem, we recall one more fact: If $(A, \mathfrak{m}, k)$ is a complete Noetherian local ring with a residue field k, and M a finitely generated A-module, then $\mu(M) = \dim_k Soc(M^\vee)$, where $\mu(M)$ is the minimal number of generators of M, and \$M^$M^\vee$ means the Matlis dual of M. The proof of this fact is quite elementary. Indeed, let $\mu(M) = m$ for some positive integer m. Then we have a surjection $A^m \to M$, and so we have an injection $M^\vee \to E(k)^m$ since $A^\vee = E(k)$. This implies that $\dim_k Soc(M^\vee) \leq m = \mu(M)$. On the other hand, since M is finitely generated, its Matlis dual $M^\vee$ is Artinian. Thus we have an injection $M^\vee \to E(k)^t$, where $t = \dim_k Soc(M^\vee)$. Applying $Hom_k(-, E(k))$, we have a surjection $A^t \to M$ since $M^{\vee\vee} = M$ and $E(k)^\vee = A$. Thus we have $\mu(M) \leq t = \dim_k Soc(M^\vee)$, and hence $\mu(M) = \dim_k Soc(M^\vee)$. ■

Now, we state and prove our following main theorem:

Theorem 2.5. Let $S=K[x_1, \cdots, x_r]$ and let $A=S/I$, where K is a field, and I is a homogeneous ideal. Then

(1) $crs(A) \leq 1+reg(A)$.

(2) If A is Cohen-Macaulay, then $drs(A) \leq 1+reg(A)$.

(3) If A is Cohen-Macaulay and K is infinite, then $drs(A)=1+reg(A)$, and $reg(A)=red(A)$.

Proof. (1) If x is a non zero-divisor of A, then $crs(A) \leq crs(A/xA)$ by Proposition 2.3. By Lemma 2.4(2) above, we may assume that $depth(A)=0$, and hence $projdim_S A=r$ by Auslander-Buchsbaum formula. By a graded local duality, we know

$$Hom_K(Ext_S^r(A, S(-r)), K) \cong H^0{}_{\mathfrak{m}}(A).$$

Also by Lemma 2.4(1) and a fact stated above, we have

$\rho_r(A)$

$=-\min\{\deg(v):$ v is a part of a minimal generators of $Ext_S^{\ell}(A, S(-r))$

$=\max\{\deg(v):$ v is a part of a minimal generators of $Soc(H_{\mathfrak{m}}^0(A))\}$

$=\max\{t: H^0{}_{\mathfrak{m}}(A)_t \neq 0\}$.

We note that $H^0{}_{\mathfrak{m}}(A)=\cup_i Hom_A(A/\mathfrak{m}^i, A)$ and $crs(A)=inf\{i: Soc(A) \not\subseteq \mathfrak{m}^i\}$, and so $\max\{t: H^0{}_{\mathfrak{m}}(A)_t \neq 0\} > crs(A)$. Since $reg(A) \geq \rho_r(A)$, we have the inequality $crs(A) \leq 1+reg(A)$.

For (2), we may also assume that $depth(A)=0$ since $drs(A) < drs(A/xA)$ for a nonzero divisor x of A by Proposition 2.2, and so if A is Cohen-Macaulay, we have $\dim(A)=depth(A)=0$, which implies $H^{r}{}_{\mathfrak{m}}(A)=A$. Thus $reg(A) \geq \max\{t: A_t \neq 0\}=\ell\ell(A)-1$ by the definition of $\ell\ell(A)$. Since $\ell\ell(A)=drs(A)$ when A is Cohen-Macaulay, we have $drs(A) \leq 1+reg(A)$.

To prove (3,) we first note that if K is infinite, then there is a homogeneous system of parameter $\boldsymbol{x}$ of A of degree 1 such that $\ell\ell(A)=\ell\ell(A/(\boldsymbol{x}))$ by Fact 2.6 below. Let $\overline{A}=A/\boldsymbol{x}$. Since A is Cohen-Macaulay, $\dim(\overline{A})=depth(\overline{A})=0$, and we have $reg(\overline{A})=\max\{t: \overline{A}_t \neq 0\}$ has finite length (see the book for Exercise 20.18[6]). Hence by Lemma 2.4, $reg(A)+1 = reg(\overline{A})+1 = \max\{t: \overline{A}_t \neq 0\}+1 = \ell\ell(\overline{A}) = \ell\ell(A) = drs(A)$, and $reg(A)=red(A)$ by Remark 2.6 below. ■

Remark 2.6.[8,Lemma2.2] Let A be a homogeneous Cohen-Macaulay ring over an infinite field K. For each homogeneous system of parameters $\boldsymbol{x}=x_1, \cdots, x_d$ consisting of forms of degree 1, $\ell\ell(A) = \ell\ell(A/\boldsymbol{x}) = 1+red(A)$, i.e., $\ell\ell(A)$ is attained for any system of parameters which is generated by elements of degree 1.

Remark 2.7. In the paper[9], he raised a question if $\ell\ell(A)$ is attained by a system of parameters that generates a reduction of $\boldsymbol{m}$ or not, for a Gorenstein local ring $(A, \boldsymbol{m})$ with an infinite residue field. Remark 2.6 shows that this question is true for a homogeneous Cohen-Macaulay K-algebra, where K is an infinite field. However, the question is not necessarily true for a Cohen-Macaulay local ring. For example, if $R=k[[t^e, t^{e+1}, t^{(e-1)e-1}]]$, then R is a one dimensional Cohen-Macaulay local ring and it is known that $crs(A)=2$, and $\ell\ell(R)=e-1=red(R)$[10,11].

We now recall the definition of $index(A)$ for a ring A. To define the index of a ring A, we recall the Auslander δ-invariant[4,12,13]: Let $(A, \boldsymbol{m})$ be a Cohen-Macaulay local ring with a canonical module ω. For a finitely generated A-module X, define $f-rank(X)=r$ if $X=A^r \oplus U$, where U has no free summands. Then $\delta(M)$ is defined as $\delta(M)=\inf\{f-rank(X)$: X is a maximal Cohen-Macaulay module and M is a homomorphic image of $X\}$. Then $index(A) = \inf\{\ t \geq 1: \delta(A/\boldsymbol{m}^t)>0\ \}$. It was shown in the paper[5] that $index(A)<\infty$ if and only if A is isolated non-Gorenstein, i.e., $A_{\boldsymbol{p}}$ is Gorenstein for all non maximal ideals $\boldsymbol{p}$ of A. For such rings, we note that if $\boldsymbol{m}^t \subseteq (\boldsymbol{x})$ for some system of parameters $\boldsymbol{x}=x_1, \cdots, x_d$, then $\delta(A/\boldsymbol{m}^t) \geq \delta(A/(\boldsymbol{x}))>0$ and so $index(A) \leq \ell\ell(A)$.

We say that a ring A is of minimal multiplicity if $mult(A)=1+edim(A)-\dim(A)$, where $mult(A)$ denotes the multiplicity of A, and $edim(A)$ denotes an embedding dimension of A.

Corollary 2.8. Let A be a homogeneous Cohen-Macaulay ring over an infinite field K

(1) A is of minimal multiplicity if and only if $reg(A)=1$.

(2) If A is isolated non-Gorenstein, then A is of minimal multiplicity if and only if $index(A)=2$.

Proof. The part (1) follows from Theorem 3.1 (3) and the paper[2,Corollary3.7]. The part (2) follows from (1) because $index(A) \le \ell\ell(A)$. ■

We close this section with an following example of small index: Let X be a nonsingular irreducible projective curve of genus g over an algebraically closed field K. Let L be a base point free line bundle on X and let $\varphi_L : X \to P(H^0(X,L)^*)$ be the morphism defined by L. Then there is a natural inclusion map $(*)\quad S/I \to \oplus_{n\in \mathbf{Z}} H^0(X, L^{\otimes n})$, where $S = Sym H^0(X,L)$ and I is the ideal of $\varphi_L(X) \subset P(H^0(X,L)^*)$. Let $A = S/I$, the homogeneous coordinate ring of $\varphi_L(X)$.

Proposition 2.9. With notation as above, $index(A)=3$, and $reg(A)=2$.

Proof. By Mumford[14] and others (see the papers[15,16]), if $\deg(L) \ge 2g+1$, then the map in (*) is an isomorphism, i.e. $A = \oplus_{n\in \mathbf{Z}} H^0(X, L^{\otimes n})$, which is a 2-dimensional normal domain. We now recall 'Duality Theorem'(see the papers[17,18]): Let $V \subseteq H^0(X,L)$ be a base point free linear series. Let $T = Sym V$ and $v = \dim(V)$. Then for any line bundle N on X,

$$
\begin{aligned}
&Tor_i^T(\oplus_{n\in \mathbf{Z}} H^0(X, L^{\otimes n}\otimes N), K)_j \\
&\cong Tor_{v-2-i}^T(\oplus_{n\in \mathbf{Z}} H^0(X, L^{\otimes n}\otimes N^{-1}\otimes \omega), K)_{v-j},
\end{aligned}
$$

where ω is the canonical bundle on X.

Applying Duality Theorem with $V = H^0(X,L)$ and $N = O_X$, the structure sheaf, we see that $reg(A) \le 2$ because $\oplus_{n\in \mathbf{Z}} H^0(X, L^{\otimes n}\otimes \omega)$ is generated by elements of degree ≥ 0. If we assume $g>0$ to exclude the rational normal curves (which is the only curve among the varieties of minimal degree), $reg(A)=2$. Since A is Cohen-Macaulay, we get $drs(A) = \ell\ell(A) = 3$ and $red(A)=2$. Since A is isolated non-Gorenstein, $index(A)$ is finite, and $index(A) \le \ell\ell(A)$. Since $reg(A)=2$ which implies that A is not of minimal multiplicity by Corollary 2.8, we know $index(A) > 2$, and so $index(A) = 3 = \ell\ell(A)$. ■

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