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COMBINED LAPLACE TRANSFORM WITH ANALYTICAL METHODS FOR SOLVING VOLTERRA INTEGRAL EQUATIONS WITH A CONVOLUTION KERNEL

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ABSTRACT. In this article, a homotopy perturbation transform method (HPTM) and the Laplace transform combined with Taylor expansion method are presented for solving Volterra integral equations with a convolution kernel. The (HPTM) is innovative in Laplace transform algorithm and makes the calculation much simpler while in the Laplace transform and Taylor expansion method we first convert the integral equation to an algebraic equation using Laplace transform then we find its numerical inversion by power series. The numerical solution obtained by the proposed methods indicate that the approaches are easy computationally and its implementation very attractive. The methods are described and numerical examples are given to illustrate its accuracy and stability.

1. INTRODUCTION

In recent years, many different methods have been used to approximate the solution of Volterra integral equations of convolution kernel [1, 4, 18, 21]. Block-pulse functions (BPFs) have been studied and applied for solving different problems [2, 17]. Integral equations have many applications in various areas, including mathematical physics, electrochemistry, chemistry, semi-conductors, heat conduction, seismology, metallurgy, fluid flow, scattering theory, chemical reaction and population dynamics [6, 9, 19]. Numerical methods are available for approximating the Volterra integral equation and Abel integral equation. In particular, Yang [20] proposed a method for the solution of integral equation using the Chebyshev polynomials, Hamoud and Ghadle [5] used the reliable modified of Laplace Adomian decomposition method to solve nonlinear interval Volterra-Fredholm integral equations. Huang [7] used the Taylor expansion of unknown function and obtained an approximate solution. Khodabin [13] numerically solved the stochastic Volterra integral equations using triangular functions and

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their operational matrix of integration. Yousefi [22] presented a numerical method for the Abel integral equation by Legendre wavelets. Kamyad [11] proposed a new algorithm based on the calculus of variations and discretisation method. While the homotopy perturbation method was proposed first by J. He in 1998 and was further developed and improved by himself in 2000. [7, 8]. The HPTM has been successfully applied by many researchers for solving differential equations and integral equation [3, 16]. Recently, many authors have paid attention to study the solutions of linear and nonlinear integral equation and differential equation by using various methods with combined the Laplace transform. Among these are homotopy perturbation transform method [14, 15] and the Laplace decomposition methods [10, 12]. In this article we consider Volterra integral equation with a convolution kernel given by

$$u(x) = f(x) + \int_0^x k(x-t)u(t)dt, \quad 0 \le x \le T, \tag{1.1}$$

where the source function f and the kernel function k are given, and $u(x)$ is the unknown function. The article is organized as follows: In Section 2, we present the introduction of Laplace transform and its properties. In Section 3, we describe the solution of Eq. (1.1) by using Laplace transform and Taylor expansion method. Section 4 is devoted to the solution of Eq. (1.1) by using the (HPTM) method. In Section 5, we report our numerical findings and demonstrate the accuracy of the proposed methods by considering numerical examples. Section 6 ends this article with a brief conclusion.

2. Preliminaries

We begin our article by giving the definition of Laplace transform and its properties [21], the convolution theorem and the Volterra integral equations which will be used in this article.

Definition 2.1. *The Laplace transform of a function $f(x)$ which is defined for all $x \ge 0$, is :*

$$\pounds[f(x)] = F(s) = \int_0^{+\infty} e^{-sx} f(x)dx, \tag{2.1}$$

for all values of s for which the improper integral converges. The Laplace transform has several properties, as explained below:
1) Linearity Property

$$\pounds[af(x) + bg(x)] = a\pounds[f(x)] + b\pounds[g(x)], \tag{2.2}$$

where a, b are constants. ***2) The Convolution Theorem***
The Laplace transforms for the functions $f(x)$ and $g(x)$ be given by

$$\pounds[f(x)] = F(s), \quad \pounds[g(x)] = G(s).$$

Then the Laplace convolution product of these two functions is defined by

$$\pounds\left[\int_0^x f(x-t)g(t)dt\right] = F(s)G(s),$$

Theorem 2.2. [21] *Suppose $F(s)$ is the Laplace transform of $f(x)$, which has a Maclaurin power series expansion in the form*

$$f(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}.$$

Applying the Laplace transform, it is possible to be written formally

$$F(S) = \sum_{i=0}^{\infty} \frac{a_i}{s^{i+1}}.$$

3. Solution of Volterra integral equation by using Laplace transform and Taylor series

First, the Laplace transform is applied to both sides of Eq. (1.1)

$$\pounds[u(x)] = \pounds[f(x)] + \pounds[\int_0^x k(x-t)u(t)dt].$$

By using the Laplace transform property (2) the equation below can be obtained

$$\pounds[u] = \pounds[f] + \pounds[k]\pounds[u].$$

Thus, the given equation is equivalent to

$$\pounds[u] = \frac{\pounds[f]}{1 - \pounds[k]} = F(s).$$

Applying Theorem (2.2), $F(s)$ can be expanded as an absolutely convergent series, which is given by

$$\pounds[u] = \frac{c_1}{s} + \frac{c_2}{s^2} + \frac{c_3}{s^3} + ...$$

where $c_1, c_2, c_3, ...$ are the known constants. Taking the inverse Laplace transform on both sides of the above equation, we can obtain

$$u(x) = c_1 + \frac{c_2}{\Gamma(2)}x + \frac{c_3}{\Gamma(3)}x^2 + \frac{c_4}{\Gamma(4)}x^3 + ...$$

which is uniformly convergent to the exact solution. So we approximate the solution $u(x)$ by using

$$u(x) = c_1 + \frac{c_2}{\Gamma(2)}x + \frac{c_3}{\Gamma(3)}x^2 + ... + \frac{c_n}{\Gamma(n)}x^{n-1},$$

with the error function $e_n = u(x) - u_n(x)$.

4. Solution of Volterra integral equation by using Homotopy perturpation transform method :

We solved spatial case from Eq. (1.1) namely singular integral equation of Abel type, so we consider the following Abel's integral equation of second kind as

$$u(x) = f(x) + \int_0^x \frac{u(t)}{\sqrt{x-t}}dt, \quad 0 \le x \le 1, \tag{4.1}$$

Applying the Laplace transform on both sides in Eq. (4.1), we get

$$\pounds[u(x)] = \pounds[f(x)] + \pounds[\int_0^x \frac{u(t)}{\sqrt{x-t}}dt], \tag{4.2}$$

By using the Laplace transform convolution property, Eq. (4.2) becomes

$$\pounds[u(x)] = \pounds[f(x)] + \sqrt{\frac{\pi}{s}}\pounds[u(x)], \tag{4.3}$$

Applying the inverse Laplace transform on both sides in Eq. (4.3), we get

$$u(x) = f(x) + \pounds^{-1}\left\{\sqrt{\frac{\pi}{s}}\pounds[u(x)]\right\}.$$

Abel integral Eq. (4.1) has the solution in the following series form as

$$\psi(x) = \sum_{n=0}^{\infty} p^n \psi_n(x), \tag{4.4}$$

where $\psi_n(x), n = 0, 1, 2, 3, ...$ are functions to be determined. We use the following iterative scheme to evaluate $\psi_n(x)$.
By using HPTM to solve Eq. (4.1), we consider the following convex homotopy

$$\sum_{n=0}^{\infty} p^n \psi_n(x) = f(x) + p\left\{\pounds^{-1}\left(\sqrt{\frac{\pi}{s}}\pounds\left(\sum_{n=0}^{\infty} p^n \psi_n(x)\right)\right)\right\} \tag{4.5}$$

This is coupling of the Laplace transform and homotopy perturbation Method. Now, equating the coefficient of corresponding power of p on both sides, the following approximations are obtained as:

$$p^0 : \psi_0(x) = f(x), \qquad p^n : \psi_n(x) = \pounds^{-1}\left\{\sqrt{\frac{\pi}{s}}\pounds\left(\psi_{n-1}(x)\right)\right\} \qquad n = 1, 2, 3, ... \tag{4.6}$$

Hence the solution of the Eq. (4.1) is given as

$$u(x) = \lim_{p\to 1} \psi(x) = \sum_{n=0}^{\infty} \psi_n(x). \tag{4.7}$$

It is worth to note that the major advantage of homotopy perturbation transform method is that the perturbation equation can be freely constructed in many ways (therefore problem is dependent) by homotopy in topology and the initial approximation can also be freely selected.

It is to be noted that the rate of convergence of the series representing the solution in Eq. (4.7) depends upon the initial choices $\psi_0(x)$

5. NUMERICAL EXAMPLES:

In this section we shall demonstrate the effectiveness of the proposed methods by several examples. All the results are calculated using the symbolic calculus software MATLAB.

Example 1. Consider the Abel integral equation of second kind

$$u(x) = 2\sqrt{x} - \int_0^x \frac{u(t)}{\sqrt{x-t}}dt, \qquad 0 \leq x \leq 1 \tag{5.1}$$

with the exact solution $u(x) = 1 - e^{\pi x} erfc(\pi x)$, where the complimentary error function $erfc(\pi x)$ defined as

$$erfc(\pi x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt.$$

a) Applying Laplace transform and Taylor series

Using the Laplace transform and convolution properties, we get

$$\pounds[u] = \pounds[2\sqrt{x}] - \pounds[x^{-\frac{1}{2}}]\pounds[u].$$

Hence,

$$\frac{\pounds[2\sqrt{x}]}{1 + \pounds[x^{-\frac{1}{2}}]} = \pounds[u],$$

Or equivalently,

$$F(s) = \frac{\sqrt{\pi}}{s(\sqrt{s} + \pi)} = \pounds[u].$$

The left hand side of $F(s)$ in the power of $\frac{1}{s}$ expanded as in

$$\begin{aligned} F(s) &= \pi^{\frac{1}{2}}(\frac{1}{s})^{\frac{3}{2}} - \pi(\frac{1}{s})^2 + \pi^{\frac{3}{2}}(\frac{1}{s})^{\frac{5}{2}} - \pi^2(\frac{1}{s})^3 \\ &\quad + \pi^{\frac{5}{2}}(\frac{1}{s})^{\frac{7}{2}} - \pi^3(\frac{1}{s})^4 + \pi^{\frac{7}{2}}(\frac{1}{s})^{\frac{9}{2}} - \pi^4(\frac{1}{s})^5 ... \end{aligned} \tag{5.2}$$

Now applying the inverse Laplace transform to (5.2), we obtain

$$\begin{aligned} u(x) &= 2x^{\frac{1}{2}} - \pi x + \frac{4\pi}{3}x^{\frac{3}{2}} - \frac{\pi^2}{2}x^2 + \frac{8\pi^2}{15}x^{\frac{5}{2}} \\ &\quad - \frac{\pi^3}{6}x^3 + \frac{16\pi^3}{105}x^{\frac{7}{2}} - \frac{\pi^4}{24}x^4 ... \end{aligned} \tag{5.3}$$

b) Applying (HPTM)

Applying the aforesaid homotopy perturbation transform method, we get

$$\sum_{n=0}^{\infty} p^n \psi_n(x) = 2\sqrt{x} - p\left\{ \pounds^{-1}\left(\sqrt{\frac{\pi}{s}}\pounds\left(\sum_{n=0}^{\infty} p^n\psi_n(x)\right)\right)\right\} \tag{5.4}$$

The various $\psi_n(x), n = 0, 1, 2, 3, ...$ are given as

$$\begin{aligned} \psi_0(x) &= 2\sqrt{x}, \qquad \psi_1(x) = -\pi x, \\ \psi_2(x) &= \frac{4\pi}{3}x^{\frac{3}{2}}, \qquad \psi_3(x) = -\frac{\pi^2}{2}x^2, \\ \psi_4(x) &= \frac{8\pi^2}{15}x^{\frac{5}{2}}, ... \end{aligned} \tag{5.5}$$

Hence the solution of the given problem Eq. (5.1) is given as

$$\begin{aligned} u(x) &= \sum_{n=0}^{\infty} \psi_n(x) \\ &= 2\sqrt{x} - \pi x + \frac{4\pi}{3}x^{\frac{3}{2}} - \frac{\pi^2}{2}x^2 + \frac{8\pi^2}{15}x^{\frac{5}{2}} - \frac{\pi^3}{6}x^3 + ... \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(\pi x)^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})} = 1 - E_{\frac{1}{2}}(-\sqrt{\pi x}) \\ &= 1 - e^{\pi x} erfc(\sqrt{\pi x}). \end{aligned} \tag{5.6}$$

This is the exact solution of the Abel integral Eq. (5.1)

Example 2. Consider the singular Volterra integral equation

$$u(x) = x^2 + \frac{16}{15}x^{\frac{5}{2}} - \int_0^x \frac{u(t)}{\sqrt{x-t}}dt, \quad 0 \leq x \leq 1, \tag{5.7}$$

with exact solution $u(x) = x^2$. A homotopy perturbation transform method can be constructed as follows

$$\sum_{n=0}^{\infty} p^n \psi_n(x) = x^2 + \frac{16}{15}x^{\frac{5}{2}} - p\left\{ \pounds^{-1}\left(\sqrt{\frac{\pi}{s}}\pounds\left(\sum_{n=0}^{\infty} p^n\psi_n(x)\right)\right)\right\} \tag{5.8}$$

giving various $\psi_n(x), n = 0, 1, 2, 3, ...$ as follows

$$\begin{aligned} \psi_0(x) &= x^2 + \frac{16}{15}x^{\frac{5}{2}}, \qquad \psi_1(x) = -\frac{16}{15}x^{\frac{5}{2}} - \frac{\pi}{3}x^3, \\ \psi_2(x) &= \frac{\pi}{3}x^3 + \frac{32\pi}{105}x^{\frac{7}{2}}, \qquad \psi_3(x) = -\frac{\pi^2}{12}x^4 - \frac{32\pi}{105}x^{\frac{7}{2}}, ... \\ \psi_{31}(x) &= -\frac{524288\pi^{15}x^{\frac{35}{2}}}{221643095476699771875} - \frac{\pi^{16}}{3201186852864000}x^{18}. \end{aligned} \tag{5.9}$$

Hence the solution of the Eq. (5.7) is given as

$$u(x) = \lim \psi(x) = \sum_{i=0}^{\infty} \psi_n(x) = \sum_{i=0}^{n} \psi_n(x) + O(x^{3+\frac{n}{2}}) \to x^2 \quad as \quad n \to \infty. \tag{5.10}$$

TABLE 1. Comparison between exact and approximate solution for Example 2.

x	$u_{Exact}(x)$	$u_{Appr.}(x)$	$E_{31}(u)$
0.2	0.04	0.0399999999999999999	7.37271×10^{-21}
0.4	0.16	0.1599999999999998067	1.93271×10^{-15}
0.6	0.36	0.3599999999997143663	2.85634×10^{-12}
0.8	0.64	0.6399999999493351566	5.06648×10^{-10}
1.0	1.00	0.9999999971875362176	2.81246×10^{-8}

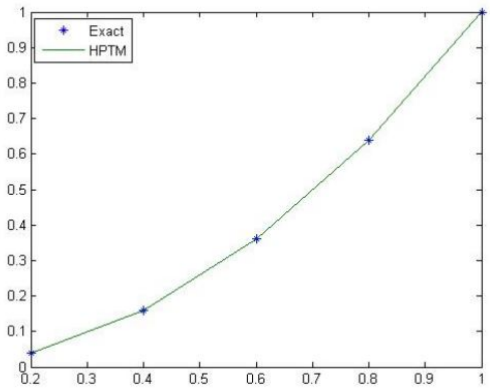

FIGURE 1. Comparison of approximate and exact solutions for Example 2.

Fig. 1 shows the comperison between the exact solution and the approximate solution obtained by the proposed method. It is seen from Fig. 1 that the solution obtained by the proposed method nearly identical to the exact solution. The accuracy of the result can be improved by introducing more terms of the approximate solutions. The HPTM solution is compared with the exact solution of the Abel integral equation at the different value of x in Table 1. for Example 2.

Example 3. Consider the Volterra integral with a convolution kernel

$$u(x) - \sin x = -\int_0^x \cos(x-t)u(t)dt. \tag{5.11}$$

which has $u(x) = \frac{2\sqrt{3}}{3}\sin(\frac{\sqrt{3}x}{2})e^{\frac{-x}{2}}$ as exact solution. Taking the Laplace transforms on both sides of (5.11) we get

$$\pounds[u(x)] - \pounds[\sin x] = -\pounds\left[\int_0^x \cos(x-t)u(t)dt\right]. \tag{5.12}$$

Applying the Laplace convolution product of two function the above Eq. becomes

$$\pounds[u(x)] - \pounds[\sin x] = -\pounds[\cos(x)]\pounds[u(x)], \tag{5.13}$$

which provides

$$\pounds[u(x)] = \frac{\pounds[\sin x]}{1+\pounds[\cos x]} = \frac{1}{s^2+s+1}. \tag{5.14}$$

Now expanding in power of $\frac{1}{s}$ the right hand side of Eq.(5.14) we obtain

$$\pounds[u] = \frac{1}{s^2} - \frac{1}{s^3} + \frac{1}{s^5} - \frac{1}{s^6} + \frac{1}{s^8} - \frac{1}{s^9} + \frac{1}{s^{11}}...$$

By taking the inverse Laplace transform to the above equation, we obtain

$$u(x) = x - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \frac{x^8}{8!} + \frac{x^{10}}{10!}\cdots. \tag{5.15}$$

which is convergent to the exact solution

Example 4. Consider the singular Volterra integral equation

$$u(x) = \sqrt{x} + \frac{\pi x}{2} + \int_0^x \frac{u(t)}{\sqrt{x-t}}dt, \qquad 0 \le x \le 1 \tag{5.16}$$

which has $u(x) = \sqrt{x}$ as the exact solution.

A homotopy perturbation transform method can be constructed as follows (from Eq. (5.16):

$$\sum_{n=0}^{\infty} p^n\psi_n(x) = \sqrt{x} + \frac{\pi x}{2} - p\left\{\pounds^{-1}\left(\sqrt{\frac{\pi}{s}}\pounds\left(\sum_{n=0}^{\infty} p^n\psi_n(x)\right)\right)\right\} \tag{5.17}$$

The various iterates $\psi_n(x), n = 0, 1, 2, 3, ...$ are given as:

$$\begin{aligned}
\psi_0(x) =& \sqrt{x} + \frac{\pi x}{2}, \\
\psi_1(x) =& -\frac{\pi x}{2} - \frac{2}{3}\pi x^{\frac{3}{2}}, \\
\psi_2(x) =& \frac{2}{3}\pi x^{\frac{3}{2}} + \frac{1}{4}\pi^2 x^2, \\
\psi_3(x) =& -\frac{1}{4}\pi^2 x^2 - \frac{4}{15}\pi^2 x^{\frac{5}{2}}, ... \\
\psi_{31}(x) =& -\frac{65536\pi^{16}x^{\frac{33}{2}}}{6332659870762850625} - \frac{\pi^{16}}{39916800}x^{16}
\end{aligned} \tag{5.18}$$

Hence, the solution of the Eq.(5.16) is given as

$$u(x) = \lim_{p} \to 1\psi(x) = \sum_{i=0}^{\infty} \psi_n(x) \to \sqrt{x} \quad as \quad n \to \infty. \tag{5.19}$$

TABLE 2. Comparison between exact and approximate solutions for example 4

x	$u_{Exact}(x)$	$u_{Appr.}(x)$	$E_{31}(u)$
0.2	0.447213595499957939	0.447213595499957936	2.73078×10^{-18}
0.4	0.632455532033675866	0.632455532033422772	2.53094×10^{-13}
0.6	0.774596669241483377	0.774596669037878881	2.03604×10^{-10}
0.8	0.894427190999915856	0.894427167542697200	2.34572×10^{-8}
1.0	1.0	0.999999068266455868	9.31734×10^{-7}

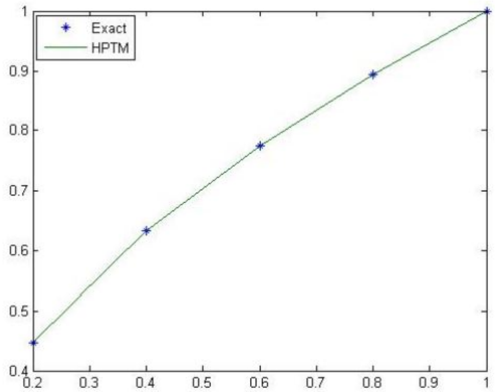

FIGURE 2. Comparison of approximate and exact solutions for Example 4

Fig. 2 shows the comperison between the exact solution and the approximate solution obtained by the HPTM. It is seen from Fig. 2. the solution obtained by the proposed method nearly identical to the exat solution. The above result is in complete agreement with Pandey et al.[21]. In this example, the simplicity and accuracy of the proposed method is illustrated by computing the absolute error $E_{31}(x) = u_{exact}(x) - u_{appr.}(x)$ for the Example 4.The accuracy of the result can be improved by introducing more terms of the approximate solutions. In Table 2. HPTM solution are compared with the exact solution of the Abel integral Eq. (5.16). There is good agreement between exact and approximate solution obtained by proposed method. The table also shows the absolute error between the exact and approximate solution.

Example 5. Consider the singular Volterra integral equation

$$u(x) = x + \frac{4}{3}x^{\frac{3}{2}} - \int_0^x \frac{u(t)}{\sqrt{x-t}}dt, \qquad 0 \le x \le 1. \tag{5.20}$$

which has x as the exact solution.

A homotopy perturbation transform method can be constructed as

$$\sum_{n=0}^{\infty} p^n \psi_n(x) = x + \frac{4}{3}x^{\frac{3}{2}} - p\left\{ \pounds^{-1}\left(\sqrt{\frac{\pi}{s}}\pounds\left(\sum_{n=0}^{\infty} p^n \psi_n(x))\right)\right)\right\} \tag{5.21}$$

The various iterates $\psi_n(x), n = 0, 1, 2, 3, ...$ are given as:

$$\begin{aligned} \psi_0(x) &= x + \frac{4}{3}x^{\frac{3}{2}}, \qquad \psi_1(x) = -\frac{4}{3}x^{\frac{3}{2}} - \frac{\pi}{2}x^2, \\ \psi_2(x) &= \frac{\pi}{2}x^2 + \frac{8\pi}{15}x^{\frac{5}{2}}, \qquad , \psi_3(x) = -\frac{8\pi}{15}x^{\frac{5}{2}} - \frac{\pi^2}{6}x^3, ... \\ \psi_{31}(x) &= -\frac{131072\pi^{15}}{6332659870762850625}x^{\frac{33}{2}} - \frac{\pi^{16}}{355687428096000}x^{17} \end{aligned} \tag{5.22}$$

Hence, from Eq. (4.7), the solution is

$$u(x) = \lim \psi(x) = \sum_{i=0}^{\infty} \psi_n(x) = \sum_{i=0}^{n} \psi_n(x) + O(x^{2+\frac{n}{2}}) \rightarrow x \quad as \quad n \rightarrow \infty \tag{5.23}$$

TABLE 3. Comparison between exact and approximate solutions for Example 5.

x	$u_{Exact}(x)$	$u_{Appr.}(x)$	$E_{31(u)}$
0.2	0.2	0.199999999999999999	3.31772×10^{-19}
0.4	0.4	0.399999999999956514	4.34860×10^{-14}
0.6	0.6	0.599999999957154945	4.28451×10^{-11}
0.8	0.8	0.799999994300205120	5.69979×10^{-9}
1.0	1.0	0.999999746878259586	2.53122×10^{-7}

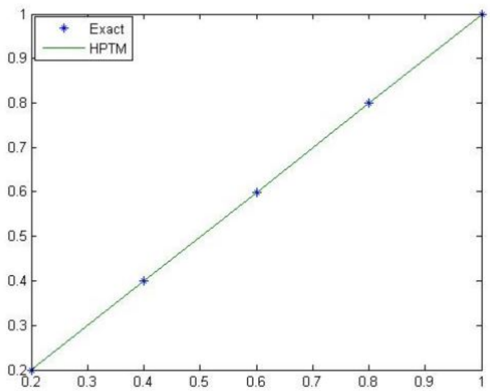

FIGURE 3. Comparison of approximate and exact solutions for Example 5.

6. Conclusion

In this article, we applied Laplace transform with Taylor Series and homotopy perturbation transform methods to solve the Volterra integral equation with a convolution kernel. From the above examples, it is obvious that our proposed methods give the same approximate solutions, and they are employed to obtain quick and accurate solution of the integral equation of a convolution type. The methods require much less computational work compared with traditional methods. We observed that our developed mechanism is straight forward and easy to apply. The proposed approaches can be further implemented to solve other linear and nonlinear problems arising in science and engineering. Numerical results show that the methods are working well and the accuracy is comparable with exact solutions.

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