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FINITE ELEMENT DUAL SINGULAR FUNCTION METHODS FOR HELMHOLTZ AND HEAT EQUATIONS

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ABSTRACT. The dual singular function method(DSFM) is a numerical algorithm to get optimal solution including corner singularities for Poisson and Helmholtz equations. In this paper, we apply DSFM to solve heat equation which is a time dependent problem. Since the DSFM for heat equation is based on DSFM for Helmholtz equation, it also need to use Sherman-Morrison formula. This formula requires linear solver $n+1$ times for elliptic problems on a domain including n reentrant corners. However, the DSFM for heat equation needs to pay only linear solver once per each time iteration to standard numerical method and perform optimal numerical accuracy for corner singularity problems. Because the Sherman-Morrison formula is rather complicated to apply computation, we introduce a simplified formula by reanalyzing the Sherman-Morrison method.

1. INTRODUCTION

In order to deal with corner singularity solution, let the computational domain Ω be an open and bounded concave polygon in $\mathbb{R}^2$. We assume that Ω has one reentrant corner for simple explanation. It can be readily applied on multiple reentrant corner domain and is performed in §4. The goal of this paper is to construct the dual singular function method to solve heat equation:

$$\begin{aligned} u_t - \mu\triangle u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

with given initial value $u(t=0,x) = h(x)$ in Ω and thermal diffusivity $\mu > 0$. Although the given function f is very smooth, the solution of (1.1) have singular behavior near the reentrant corner. This corner singularity make lose accuracy of numerical solution throughout the whole domain. To overcome this difficulty, there are several primary ways. One is locally adaptive mesh refinement or moving mesh technique. Other ways are so called Singular Function Method (SFM) (see, e.g., [9]) and Dual Singular Function Method (DSFM) constructed in

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[1, 2, 7, 8]. DSFM for Poisson equation has been introduced in [3, 4] and then it is extended to solve Helmholtz equations in [10]:

$$\begin{aligned} -\mu\triangle u + ku &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega. \end{aligned} \tag{1.2}$$

Because it is known in [5] that the singular function of heat equation (1.1) can be written by a linear combination of that of Helmholtz equation (1.2), we can construct DSFM to solve heat equation (1.1) by combining a time discrete scheme and results in [10]. DSFM has to use Sherman-Morrison formula which requests linear solver 2 times for any elliptic problem, but DSFM for heat equation does not need linear solvers per time iteration cost to standard numerical algorithm. Because the Sherman-Morrison formula is rather complicated to apply real computation, we introduce a simplified version by reanalyzing the Sherman-Morrison formula.

This paper organized as follows. In section 2, we summarize the results of DSFM for Helmholtz equation in [10], and we establish the relationship between regular part of the solution and discrete singular functions by analyzing Sherman-Morrison formulation (see Theorem 2.4). We will construct DSFM to solve heat equation (1.1) in section 3 and present several numerical results in section 4.

2. Dual singular function method to solve Helmholtz equation

We denote $\partial\Omega = \Gamma_{in} \cup \Gamma_{out}$, where Γ_{in} is part of boundary including reentrant corner and $\Gamma_{out} = \partial\Omega - \Gamma_{in}$. Let ω be the internal angle of Γ_{in} satisfying $\pi < \omega < 2\pi$. The singular and the dual singular functions of Poisson equation are summarized in [3, 4] as

$$s_P(r,\theta) := r^{\pi/\omega}\Theta(\theta) \quad \text{and} \quad s_{Pd}(r,\theta) := r^{-\pi/\omega}\Theta(\theta), \tag{2.1}$$

where

$$\Theta(\theta) = c\cos(\alpha\theta) + d\sin(\alpha\theta). \tag{2.2}$$

We note that c and d will be determined to hold the boundary conditions of u near the corner on Γ_{in}. We introduce the following lemma in [3].

Lemma 2.1. *The functions s_P and s_{Pd} is harmonic functions.*

The singular function $s_H \in H^1(\Omega)$ and the dual singular function $s_{Hd} \in L^2(\Omega)$ of Helmholtz equation (1.2) have to satisfy

$$-\mu\triangle s_H + ks_H = 0 \quad \text{and} \quad -\mu\triangle s_{Hd} + ks_{Hd} = 0. \tag{2.3}$$

The functions are written by power series

$$s_H(r,\theta) = r^{\pi/\omega}\left(1 + \sum_{i=1}^{\infty} \frac{k^i}{\nu_i 4^i i! \prod_{m=1}^{i}(m+\alpha)} r^{2i}\right)\Theta(\theta) \tag{2.4}$$

and

$$s_{Hd}(r,\theta) = r^{-\pi/\omega}\left(1 + \sum_{i=1}^{\infty} \frac{k^i}{\nu_i 4^i i! \prod_{m=1}^{i}(m-\alpha)} r^{2i}\right)\Theta(\theta), \tag{2.5}$$

where $\Theta(\theta)$ is defined in (2.2). We note here $s_{Hd} \notin H^1(\Omega)$. Because r^β is in $H^2(\Omega)$ if $\beta \geq 1$, the singular parts of (2.4) and (2.5) are only

$$S(r,\theta) := r^{\pi/\omega}\Theta(\theta) \quad \text{and} \quad S_d(r,\theta) := r^{-\pi/\omega}\Theta(\theta). \tag{2.6}$$

So we can define singular and dual singular functions as in (2.1). We note here that $S(r,\theta)$ and $S_d(r,\theta)$ are not satisfying (2.3) and it is an obstacle to construct DSFM for Helmholtz equation.

We need to use a cut-off function to be 0 boundary condition, because $S(r,\theta)$ and $S_d(r,\theta)$ of (2.6) is not 0 on Γ_{out}. To do this, we use notation

$$B(r_1;r_2) = \{(r,\theta) : r_1 < r < r_2 \text{ and } 0 < \theta < \omega\} \cap \Omega.$$

We now define smooth cut-off function η_ρ as

$$\eta_\rho(r) = \begin{cases} 1 & \text{in } B(0;\frac{1}{2}\rho), \\ p(r) & \text{in } B(\frac{1}{2}\rho;\rho), \\ 0 & \text{in } \Omega\backslash B(0;\rho), \end{cases} \tag{2.7}$$

with $p(r)$ is a very smooth function and ρ is a parameter determining the range of $p(x)$. Then the solution u of (1.2) can be rewritten by

$$u = w + \alpha\eta_\rho S, \tag{2.8}$$

where α is called the stress intensity factor and $w \in H^2(\Omega)$ is regular part of solution. The main idea of DSFM is to find α and w instead of computing $u \notin H^2(\Omega)$. We start to construct DSFM with rewriting the Helmholtz equation by inserting (2.8) into (1.2)

$$\begin{aligned} -\mu\triangle w + kw + \alpha(-\mu\triangle\eta_\rho S + k\eta_\rho S) = f \quad &\text{in } \Omega, \\ w = g \quad &\text{on } \partial\Omega. \end{aligned} \tag{2.9}$$

Since η_ρ equals 0 identically in Γ_{out} and $S = 0$ on Γ_{in}, the function $\eta_\rho u_S$ becomes 0 on $\partial\Omega$. So u and w have same boundary values. To derive DSFM for heat equation, we test (2.9) by $v \in H_0^1$ to get

$$\langle\nabla w\,,\,\nabla v\rangle + \langle kw\,,\,v\rangle + \alpha\,\langle -\mu\triangle\eta_\rho S + kS\,,\,v\rangle = \langle f\,,\,v\rangle\,. \tag{2.10}$$

The variational formulation (2.10) has 2 unknowns (w,α), so we need one more equation and the equation is

- a single equation, because α is a real number.
- linearly independent to above system.
- not disappeared α.
- computable near the singularity corner.

So we test with the dual singular function $S_d(r,\theta) \notin H^1(\Omega)$ to obtain

$$\langle -\mu\triangle w + kw\,,\,\eta_{2\rho}S_d\rangle + \alpha\,\langle -\mu\triangle\eta_\rho S + k\eta_\rho S\,,\,\eta_{2\rho}S_d\rangle = \langle f\,,\,\eta_{2\rho}S_d\rangle\,. \tag{2.11}$$

We have a crucial lemma which is proved in [10].

Lemma 2.2. *Dual Singular function S_d has the following properties.*

1. *$S_d \in L^2$ and $S_d \notin H^1$.*
2. *$\langle -\triangle\phi\,,\,\eta_{2\rho}S_d\rangle = \langle \phi\,,\,-\triangle\eta_{2\rho}S_d\rangle$, for all $\phi \in H_0^1$.*

In conjunction with Lemma 2.2, (2.11) can be re written by

$$\alpha = \frac{1}{\beta_s}(\beta_f - \langle w\,,\,-\mu\triangle\eta_{2\rho}S_d + kS_d\rangle), \tag{2.12}$$

where $\beta_s = \langle -\mu\triangle\eta_\rho S + kS\,,\,\eta_{2\rho}S_d\rangle$ and $\beta_f = \langle f\,,\,\eta_{2\rho}S_d\rangle$. DSFM is an algorithm to solve the system (2.11) and (2.12).

In order to construct fully discrete algorithm of the system, we introduce finite element space

$$V_h := \{v_h \in H_0^1(\Omega) : v_h|_K \in \mathcal{P}(K) \quad \forall K \in \mathfrak{T}\},$$

where $\mathcal{P}(K)$ is a polynomial function space degree $\leq p$. In light of inserting α in (2.12) into (2.11), the discrete system becomes $w_h \in V_h$ and $\alpha_h \in \mathbb{R}$ satisfying, for all $v_h \in V_h$,

$$\begin{aligned}\langle \nabla w_h\,,\,\nabla v_h\rangle - \frac{1}{\beta_s}\langle w_h\,,\,-\mu\triangle\eta_{2\rho}S_d + k\eta_{2\rho}S_d\rangle \langle -\mu\triangle\eta_\rho S + k\eta_\rho S\,,\,v_h\rangle \\ + \langle kw_h\,,\,v_h\rangle = \langle f\,,\,v_h\rangle - \frac{\beta_f}{\beta_s}\langle -\mu\triangle\eta_\rho S + k\eta_\rho S\,,\,v_h\rangle .\end{aligned} \tag{2.13}$$

The matrix form of (2.13) is

$$(A + ab^T)w_h = F, \tag{2.14}$$

where the matrix A is the discrete linear operator of the Hehmholtz equation. Also a, b^T and F in are generated by $\langle -\mu\triangle\eta_\rho S + k\eta_\rho S\,,\,v_h\rangle$, $\frac{1}{\beta_s}\langle w_h\,,\,-\mu\triangle\eta_{2\rho}S_d + k\eta_{2\rho}S_d\rangle$ and $\langle f\,,\,v_h\rangle - \frac{\beta_f}{\beta_s}\langle -\mu\triangle\eta_\rho S + k\eta_\rho S\,,\,v_h\rangle$, respectively. The linear system (2.14) can be solved by the Sherman-Morrison formulation in [6]:

$$(A + ab^T)^{-1} = A^{-1} - \frac{A^{-1}ab^TA^{-1}}{a + b^TA^{-1}a}. \tag{2.15}$$

The well posedness of (2.13) has been proved and obtain the following finite element approximation in [10].

Theorem 2.3. *Let (w, α) be the solution of* (2.9) *and $w \in H^2(\Omega) \cap H_0^1(\Omega)$. And let (w_h, α_h) be the solution of the system* (2.12) *and* (2.13). *Then there exists a positive constant h_0 such that $h \leq h_0$, for all h, satisfying*

$$\|w - w_h\|_1 \leq Ch\|f\|_0, \quad \|w - w_h\|_0 \leq Ch^{1+\alpha}\|f\|_0,$$

$$|\alpha - \alpha_h| \leq Ch^{1+\alpha}\|f\|_0.$$

The Sherman-Morrison formulation (2.15) is crucial to solve (2.14), but (2.15) requires linear solver 2 times and is rather complicated to apply to real computation, especially multiple singularities problems and time evolution problems like heat equation. So we reanalyze the Sherman-Morrison formulation (2.15) with solutions u_h and z_h of

$$\langle \nabla u_h\,,\,\nabla v_h\rangle + \langle ku_h\,,\,v_h\rangle = \langle f\,,\,v_h\rangle\,,\ \forall v_h \in V_h, \tag{2.16}$$

and

$$\langle \nabla z_h \,,\, \nabla v_h \rangle + \langle k z_h \,,\, v_h \rangle = \langle -\mu \triangle \eta_\rho S + k \eta_\rho S \,,\, v_h \rangle \,,\ \forall v_h \in V_h, \tag{2.17}$$

receptively. Then we arrive at the following result.

Theorem 2.4. *If we define $\zeta_{v_h} = \langle v_h \,,\, -\mu \triangle \eta_{2\rho} S_d + k \eta_{2\rho} S_d \rangle$, for any function $v_h \in V_h$, then w_h which is the solution of* (2.13) *is*

$$w_h = u_h - \left(\frac{\beta_f - \zeta_{u_h}}{\beta_s - \zeta_{z_h}} \right) z_h. \tag{2.18}$$

Proof. In light of definition of ζ_{w_h}, (2.13) can be rewritten by, for all $v_h \in V_h$,

$$\begin{aligned} \langle \nabla w_h \,,\, \nabla v_h \rangle + \langle k w_h \,,\, v_h \rangle - \frac{\zeta_{w_h}}{\beta_s} \langle -\mu \triangle \eta_\rho S + k \eta_\rho S \,,\, v_h \rangle \\ = \langle f \,,\, v_h \rangle - \frac{\beta_f}{\beta_s} \langle -\mu \triangle \eta_\rho S + k \eta_\rho S \,,\, v_h \rangle \,. \end{aligned}$$

Since $b^T A^{-1} F \in \mathbb{R}$, the Sherman-Morrison formulation (2.15) yields

$$\begin{aligned} w_h &= A^{-1} F - \frac{A^{-1} a b^T A^{-1}}{a + b^T A^{-1} a} F \\ &= A^{-1} F - \frac{b^T A^{-1} F}{1 + b^T A^{-1} a} A^{-1} a. \end{aligned} \tag{2.19}$$

From (2.16) and (2.17),

$$A^{-1} F = u_h - \frac{\beta_f}{\beta_s} z_h.$$

Also, $A^{-1} a = z_h$ and $b^T v_h = -\zeta_{v_h} / \beta_s$, for $v_h \in V_h$ implies that

$$b^T A^{-1} a = -\zeta_{z_h} / \beta_s \quad \text{and} \quad b^T A^{-1} F = \frac{-\zeta_{u_h}}{\beta_s} + \frac{\beta_f}{\beta_s^2} \zeta_{z_h}.$$

Therefore, (2.19) becomes

$$\begin{aligned} w_h &= u_h - \frac{\beta_f}{\beta_s} z_h - \frac{-\frac{\zeta_{u_h}}{\beta_s} + \frac{\beta_f}{\beta_s^2} \zeta_{z_h}}{1 - \frac{\zeta_{z_h}}{\beta_s}} z_h \\ &= u_h - \left(\frac{\beta_f}{\beta_s} - \frac{\beta_s \zeta_{u_h} - \beta_f \zeta_{z_h}}{\beta_s (\beta_s - \zeta_{z_h})} \right) z_h \\ &= u_h - \left(\frac{\beta_f (\beta_s - \zeta_{z_h}) - \beta_s \zeta_{u_h} + \beta_f \zeta_{z_h}}{\beta_s (\beta_s - \zeta_{z_h})} \right) z_h = u_h - \left(\frac{\beta_f - \zeta_{u_h}}{\beta_s - \zeta_{z_h}} \right) z_h. \end{aligned}$$

the proof is complete. □

We so construct the following DSFM using (2.4).

Algorithm 1 (FE-DSFM for Helmholtz equation)**.** Let w_h be discrete smooth part and let α_h be discrete stress intensity factors in (2.13).

Step 1: Find (u_h, z_h) as the solution of (2.16) and (2.17).
Step 2: Compute w_h by (2.18).
Step 3: Compute α_h by second part of (2.13).

We now apply Algorithm 1 to more complicated problem including several reentrant corners. The solution u can be expressed by

$$u = w + \sum_{i=1}^{n} \alpha_i \eta_{\rho i} S_i,$$

and the Helmholtz equation becomes

$$\begin{aligned} -\mu\triangle w + kw + \sum_{i=1}^{n} \alpha_i(-\mu\triangle\eta_{\rho i} S_i + k\eta_{\rho i} S_i) = f \quad \text{in } \Omega, \\ w = g \quad \text{on } \partial\Omega. \end{aligned} \tag{2.20}$$

We can choose cut off functions $\eta_{\rho i}$ and $\eta_{2\rho j}$ to have mutually disjoint support to hold $\langle -\mu\triangle\eta_{\rho i} S_i + kS_i \, , \, \eta_{2\rho j} S_{dj} \rangle = 0$ if $i \neq j$. In light of testing (2.20) by $\eta_{2\rho i} S_{di}$, we readily obtain

$$\alpha_i = \frac{1}{\beta_{si}}(\beta_{fi} - \langle w \, , \, -\mu\triangle\eta_{2\rho i} S_{di} + kS_{di} \rangle). \tag{2.21}$$

The finite element weak form of system (2.20) and (2.21) is to find $w_h \in V_h$ and $\alpha_{hi} \in \mathbb{R}$, $i = 1 \cdots n$, satisfying, for all $v_h \in V_h$,

$$\begin{aligned} \langle \nabla w_h \, , \, \nabla v_h \rangle + \langle k w_h \, , \, v_h \rangle + \sum_{i=1}^{n} \alpha_{hi} \langle -\mu\triangle\eta_{\rho i} S_i + k\eta_{\rho i} S_i \, , \, v_h \rangle = \langle f \, , \, v_h \rangle \, , \\ \alpha_{hi} = \frac{1}{\beta_{si}}(\beta_{fi} - \langle w_h \, , \, -\mu\triangle\eta_{2\rho i} S_{di} + k\eta_{2\rho i} S_{di} \rangle), \end{aligned} \tag{2.22}$$

and its matrix form becomes

$$(A + \sum_{i=1}^{n} a_i {b_i}^T) w_h = F.$$

Let $\zeta_{i,v_h} = \langle v_h \, , \, -\mu\triangle\eta_{2\rho i} S_{di} + kS_{di} \rangle$ and z_{ih} be the solution of

$$\langle \nabla z_{ih} \, , \, \nabla v_h \rangle + \langle k z_{ih} \, , \, v_h \rangle = \langle -\mu\triangle\eta_{\rho i} S_i + k\eta_{\rho i} S_i \, , \, v_h \rangle \, , \, \forall v_h \in V_h. \tag{2.23}$$

In conjunction with Theorem 2.4, w_h would be

$$w_h = u_h - \sum_{i=1}^{n} \left(\frac{\beta_{fi} - \zeta_{u_h}}{\beta_{si} - \zeta_{z_{ih}}} \right) z_{ih}. \tag{2.24}$$

Finally we arrive at DSFM to solve Helmholtz equations (1.2) on domain Ω including several reentrant corners.

Algorithm 2 (Multiple singularities version of Algorithm 1)**.** Let w_h be discrete smooth part and let α_{ih} be discrete stress intensity factors of α_i

Step 1: Find u_h and z_{ih}, $i = 1 \ldots n$, as the solution of (2.16) and (2.23).

Step 2: Compute w_h by (2.24).
Step 3: Compute α_h by second part of (2.22).

3. DUAL SINGULAR FUNCTION METHOD FOR THE HEAT EQUATION

In this section, we extend Algorithm 1 to solve the heat equation (1.1) using the backward Euler time discrete formula (BDF1) and the second order backward Euler time discrete formula (BDF2). We first consider BDF1 for heat equation:

$$\frac{u^{n+1}-u^n}{\tau} - \mu\triangle u^{n+1} = f^{n+1}. \tag{3.1}$$

We already know in [5] that the form of singular solution is same to that of Helmholtz equation. It means that the solution of (1.1) can be expressed by

$$u(x,t) = w(x,t) + \alpha(t)\eta_\rho S, \tag{3.2}$$

and (3.1) can be rewritten by

$$(w^{n+1} - \mu\tau\triangle w^{n+1}) + \alpha^{n+1}(\eta_\rho S - \mu\tau\triangle\eta_\rho S) = \tau f^{n+1} + w^n + \alpha^n\eta_\rho S. \tag{3.3}$$

Since (3.3) and (2.9) have similar expression, the same strategy to Algorithm 1 leads us the solver of (3.3): find $w_h^{n+1} \in V_h$ and $\alpha_h^{n+1} \in \mathbb{R}$ satisfying, for all $v_h \in V_h$,

$$\begin{aligned}
&\mu\tau \left\langle \nabla w_h^{n+1} , \nabla v_h \right\rangle + \left\langle w_h^{n+1} , v_h \right\rangle + \alpha_h^{n+1} \left\langle \eta_\rho S - \mu\tau\triangle\eta_\rho S , v_h \right\rangle \\
&\quad = \tau \left\langle f^{n+1} , v_h \right\rangle + \left\langle \nabla w_h^n , \nabla v_h \right\rangle + \alpha_h^n \left\langle \eta_\rho S , v_h \right\rangle ,
\end{aligned} \tag{3.4}$$

$$\alpha^{n+1} = \frac{1}{\beta_s}\left(\tau\beta_f^{n+1} + \left\langle w^n + \alpha^n\eta_\rho S , \eta_{2\rho}S_d \right\rangle - \zeta_{w^{n+1}}\right),$$

where

$$\beta_s = \left\langle \eta_\rho S - \mu\tau\triangle\eta_\rho S , \eta_{2\rho}S_d \right\rangle , \quad \beta_f^{n+1} = \left\langle f(t^{n+1}) , \eta_{2\rho}S_d \right\rangle ,$$
$$\zeta_{w^{n+1}} = \left\langle w^{n+1} , \eta_{2\rho}S_d - \mu\tau\triangle\eta_{2\rho}S_d \right\rangle .$$

The matrix form of the system becomes (2.14) and it can be solved by (2.15). To construct algorithm for heat equations, we denote z_h be the solution of

$$\mu\tau \left\langle \nabla z_h , \nabla v_h \right\rangle + \left\langle z_h , v_h \right\rangle = \left\langle -\mu\tau\triangle\eta_\rho S + \eta_\rho S , v_h \right\rangle , \ \forall v_h \in V_h. \tag{3.5}$$

Then we arrive at DSFM with BDF1 scheme to solve heat equation.

Algorithm 3 (FE-DSFM with backward Euler for Heat equation)**.** Let z_h be the solution of (3.5) and set $\widehat{u}_h^0 = u^0$. Repeat for $1 \le n \le N$:

Step 1: Find u_h^{n+1} as the solution of

$$\mu\tau \left\langle \nabla u_h^{n+1} , \nabla v_h \right\rangle + \left\langle u_h^{n+1} , v_h \right\rangle = \tau \left\langle f^{n+1} , v_h \right\rangle + \left\langle \widehat{u}_h^n , v_h \right\rangle , \ \forall v_h \in V_h.$$

Step 2: Compute w_h^{n+1}

$$w_h^{n+1} = u_h^{n+1} - \left(\frac{\tau\beta_f + \left\langle \widehat{u}_h^n , \eta_{2\rho}S_d \right\rangle - \zeta_{u_h^{n+1}}}{\beta_s - \zeta_{z_h}}\right) z_h.$$

Step 3: Compute α_h^{n+1} by second part of (3.4).
Step 4: Update $\widehat{u}_h^{n+1} = w_h^{n+1} + \alpha_h^{n+1}\eta_\rho S$.

Remark 1. The Sherman-Morrison formulation (2.15) requires to solve linear system 2 times for Poisson and Helmholtz equations, but Algorithm 3 need to apply linear solver only 1 at each iteration, so this algorithm use same linear solvers per each time iteration to standard method. And Algorithm 3 has optimal accuracy behavior for corner singularities problem, in contrast standard numerical method.

We now construct DSFM for heat equation with backward Euler formula (BDF2) time discretization:

$$\frac{3u^{n+1} - 4u^n + u^{n-1}}{2\tau} - \mu\triangle u^{n+1} = f^{n+1}.$$

By the same manner with the case BDF1, (3.2) leads us

$$\begin{aligned} 3w^{n+1} - 2\mu\tau\triangle w^{n+1} + \alpha^{n+1}(3\eta_\rho S - 2\mu\tau\triangle\eta_\rho S) \\ = 2\tau f^{n+1} + 4(w^n + \alpha^n\eta_\rho S) - (w^{n-1} + \alpha^{n-1}\eta_\rho S), \end{aligned} \tag{3.6}$$

and testing (3.6) with $v_h \in V_h$ and with $\eta_{2\rho i}S_{di}$ yield

$$\begin{aligned} &2\mu\tau\left\langle \nabla w_h^{n+1}, \nabla v_h\right\rangle + 3\left\langle w_h^{n+1}, v_h\right\rangle + \alpha_h^{n+1}\left\langle 3\eta_\rho S - 2\mu\tau\triangle\eta_\rho S, v_h\right\rangle \\ &= 2\tau\left\langle f^{n+1}, v_h\right\rangle + 4(\langle\nabla w_h^n, \nabla v_h\rangle + \alpha_h^n\langle\eta_\rho S, v_h\rangle) \\ &\qquad - (\left\langle\nabla w_h^{n-1}, \nabla v_h\right\rangle + \alpha_h^{n-1}\langle\eta_\rho S, v_h\rangle), \ \forall v_h \in V_h, \\ &\alpha_h^{n+1} = \frac{1}{\beta_s}(F_h^{n+1} - \zeta_{w_h^{n+1}}), \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} F^{n+1} &= 2\tau\left\langle f^{n+1}, \eta_{2\rho}S_d\right\rangle + 4(\langle w^n, \eta_{2\rho}S_d\rangle + \alpha^n\langle\eta_\rho S, \eta_{2\rho}S_d\rangle) \\ &\qquad - (\left\langle w^{n-1}, \eta_{2\rho}S_d\right\rangle + \alpha^{n-1}\langle\eta_\rho S, \eta_{2\rho}S_d\rangle), \\ \zeta_{w^{n+1}} &= \left\langle w^{n+1}, 3\eta_{2\rho}S_d - 2\mu\tau\triangle\eta_{2\rho}S_d\right\rangle \text{ and } \beta_s = \langle 3\eta_\rho S - 2\mu\tau\triangle\eta_\rho S, \eta_{2\rho}S_d\rangle. \end{aligned}$$

Since this system also constructs a linear system of the form

$$(A + ab^T)w_h = F,$$

we can find w_h by Theorem 2.4, that is

$$w_h^{n+1} = u_h^{n+1} - \left(\frac{F_h^{n+1} - \zeta_{u_h^{n+1}}}{\beta_s - \zeta_{z_h}}\right) z_h, \tag{3.8}$$

where z_h is the solution of

$$2\mu\tau\langle\nabla z_h, \nabla v_h\rangle + 3\langle z_h, v_h\rangle = \langle -2\mu\tau\triangle\eta_\rho S + 3\eta_\rho S, v_h\rangle, \ \forall v_h \in V_h.$$

If we denote $\widehat{u}_h^n = w_h^n + \alpha_h^n \eta_\rho S$ and $\widehat{u}_h^{n-1} = w_h^{n-1} + \alpha_h^{n-1}\eta_\rho S$, then u_h^{n+1} becomes the solution of, for all $v_h \in V_h$,

$$\begin{aligned} 2\mu\tau \left\langle \nabla u_h^{n+1}\,,\, \nabla v_h \right\rangle + 3\left\langle u_h^{n+1}\,,\, v_h\right\rangle \\ = 2\tau\left\langle f^{n+1}\,,\, v_h\right\rangle + 4\left\langle \widehat{u}_h^{n}\,,\, v_h\right\rangle - \left\langle \widehat{u}_h^{n-1}\,,\, v_h\right\rangle . \end{aligned} \tag{3.9}$$

Finally, we arrive at BDF2 DSFM to solve heat equation (1.1).

Algorithm 4 (FE-DSFM with BDF2 for Heat equation)**.** Let w_h^{n+1} be discrete smooth part and α_h^{n+1} be discrete stress intensity factors of $\alpha(t)$ of $n+1$ step. Initially given u^0, set w_h^1 and α_h^1 using the backward FE-DSFM. Repeat for $1 \le n \le N$

Step 1: Find u_h^{n+1} as the solution of (3.9).
Step 2: Compute w_h^{n+1} by (3.8).
Step 3: Compute α_h^{n+1} by second part of (3.7).

4. Numerical test

In this section, we document the computational performance of each algorithm within a polygonal domain with reentrant corners. We use cut-off function in (2.7) with

$$p(r) = \frac{15}{16}\Big\{\frac{8}{15} - \Big(\frac{4r}{\rho} - 3\Big) + \frac{2}{3}\Big(\frac{4r}{\rho} - 3\Big)^3 - \frac{1}{5}\Big(\frac{4r}{\rho} - 3\Big)^5\Big\}.$$

All computations are performed with the conforming P_1 finite element and 6 points quadrature rule.

Example 4.1. We consider Helmholtz equations (1.2) on the Γ shape computational domain $([-1,1]\times[-1,1]) \setminus ([0,1]\times[-1,0])$. We choose the smooth part of the solution as

$$w = \begin{cases} \sin(2\pi x)(1/2y^2 + y)(y^2 - 1), & (y < 0) \\ \sin(2\pi x)(-1/2y^2 + y)(y^2 - 1), & (y \ge 0), \end{cases}$$

and set the exact solution

$$u = w + \alpha\eta_\rho S$$

with $\alpha = 3.0$ and $\rho = 0.4$. Also, the singular function S is given by

$$S(r,\theta) = r^{\pi/\omega}\sin(\frac{\pi}{\omega}\theta)$$

with $\omega = \frac{3}{2}\pi$. The forcing term f is determined to become $k = 1.0$ and $\nu = 1.0$.

Table 1 is the error decay of the standard finite element method for Example 4.1. The losing convergence orders is natural behavior because of $u \notin H^2(\Omega)$. Table 2 is the error decay of the DSFM with Algorithm 1 and displays optimal error decay in contrast with results in Table 1

The next example is a multiple corner singularities case.

TABLE 1. Error table for Example 4.1 with Standard FEM.

	$\|u-u_h\|_0$		$\|Iu-u_h\|_\infty$		$\|u-u_h\|_1$	
h	Errors	**Order**	Errors	**Order**	Errors	**Order**
1/8	7.162e-02		1.065e-01		1.604e-00	
1/16	2.502e-02	**1.52**	3.922e-02	**1.44**	6.641e-01	**1.27**
1/32	7.051e-03	**1.83**	2.460e-02	**0.67**	2.183e-01	**1.60**
1/64	2.120e-03	**1.73**	1.573e-02	**0.64**	8.899e-02	**1.29**
1/128	6.745e-04	**1.65**	9.932e-03	**0.66**	4.868e-02	**0.87**
1/256	2.372e-04	**1.51**	6.263e-03	**0.67**	2.980e-02	**0.71**

TABLE 2. Error table for Example 4.1 with Algorithm 1.

	$\|w-w_h\|_0$		$\|Iw-w_h\|_\infty$		$\|w-w_h\|_1$		$\|\alpha-\alpha_h\|$	
h	Errors	**Order**	Errors	**Order**	Errors	**Order**	Errors	**Order**
1/8	2.312e-02		1.085e-02		1.879e-01		3.639e-03	
1/16	5.974e-03	**1.95**	2.713e-03	**2.00**	5.211e-02	**1.85**	2.081e-03	**0.81**
1/32	1.506e-03	**1.99**	6.785e-04	**2.00**	1.502e-02	**1.80**	5.095e-04	**2.03**
1/64	3.773e-04	**2.00**	1.696e-04	**2.00**	4.567e-03	**1.72**	9.663e-05	**2.40**
1/128	9.437e-05	**2.00**	4.239e-05	**2.00**	1.464e-03	**1.64**	2.484e-05	**1.96**
1/256	9.360e-05	**2.00**	1.060e-05	**2.00**	4.884e-04	**1.58**	5.627e-06	**2.14**

Example 4.2. Let the computational domain be $([-2,2]\times[-2,2])\setminus([-1,1]\times[-1.1])$ including 4 reentrant corners. Let smooth part of the solution be given by

$$w=\begin{cases}\sin(\pi x)(1/2y^2+y)(y+1)(y+2), & (y<0)\\ \sin(\pi x)(-1/2y^2+y)(y-1)(y-2), & (y\geq 0).\end{cases}$$

In this experiment, we choose $k=1.0$, $\nu=1.0$ and set the forcing term f to be the exact solution

$$u=w+\alpha_1\eta_{\rho1}S_1+\alpha_2\eta_{\rho2}S_2+\alpha_3\eta_{\rho3}S_3+\alpha_4\eta_{\rho4}S_4,$$

where $\rho_1=\rho_2=\rho_3=\rho_4=0.4$ and stress intensity factors $\alpha_1=-1.0$, $\alpha_2=2.0$, $\alpha_3=-3.0$ and $\alpha_4=4.0$.

Table 3 is the mesh analysis of Algorithm 2 which is based on (2.24). This results also display optimal error decay rates.

We now perform numerical simulations for the heat equations.

Example 4.3. We consider heat equation with Γ shape computational domain $([-1,1]\times[-1,1])\setminus([0,1]\times[-1,0])$. We use t as time variable and we perform Algorithms 3 and

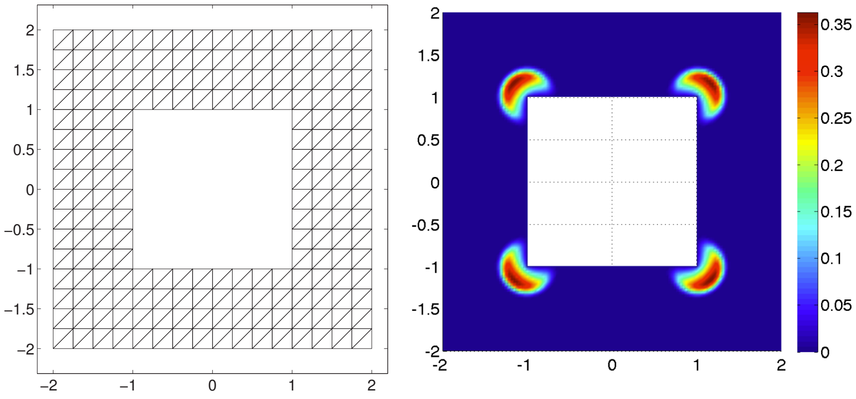

FIGURE 1. Domain, mesh and $\eta_{\rho i} S_i$, $(i = 1 \cdots 4)$ for Example 4.2

TABLE 3. Error table for Example 4.2 with Algorithm 2.

	$\|w - w_h\|_0$		$\|Iw - w_h\|_\infty$		$\|w - w_h\|_1$		$\sum_{i=1}^4 \lvert\alpha_i - \alpha_{ih}\rvert$	
h	Errors	**Order**	Errors	**Order**	Errors	**Order**	Errors	**Order**
1/4	5.984e-02		1.442e-02		3.938e-01		4.322e-02	
1/8	1.642e-02	**1.87**	3.976e-03	**1.86**	1.280e-01	**1.62**	2.878e-02	**0.59**
1/16	4.175e-03	**1.98**	9.890e-04	**2.01**	4.058e-02	**1.66**	4.921e-03	**2.55**
1/32	1.047e-03	**2.00**	2.516e-04	**1.98**	1.328e-02	**1.61**	8.649e-04	**2.51**
1/64	2.626e-04	**2.00**	6.315e-05	**1.99**	4.485e-03	**1.57**	2.715e-04	**1.67**
1/128	6.562e-05	**2.00**	1.582e-05	**2.00**	1.545e-03	**1.54**	6.048e-05	**2.17**

4 with smooth part of solution

$$w = \begin{cases} \sin(t)\sin(2\pi x)(1/2y^2 + y)(y^2 - 1), & (y < 0) \\ \sin(t)\sin(2\pi x)(-1/2y^2 + y)(y^2 - 1), & (y \geq 0), \end{cases}$$

and exact solution

$$u = w + \exp(t)\eta_\rho S \quad \text{in } [0,1] \times \Omega,$$

where $\rho = 0.4$ and forcing term is also determined by $\nu = 1.0$. In this example, the stress intensity factor depends on variable t and Tables 4$\sim$6 are mesh analysis at $t = 1$.

Since backward Euler method is only 1st order scheme for time, we set $\tau = 16h^2$ to check the optimal convergence rates and Table 4 shows the optimal results of Algorithm 3.

We apply the BDF2 time discretization scheme, Algorithm 4, for the same example. Since it is second order method, we set $\tau = \frac{1}{2}h$, and so this computational cost is much more effective than Algorithm 3.

Table 5 is results of the standard finite element method with BDF2 time discretization for Example 4.3. Since $u \notin L^2([0,1] : H^2)$, the errors do not decay with optimal order.

TABLE 4. Error table for Example 4.3 with Algorithm 3.

		$\|w-w_h\|_0$		$\|Iw-w_h\|_\infty$		$\|w-w_h\|_1$		$\|\alpha-\alpha_h\|$	
h	τ	Errors	**Order**	Errors	**Order**	Errors	**Order**	Errors	**Order**
1/8	1/4	2.009e-02		1.032e-02		1.615e-01		6.415e-03	
1/16	1/16	5.205e-03	**1.95**	2.600e-03	**1.99**	4.475e-02	**1.85**	6.552e-04	**3.29**
1/32	1/64	1.314e-03	**1.99**	6.520e-04	**2.00**	1.284e-02	**1.80**	1.798e-04	**1.87**
1/64	1/256	3.293e-04	**2.00**	1.634e-04	**2.00**	3.887e-03	**1.72**	7.150e-05	**1.33**
1/128	1/1024	8.236e-05	**2.00**	4.085e-05	**2.00**	1.240e-03	**1.65**	1.733e-05	**2.04**
1/256	1/4096	2.060e-05	**2.00**	1.021e-05	**2.00**	4.126e-04	**1.59**	4.825e-06	**1.84**

TABLE 5. Error table for Example 4.3 with Standard FEM with BDF2 time discretization.

		$\|u-u_h\|_0$		$\|Iu-u_h\|_\infty$		$\|u-u_h\|_1$	
h	τ	Errors	**Order**	Errors	**Order**	Errors	**Order**
1/8	1/16	6.447e-02		9.646e-02		1.452e-00	
1/16	1/32	2.254e-02	**1.52**	3.551e-02	**1.44**	6.013e-01	**1.27**
1/32	1/64	6.361e-03	**1.83**	2.228e-02	**0.67**	1.977e-01	**1.60**
1/64	1/128	1.916e-03	**1.73**	1.425e-02	**0.64**	8.061e-02	**1.29**
1/128	1/256	6.101e-04	**1.65**	8.999e-03	**0.66**	4.410e-02	**0.87**
1/256	1/512	2.148e-04	**1.51**	5.675e-03	**0.67**	2.700e-02	**0.71**

TABLE 6. Error table for Example 4.3 with Algorithm 4.

		$\|w-w_h\|_0$		$\|Iw-w_h\|_\infty$		$\|w-w_h\|_1$		$\|\alpha-\alpha_h\|$	
h	τ	Errors	**Order**	Errors	**Order**	Errors	**Order**	Errors	**Order**
1/8	1/16	1.956e-02		9.283e-03		1.587e-01		2.984e-03	
1/16	1/32	5.057e-03	**1.95**	2.324e-03	**2.00**	4.402e-02	**1.85**	1.738e-03	**0.78**
1/32	1/64	1.275e-03	**1.99**	5.813e-04	**2.00**	1.267e-02	**1.80**	4.257e-04	**2.03**
1/64	1/128	3.195e-04	**2.00**	1.453e-04	**2.00**	3.851e-03	**1.72**	8.058e-05	**2.40**
1/128	1/256	7.990e-05	**2.00**	3.632e-05	**2.00**	1.233e-03	**1.64**	2.072e-05	**1.96**
1/256	1/512	1.998e-05	**2.00**	9.079e-06	**2.00**	4.113e-04	**1.58**	4.690e-06	**2.14**

Table 6 is error table for Algorithm 4 for Example 4.3 and the errors converge to 0 with optimal rate. So we can conclude that DSFM for heat equation has optimal convergence rate for corner singularity problems.

Finally, Figure 2 is error evolutions of smooth part of u on the L^2 space and errors of stress intensity factors. It displays smooth error propagation on time.

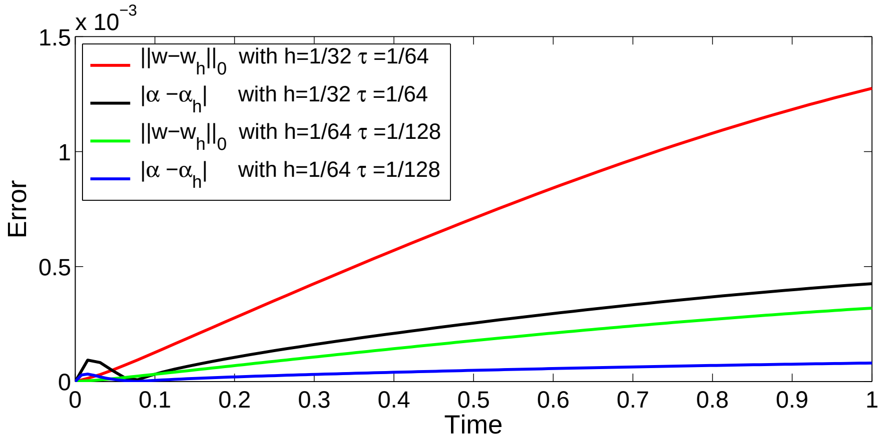

FIGURE 2. Error evolution of Example 4.3 with Algorithm 4.

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