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POLYGONAL PARTITIONS

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Abstract. By acting the dihedral group D_k on the set of k-tuple multi-partitions, we introduce k-gonal partitions for all positive integers k. We give generating functions for these new partition functions and investigate their arithmetic properties.

1. Introduction

Let $\mathcal{P}_k$ be the set of k-tuple multi-partitions

$$\mathcal{P}_k := \{(\lambda_1, \lambda_2, \dots, \lambda_k) : \lambda_i\text{'s are ordinary partitions}\}.$$

In a recent paper [1], the author introduced a new way to obtain new partition classes by applying group actions on $\mathcal{P}_k$ for $k = 2$ or 3. For example, when $k = 3$, the author defined a group action on $\mathcal{P}_3$ by

$$\sigma(\lambda_1, \lambda_2, \lambda_3) = (\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \lambda_{\sigma(3)})$$

for $\sigma \in S_3$, where S_3 is the symmetric group. Then, it is natural to define the number of orbits of weight n, where the weight of multi-partitions is defined by the sum of the parts appeared in the partitions. Recall that if $\sigma(\lambda) = \pi$, then λ and π are in the same orbit. For a given tri-partition $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, suppose that each partition λ_i is on the i-th vertex of a regular triangle. Two tri-partitions $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and $\pi = (\pi_1, \pi_2, \pi_3)$ are in the same orbits if λ can be obtained by rotating or

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reflecting π on the regular triangle. Thus, we can understand $|\mathcal{P}_3/S_3|(n)$, the number of orbits of weight n, counts the number of tri-partitions of n, where we regard two partitions are the same if we can obtain one tri-partition by rotating or reflecting the other tri-partition. In this light, we say $|\mathcal{P}_3/S_3|(n)$ is the number of triangular partitions. In the previous paper [1], the author proved that

$$\sum_{n=0}^{\infty} |\mathcal{P}_3/S_3|(n)q^n = \frac{1}{6}\left(\frac{1}{(q)_\infty^3} + \frac{3}{(q)_\infty (q^2;q^2)_\infty} + \frac{2}{(q^3;q^3)_\infty}\right). \tag{1.1}$$

Here and throughout the paper, we use a standard q-series notation:

$$(a)_\infty := (a;q)_\infty := \prod_{n=1}^{\infty}(1-aq^{n-1}).$$

In the geometric sense, it is natural to examine $|\mathcal{P}_k/D_k|(n)$, where D_k is the dihedral group of order $2k$. To this end, we define a group action on $\mathcal{P}_k$ by

$$\sigma(\lambda_1,\lambda_2,\ldots,\lambda_k) = (\lambda_{\sigma(1)},\lambda_{\sigma(2)},\ldots,\lambda_{\sigma(k)})$$

for $\sigma \in D_k$. Then, $|\mathcal{P}_k/D_k|(n)$ is the number of orbits with weight n, i.e. the number of k-tuple multi-partitions of n, where k-tuple partitions in the same orbit define the same partition. We may think the partitions λ_1, …, λ_k in a k-tuple multi-partition are on the vertices of k-gon. Therefore, $|\mathcal{P}_k/D_k|(n)$ is the number of k-tuple partitions of n, where we regard two k-tuple multi-partitions are the same if one multi-partition can be obtained by reflecting or rotating the other multi-partition. In this light, we say $P_k(n) := |\mathcal{P}_k/D_k|(n)$ is the number of k-gonal partitions of n.

The first task to examine the arithmetic properties of k-gonal partitions is finding a generating function.

Theorem 1.1. *For a positive integer k,*

$$2k\sum_{n=0}^{\infty} P_k(n)q^n = \begin{cases} \frac{k}{(q)_\infty (q^2;q^2)_\infty^{(k-1)/2}} + \sum_{j=1}^{k} \frac{1}{(q^{k/(k,j)};q^{k/(k,j)})_\infty^{(k,j)}}, & \text{if } k \text{ is odd,} \\ \frac{k}{2(q^2;q^2)_\infty^{k/2}} + \frac{k}{2(q)_\infty^2 (q^2;q^2)_\infty^{(k-2)/2}} + \sum_{j=1}^{k} \frac{1}{(q^{k/(k,j)};q^{k/(k,j)})_\infty^{(k,j)}} & \text{, if } k \text{ is even.} \end{cases}$$

For example, when $k=1$, $P_1(n)$ is an ordinary partition function $p(n)$. For $k=3$, we recover the generating function (1.1). For $k=4$, we find a square-partition generating function

$$\sum_{n=0}^{\infty} P_4(n)q^n = \frac{1}{8}\left(\frac{1}{(q)_\infty^4} + \frac{3}{(q^2;q^2)_\infty^2} + \frac{2}{(q)_\infty^2(q^2;q^2)_\infty} + \frac{2}{(q^4;q^4)_\infty}\right).$$

Our next goal is to investigate arithmetic properties of k-gonal partitions. The author [1] proved that

$$P_3(3n+2) \equiv 0 \pmod 3$$

for all non-negative integers n. In this article, we examine the k-gonal partitions modulo 3, 5, and 7. For example, for modulo 5, we prove the following congruences.

Theorem 1.2. *For all non-negative integers n,*

$$\begin{aligned} P_3(25n+22) &\equiv 0 \pmod 5, \\ P_4(25n+21) &\equiv 0 \pmod 5, \\ P_6(25n+19) &\equiv 0 \pmod 5, \\ P_7(25n+18) &\equiv 0 \pmod 5. \end{aligned}$$

One might conjecture that $P_8(25n+17) \equiv 0 \pmod 5$, but $P_8(17) = 7805082$.

The rest of the paper is organized as follows. In the next section, we give basic facts on the theory of modular forms and introduce ℓ-regular partitions, which we use to prove the congruences for polygonal partitions. In the following section, we give proofs for the generating function for k-gonal partitions and their arithmetic properties.

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2. Preliminaries

We first introduce basic properties of modular forms. For more details, consult [4] for example. Define $\Gamma := SL_2(\mathbb{Z})$ and $\Gamma_0(N) := \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod N\right\}$. For a meromorphic function f on the complex upper half plane $\mathcal{H}$ and an integer k, we define the weight k slash operator by

$$f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} := (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

We say a holomorphic function f is a modular form of weight k on $\Gamma_0(N)$ if f is invariant under the weight k slash operator. Let $\mathcal{M}_k(\Gamma_0(N))$ (resp. $\mathcal{M}_k^!(\Gamma_0(N))$) denote the vector space of holomorphic modular forms (resp. weakly holomorphic forms) on $\Gamma_0(N)$, where by weakly holomorphic, we mean that f might have poles at the cusps. For a positive integer m, we define the U_m-operator: if $f(q)$ has a Fourier expansion $f(q) = \sum a(n)q^n$ with $q = \exp(2\pi i z)$, then

$$U_m f(z) := \sum a(mn)q^n.$$

For a prime p, it is well known that if $f(z) \in \mathcal{M}_k(\Gamma_0(Np^2))$, then $U_p f(z) \in \mathcal{M}_k(\Gamma_0(Np))$. The Dedekind eta function $\eta(z)$ is defined by $\eta(z) := q^{1/24}(q;q)_\infty$, where $q = \exp(2\pi i z)$ and $z \in \mathcal{H}$. For a fixed positive integer N and integers r_i's, a function of the form

$$f(z) := \prod_{\substack{n|N \\ n>0}} \eta(nz)^{r_n}. \tag{2.1}$$

is called an η-quotient. By the famous results of Newman [3] and Ligozat [2], we can determine when an η-quotient becomes a (weakly) holomorphic form of level N and the order of the η-quotient at the cusps of $\Gamma_0(N)$.

For a given positive integer ℓ, we say a partition λ is ℓ-regular if there is no part divisible by ℓ. It is not hard to see that

$$\sum_{n=0}^{\infty} a_\ell(n) q^{n+(\ell-1)/24} = \frac{\eta(\ell z)}{\eta(z)},$$

where $a_\ell(n)$ is the number of ℓ-regular partitions of n. We will use the fact that

$$\frac{\eta^3(9z)}{\eta^3(z)} \in \mathcal{M}_0^!(\Gamma_0(9)), \quad \frac{\eta(25z)}{\eta(z)} \in \mathcal{M}_0^!(\Gamma_0(25)), \quad \frac{\eta(49z)}{\eta(z)} \in \mathcal{M}_0^!(\Gamma_0(49)).$$

3. Proofs

We start with a proof for the generating function. In the later subsections, we consider $P_k(n)$ modulo 5, 7, and 3.

3.1. Proof of Theorem 1.1. We first recall the Burnside's lemma, which says that the number of orbits $|X/G|$ is given by

$$|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|,$$

where X is a set and G is a group acting on X, and X^g is the invariant subset of X under the action of g. Thus, we need to find the invariant subset of each element in D_k. Now we assume that k is odd. Then there are k reflections and k rotations in D_k. For the reflections, all of them are the reflections along the axis connecting one vertex of k-gon and the mid of opposite edge. Thus, two partitions in the vertices in the reflection should be the same and the partition on the vertex in the reflection axis has no regulation. Thus, for a reflection $g \in D_k$, the number of k-tuple multi-partitions of weight n in the invariant subset $\mathcal{P}_k^g$ is generated by

$$\frac{1}{(q)_\infty (q^2;q^2)_\infty^{(k-1)/2}}.$$

On the other hand, for a rotation $g \in D_k$, we can think it rotate clockwise by $2\pi j/k$. To be invariant under this rotation, the partitions on $k/(k,j)$ vertices should be the same, and thus such k-tuple multi-partitions of n is generated by

$$\frac{1}{(q^{k/(k,j)};q^{k/(k,j)})_\infty^{(k,j)}},$$

which conclude the first part of Theorem 1.1.

Now we consider the even k cases. For rotations, invariant subgroups are the same as odd k cases. For the reflections, there are two kinds of reflections for even k. There are $k/2$ reflections along the axis connecting mid points of opposite edges and there are $k/2$ reflections along the axis connecting two opposite vertices. The first type reflections correspond to

$$\frac{1}{(q^2;q^2)_\infty^{k/2}}$$

and the second type reflections correspond to

$$\frac{1}{(q)_\infty^2 (q^2;q^2)_\infty^{(k-2)/2}},$$

which conclude the second part of Theorem 1.1.

3.2. Modulo 5. We only prove the first congruence of Theorem 1.2 in detail as the other congruences follow in a similar manner.

Proof of Theorem 1.2. Let $F_5(q)$ be defined by

$$F_5(q) := \left(\frac{\eta^3(25z)}{\eta^3(z)} + 2\frac{\eta(25z)\eta(50z)}{\eta(z)\eta(2z)} + 3\frac{\eta(75z)}{\eta(3z)} \right) \Delta^3(25z),$$

where $\Delta(z)$ is the unique cusp form of weight 12 and of level 1.Then, $F_5(q) \in \mathcal{M}_{36}(\Gamma_0(150))$. Since $U_n F(nz) = F(z)$, it suffices to prove that

$$U_{25} F_5 \equiv 0 \pmod{5}.$$

Since $U_{25}F_5 \in \mathcal{M}_{36}(\Gamma_0(30))$ and its dimension is 214, we can prove the congruence by checking the first 215 coefficients due to Strum's theorem. □

3.3. Modulo 7. For modulo 7, we obtain the following congruences.

Theorem 3.1. *For all non-negative integers n,*

$$\begin{aligned} P_3(49n+43) &\equiv 0 \pmod{7}, \\ P_4(49n+41) &\equiv 0 \pmod{7}, \\ P_5(49n+39) &\equiv 0 \pmod{7}, \\ P_6(49n+37) &\equiv 0 \pmod{7}, \\ P_8(49n+33) &\equiv 0 \pmod{7}. \end{aligned}$$

Since $P_9(31) = 25775333497 \equiv 6 \pmod{7}$, it is not true that $P_9(49n+31) \equiv 0 \pmod{7}$ for all non-negative integers.

Here we give a proof for the second congruence in detail and omit the other cases.

Proof. Recall that

$$\sum_{n=0}^{\infty} P_4(n) q^n = \frac{1}{8} \left(\frac{1}{(q)_\infty^4} + \frac{3}{(q^2;q^2)_\infty^2} + \frac{2}{(q)_\infty^2 (q^2;q^2)_\infty} + \frac{2}{(q^4;q^4)_\infty} \right).$$

Define $F_7(q)$ by

$$F_7(q) := \left(\frac{\eta^4(49z)}{\eta^4(z)} + 3\frac{\eta^2(98z)}{\eta^2(2z)} + 2\frac{\eta^2(49z)\eta(98z)}{\eta^2(z)\eta(2z)} + 2\frac{\eta(196z)}{\eta(4z)} \right) \Delta^8(49z).$$

We can check that $F_7 \in \mathcal{M}_{96}(\Gamma_0(196)$. Thus, to prove $U_{49}F_7 \equiv 0 \pmod 7$, one need to check the first 384 coefficients as $U_{49}F_7 \in \mathcal{M}_{96}(28)$ of which dimension is 383. □

3.4. Modulo 3. In this subsection, we prove a mod 3 congruence.

THEOREM 3.2. *For all non-negative integers n,*

$$P_6(9n+7) \equiv 0 \pmod 3$$

Proof. From Theorem 1.1, we find that

$$\sum_{n=0}^{\infty} P_6(n)q^n = \frac{1}{12}\left(\frac{1}{(q)_\infty^6} + \frac{4}{(q^2;q^2)_\infty^3} + \frac{3}{(q)_\infty^2(q^2;q^2)_\infty^2} + \frac{2}{(q^3;q^3)_\infty^2} + \frac{2}{(q^6;q^6)_\infty} \right).$$

Now we define

$$F_3(q) := \left(\frac{\eta^6(9z)}{\eta^6(z)} + 4\frac{\eta^3(18z)}{\eta^3(2z)} + 3\frac{\eta^2(9z)\eta^2(18z)}{\eta^2(z)\eta^2(2z)} + 2\frac{\eta^2(27z)}{\eta^2(3z)} + 2\frac{\eta(54z)}{\eta(6z)} \right) \Delta^2(9z).$$

Then, $F_3(q) \in \mathcal{M}_{24}(\Gamma_0(162))$. The desirable congruence follows from $U_9F_3(q) \equiv 0 \pmod 9$, which can be proved by checking the first 74 coefficients as the dimension of $\mathcal{M}_{24}(\Gamma_0(18))$ is 73. □

4. Concluding Remarks

Numerics suggest that

$$\begin{aligned}
P_3(3n+2) &\equiv 0 \pmod 3,\\
P_6(9n+7) &\equiv 0 \pmod 3,\\
P_9(9n+6) &\equiv 0 \pmod 3,\\
P_{12}(9n+5) &\equiv 0 \pmod 3,\\
P_{15}(9n+4) &\equiv 0 \pmod 3,\\
P_{18}(9n+3) &\equiv 0 \pmod 3,\\
P_{21}(3n+2) &\equiv 0 \pmod 3,\\
P_{24}(9n+7) &\equiv 0 \pmod 3,
\end{aligned}$$

$$P_{27}(3n+2) \equiv 0 \pmod{3},$$
$$P_{30}(9n+8) \equiv 0 \pmod{3},$$
$$P_{33}(9n+7) \equiv 0 \pmod{3}.$$

where the first two congruences have been proven. Most of the above congruences are U_9 congruences as in the proof of Theorem 3.2. It would be very interesting if there is a systemic way to explain all of the above congruences. In other words, is there a criterion on k when $P_{3k}(9n+9-\ell(k)) \equiv 0 \pmod{3}$ holds, where $\ell(k) \equiv k \pmod{9}$?

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