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# A CLASS OF $\varphi$-RECURRENT ALMOST COSYMPLECTIC SPACE

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**Abstract.** In this paper, we study $\varphi$-recurrent almost cosymplectic $(\kappa, \mu)$-space and prove that it is an $\eta$-Einstein manifold with constant coefficients. Next, we show that a three-dimensional locally $\varphi$-recurrent almost cosymplectic $(\kappa, \mu)$-space is the space of constant curvature.

## 1. Introduction

The notion of an almost cosymplectic manifold was introduced by Goldberg and Yano in 1969 [16]. The simplest examples of such manifolds are those being the products (possibly local) of almost Kählerian manifolds and the real line $\mathbb{R}$ or the circle $S^1$. In particular, cosymplectic manifolds in the sense of Blair [2] are of this type. However, the class of almost cosymplectic manifolds is much more wider. There are already many known examples (among others, compact, homogeneous) of such manifolds which are not products (even locally). See Cordero et al. [12] Chinea and Gonzalez [9] and Olszak ([22],[23]).

The topology of cosymplectic manifolds was studied by Blair and Goldberg [3], Chinea et al. ([10], [11]) and others. Most of the results of Libermann [20], Lichnerowicz [21], Fujimoto and Muto [15] also have applications in characterizing of topological and analytical properties of almost cosymplectic manifolds (these authors have used a different terminology).

Curvature properties of almost cosymplectic manifolds were studied mainly by Goldberg and Yano [16] ,Olszak ([22], [23]), Kirichenko [18] and Endo [14]. We relate some of them in a historical order.

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Blair et al. [4] introduced the notion of $(\kappa,\mu)$-contact metric manifolds, where $\kappa$ and $\mu$ are real numbers. The full classification of these manifolds was given by Boeckx [6]. Later Koufogiorgos and Tsichlias [19] introduced the generalized $(\kappa,\mu)$-contact metric manifolds where $\kappa$ and $\mu$ are real functions and they gave several examples. Finally, the $(\kappa,\mu,\nu)$-contact metric manifolds have been recently introduced by Koufogiorgos, Markellos and Papantoniou where $\kappa,\mu,\nu$ are are smooth functions. They proved that these manifolds exist only in the dimension 3, whereas such a manifold in dimension greater than 3 is $(\kappa,\mu)$-contact metric manifolds. Aktan studied almost $\alpha$-cosymplectic $(\kappa,\mu,\nu)$-spaces [25].

The notion of local symmetry of a Riemannian manifold has been weakened by many authors in several ways to a different extent. As a weaker version of local symmetry, [26] introduced the notion of locally $\varphi$-symmetry on a Sasakian manifold. Generalizing the notion of $\varphi$-symmetry, one of authors, [13] introduced the $\varphi$-recurrent Sasakian manifold. In the context of contact geometry the notion of $\varphi$-symmetry is introduced and studied by Boeckx, Buecken and Vanhecke [7] with several examples. In [5], Boeckx proved that every non-Sasakian $(\kappa,\mu)$-manifold is locally $\varphi$-symmetric in the strong sense. J.-B. Jun et al. studied $\varphi$-recurrent $(\kappa,\mu)$-contact metric manifolds in [17].

In this paper we consider $\varphi$-recurrent almost cosymplectic $(\kappa,\mu)$-space and we prove that $\varphi$-recurrent almost cosymplectic $(\kappa,\mu)$-space is an $\eta$-Einstein manifold with constant coefficients. Furthermore, we obtain some results on a three-dimensional locally $\varphi$-recurrent almost cosymplectic $(\kappa,\mu)$-space with constant curvature.

## 2. Preliminaries

Let $\left(M^{2n+1},\varphi,\xi,\eta,g\right)$ be a $(2n+1)$-dimensional almost contact Riemannian manifold, where $\varphi$ is a $(1,1)$-tensor field, $\xi$ is the structure vector field, $\eta$ is a 1-form and $g$ is Riemannian metric. It is well-known that $\varphi,\xi,\eta,g$ satisfy

$$\eta(\xi)=1,\quad \varphi\xi=0,\quad \eta\circ\varphi=0, \tag{1}$$

$$\varphi^2X=-X+\eta(X)\xi,\quad \eta(X)=g(X,\xi), \tag{2}$$

$$g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\eta(Y), \tag{3}$$

for any vector fields $X, Y$ on $M^{2n+1}$.

The 2-form $\Phi$ of $M^{2n+1}$ defined by $\Phi(X,Y) = g(\varphi X, Y)$, is called the fundamental 2-form. Almost contact metric manifolds such that both $\eta$ and $\varphi$ are closed are called almost cosymplectic manifolds. Finally, a normal almost cosymplectic manifold is called a cosymplectic manifold. On almost cosymplectic manifold $M^{2n+1}$, we can define a $(1,1)$-tensor field $h$ by $h = \frac{1}{2} L_\xi \varphi$, and $lX = R(X,\xi)\xi$ where $L$ and $R$ denote Lie differentiation and curvature tensor respectively. Then we may observe that $h$ is symmetric and satisfies

$$h\xi = 0, \quad l\xi = 0, \quad h\varphi = -\varphi h, \quad \nabla_X \xi = -\varphi h X, \tag{4}$$

where $\nabla$ is Levi-Civita connection [4].

## 3. Almost cosymplectic manifold with $(\kappa,\mu)$-Nullity Distribution

Let $M^{2n+1}$ be an almost cosymplectic manifold. The $(\kappa,\mu)$-nullity distribution of $M^{2n+1}$ for the pair $(\kappa,\mu)$ is a distribution

$$\begin{aligned} &N(\kappa,\mu) : p \to N_p(\kappa,\mu) = \{Z \in T_pM \mid R(X,Y)Z \\ &= \kappa [g(Y,Z)X - g(X,Z)Y] + \mu [g(Y,Z)hX - g(X,Z)hY]\}, \end{aligned} \tag{5}$$

where $\kappa, \mu \in \mathbb{R}$ and $\kappa \leq 0$.

If the characteristic vector field $\xi$ belongs to the $(\kappa,\mu)$-nullity distribution then we have

$$R(X,Y)\xi = \kappa [\eta(Y)X - \eta(X)Y] + \mu [\eta(Y)hX - \eta(X)hY]. \tag{6}$$

Then $M^{2n+1}$ is called almost cosymplectic $(\kappa,\mu)$-space. An almost cosymplectic $(\kappa,\mu)$-space satisfies the following curvature properties [25]:

$$lX = -\kappa\varphi^2 X + \mu h X, \tag{7}$$

$$l\varphi X - \varphi l X = 2\mu h \varphi X, \tag{8}$$

$$h^2 X = \kappa \varphi^2 X, \text{for } \kappa \leq 0, \tag{9}$$

$$(\nabla_\xi h) X = -\mu \varphi h X, \tag{10}$$

$$\left(\nabla_\xi h^2\right) X = 0, \tag{11}$$

$$\xi(\kappa) = 0, \tag{12}$$

$$(\nabla_X \varphi)Y = g(hX, Y)\xi - \eta(Y)hX, \tag{13}$$

$$\begin{aligned}(\nabla_X \varphi)Y - (\nabla_Y \varphi)X = &-\kappa(\eta(Y)X - \eta(X)Y) \\ &- \mu(\eta(Y)hX - \eta(X)hY).\end{aligned} \tag{14}$$

**Lemma 3.1.** *If the almost cosymplectic manifold $M^{2n+1}$ with belonging to the $(\kappa, \mu)$-nullity distribution, then:*

(i) $(\nabla_X h)Y - (\nabla_Y h)X = \kappa(\eta(Y)\varphi X - \eta(X)\varphi Y + 2g(\varphi X, Y)\xi) + \mu(\eta(Y)\varphi hX - \eta(X)\varphi hY)$,
(ii) $R(\xi, X)Y = \kappa[g(X, Y)\xi - \eta(Y)X] + \mu[g(hX, Y)\xi - \eta(Y)hX]$,
(iii) $S(X, \xi) = 2n\kappa\eta(X)$,

*where $X$ and $Y$ are vector fields on $M^{2n+1}$, $\kappa$, $\mu \in \mathbb{R}$ and $S$ is the Ricci tensor of $M^{2n+1}$* [4].

**Theorem 3.2.** *If $M$ is an almost cosymplectic $(\kappa, \mu)$-space with $\kappa < 0$, where $\kappa, \mu$ only vary in the direction of $\xi$, then*

$$R(X_\lambda, Y_\lambda)Z_{-\lambda} = \kappa\{g(\varphi Y_\lambda, Z_{-\lambda})\varphi X_\lambda - g(\varphi X_\lambda, Z_{-\lambda})\varphi Y_\lambda\},$$
$$R(X_{-\lambda}, Y{-}_\lambda)Z_\lambda = \kappa\{g(\varphi Y_{-\lambda}, Z_\lambda)\varphi X_{-\lambda} - g(\varphi X_{-\lambda}, Z_\lambda)\varphi Y_{-\lambda}\},$$
$$R(X_\lambda, Y_{-\lambda})Z_{-\lambda} = -\kappa g(X_\lambda, \varphi Z_{-\lambda})\varphi Y_{-\lambda},$$
$$R(X_\lambda, Y_{-\lambda})Z_\lambda = -\kappa g(Z_\lambda, \varphi Y_{-\lambda})\varphi X_\lambda,$$
$$R(X_\lambda, Y_\lambda)Z_\lambda = 0,$$
$$R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} = 0,$$

*where $X_{\pm\lambda}, Y_{\pm\lambda}, Z_{\pm\lambda}$ are eigenvectors of $h$ associated to the eigenvalues $\pm\lambda = \pm\sqrt{-\kappa}$* [8].

**Lemma 3.3.** *Let $M$ be an almost cosymplectic manifold with belonging to the $(\kappa, \mu)$-nullity distribution. For any vector field $X$, the Ricci operator $Q$ is given by*

$$QX = \mu hX + 2n\kappa\eta(X)\xi. \tag{15}$$

*Proof.* We can easily get the proof by the above theorem. □

**Lemma 3.4.** *If almost cosymplectic manifold with $\xi$ belonging to the $(\kappa, \mu)$-nullity distribution, then*

$$\begin{aligned}(\nabla_X S)(Y, Z) = &\mu g((\nabla_X h)Y, Z) - 2n\kappa g(Y, \varphi hX)\eta(Z) \\ &- 2n\kappa\eta(Y)g(Z, \varphi hX).\end{aligned} \tag{16}$$

*Proof.* From (15) we get

$$S(Y,Z) = \mu g(hY,Z) + 2n\kappa\eta(Y)\eta(Z).$$

Taking covariant derivative by the direction $X$, we can write

$$(\nabla_X S)(Y,Z) = \nabla_X S(Y,Z) - S(\nabla_X Y,Z) - S(Y,\nabla_X Z).$$

Using (1)-(4) and (15), we get (16). Hence, the lemma is proved completely. □

**Proposition 3.5.** *Let $M^{2n+1}$ be a $(2n+1)$-dimensional almost cosymplectic $(\kappa,\mu)$-space. Then*

$$Q\varphi X - \varphi Q X = 2\mu h\varphi X, \tag{17}$$

$$S(\varphi X,\varphi Y) = -S(X,Y) + 2n\kappa\eta(X)\eta(Y), \tag{18}$$

$$r = 2n\kappa, \tag{19}$$

$$(\nabla_X\eta)Y = -g(Y,\ \varphi hX), \tag{20}$$

*Also in $(\kappa,\mu)$-manifold, the following holds*

$$\begin{aligned}\eta(R(X,Y)Z) =& \kappa[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\\ &+ \mu[g(hY,Z)\eta(X) - g(hX,Z)\eta(Y)].\end{aligned} \tag{21}$$

## 4. $\varphi$-Recurrent Almost Cosymplectic $(\kappa,\mu)$-Spaces

**Definition 4.1** ([26])**.** *A Sasakian manifold is said to be **locally $\varphi$-symmetric** if the relation*

$$\varphi^2((\nabla_W R)(X,Y,Z)) = 0,$$

*holds for all vector fields $X,Y,Z,W$ orthogonal to $\xi$.*

**Definition 4.2** ([13])**.** *A $(\kappa,\mu)$-contact metric manifold is said to be **$\varphi$-recurrent** if and only if there exists a non-zero 1-form $A$ such that*

$$\varphi^2((\nabla_W R)(X,Y,Z)) = A(W)R(X,Y,Z), \tag{22}$$

*for all vector fields $X,Y,Z,W$. Here $X,Y,Z,W$ are arbitrary vector fields which are not necessarily orthogonal to $\xi$.*

If $X,Y,Z,W$ are orthogonal to $\xi$, then the manifold is called locally $\varphi$-recurrent. If the 1-form $A$ vanishes identically, then the manifold is said to be a locally $\varphi$-symmetric manifold.

**Definition 4.3** ([4])**.** *A contact metric manifold is said to be $\eta$-**Einstein** if the Ricci tensor $S$ of type $(0,2)$ satisfies the condition*

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{23}$$

*where $a$ and $b$ are smooth functions on $M^{2n+1}$.*

Now, we prove the main result of this paper.

**Theorem 4.4.** *A $\varphi$-recurrent almost cosymplectic $(\kappa,\mu)$-space is an $\eta$-Einstein manifold with constant coefficients.*

*Proof.* By virtue of (2) and (22) we have

$$-(\nabla_W R)(X,Y,Z) + \eta((\nabla_W R)(X,Y,Z))\xi = A(W)R(X,Y,Z), \tag{24}$$

from which it follows that

$$\begin{aligned} -g((\nabla_W R)(X,Y,Z),U) + \eta((\nabla_W R)(X,Y,Z))\eta(U) \\ = A(W)g(R(X,Y,Z),U). \end{aligned} \tag{25}$$

Let $\{e_i\}$, $i = 1,2,3,...2n+1$, be an orthonormal basis of the tangent space at any point of the manifold. Putting $X = U = \{e_i\}$ in (25) and taking summation over $i$, $1 \leq i \leq 2n+1$, we get

$$-(\nabla_W S)(Y,Z) + \sum_{i=1}^{2n+1} \eta((\nabla_W R)(e_i,Y)Z)\eta(e_i) = A(W)S(Y,Z). \tag{26}$$

The second term of (26) by putting $Z = \xi$ takes the form

$$g((\nabla_W R)(e_i,Y)\xi,\xi)g(e_i,\xi),$$

which is denoted by $E$. In this case $E$ vanishes. Since the following equation is well known,

$$\begin{aligned} g((\nabla_W R)(e_i,Y)\xi,\xi) =& g(\nabla_W R(e_i,Y)\xi,\xi) - g(R(\nabla_W e_i,Y)\xi,\xi) \\ & - g(R(e_i,\nabla_W Y)\xi,\xi) - g(R(e_i,Y)\nabla_W \xi,\xi), \end{aligned}$$

at $p \in M$. Using (2) and (6), we obtain

$$\begin{aligned} & g(R(e_i,\nabla_W Y)\xi,\xi) \\ & = g(\kappa[\eta(\nabla_W Y)e_i - \eta(e_i)\nabla_W Y] + \mu[\eta(\nabla_W Y)he_i - \eta(e_i)h\nabla_W Y],\xi) \\ & = \kappa[\eta(\nabla_W Y)\eta(e_i) - \eta(e_i)\eta(\nabla_W Y)] \\ & = 0, \end{aligned}$$

since $g(hX,Y) = g(X,hY)$. Thus, we derive

$$g((\nabla_W R)(e_i,Y)\xi,\xi) = g(\nabla_W R(e_i,Y)\xi,\xi) - g(R(e_i,Y)\nabla_W \xi,\xi).$$

In virtue of $g\left(R\left(e_i, Y\right)\xi, \xi\right) = g\left(R\left(\xi, \xi\right)e_i, Y\right) = 0$, we have

$$g\left(\left(\nabla_W R\right)\left(e_i, Y\right)\xi, \xi\right) + g\left(R\left(e_i, Y\right)\nabla_W \xi, \xi\right) = 0,$$

since $\left(\nabla_W g\right) = 0$, which implies

$$g\left(\left(\nabla_W R\right)\left(e_i, Y\right)\xi, \xi\right) = -g\left(R\left(e_i, Y\right)\xi, \nabla_W \xi\right) - g\left(R\left(e_i, Y\right)\nabla_W \xi, \xi\right) = 0.$$

Using (4) and applying skew-symmetry of $R$, we find

$$\begin{aligned} g\left(\left(\nabla_W R\right)\left(e_i, Y\right)\xi, \xi\right) &= g\left(R\left(e_i, Y\right)\xi, \varphi hW\right) + g\left(R\left(e_i, Y\right)\varphi hW, \xi\right) \\ &= g\left(R\left(\varphi hW, \xi\right)Y, e_i\right) + g\left(R\left(\xi, \varphi hW\right)Y, e_i\right). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} E &= \sum_{i=1}^{2n+1}\left[g\left(R\left(\varphi hW, \xi\right)Y, e_i\right)g\left(\xi, e_i\right) + g\left(R\left(\xi, \varphi hW\right)Y, e_i\right)g\left(\xi, e_i\right)\right] \\ &= g\left(R\left(\varphi hW, \xi\right)Y, \xi\right) + g\left(R\left(\xi, \varphi hW\right)Y, \xi\right) = 0. \end{aligned}$$

Replacing $Z$ by $\xi$ in (26) and using third relation of Lemma 3.1, we have

$$-\left(\nabla_W S\right)\left(Y, \xi\right) = 2n\kappa A\left(W\right)\eta\left(Y\right). \tag{27}$$

Now, we have

$$\left(\nabla_W S\right)\left(Y, \xi\right) = \nabla_W S\left(Y, \xi\right) - S\left(\nabla_W Y, \xi\right) - S\left(Y, \nabla_W \xi\right).$$

Using (4) and the third relation of Lemma 3.1 in the above relation, it follows that

$$\left(\nabla_W S\right)\left(Y, \xi\right) = 2n\kappa\left(\nabla_W \eta\right)Y + S\left(Y, \varphi hW\right). \tag{28}$$

By virtue of $g\left(\varphi X, Y\right) = -g\left(X, \varphi Y\right)$ and (20), we get from (28)

$$\left(\nabla_W S\right)\left(Y, \xi\right) = -2n\kappa g\left(Y, \varphi hW\right) + S\left(Y, \varphi hW\right). \tag{29}$$

From (27) and (29), we derive

$$2n\kappa A\left(W\right)\eta\left(Y\right) = 2n\kappa g\left(Y, \varphi hW\right) - S\left(Y, \varphi hW\right). \tag{30}$$

Replacing $Y$ by $\varphi Y$ in (30) and using (1), (3) and (18), we obtain

$$2n\kappa g\left(\varphi Y, \varphi hW\right) - S\left(\varphi Y, \varphi hW\right) = 0,$$

or

$$2n\kappa g\left(Y, hW\right) + S\left(Y, hW\right) = 0.$$

From the last equation, we find

$$S\left(Y, hW\right) = 2n\kappa g\left(Y, hW\right). \tag{31}$$

Replacing $W$ by $hW$ and using (9), we get

$$S\left(Y, W\right) = \alpha g\left(Y, W\right) + \beta g\left(Y, W\right),$$

where $\alpha = -2n\kappa$ and $\beta = 4n\kappa$. So, the manifold is an $\eta$-Einstein manifold with constant coefficients. Hence the theorem is proved. □

**Theorem 4.5.** *In a $\varphi$-recurrent almost cosymplectic $(\kappa, \mu)$-space $\left(M^{2n+1}, g\right)$ $(n > 1)$ the characteristic vector field $\xi$ and the vector field $\rho$ associated to the 1-form $A$ are co-directional and the 1-form $A$ is given by*

$$A(W) = \eta(W)\,\eta(\rho),$$

*provided that $(2n-1)^2\kappa + \mu^2 \neq 0$.*

*Proof.* In an almost cosymplectic $(\kappa, \mu)$-space, the relation (24) holds. Changing $W, X, Y$ cyclically in (24) and then adding the results, thus we derive

$$\begin{aligned}
&-\left[(\nabla_W R)(X,Y)Z + (\nabla_X R)(Y,W)Z + (\nabla_Y R)(W,X)Z\right] \\
&+\left[\eta\left((\nabla_W R)(X,Y)Z\right) + \eta\left((\nabla_X R)(Y,W)Z\right) + \eta\left((\nabla_Y R)(W,X)Z\right)\right] \\
&= A(W)R(X,Y)Z + A(X)R(Y,W)Z + A(Y)R(W,X)Z,
\end{aligned}$$

which yields by virtue of Bianchi's identity that

$$A(W)\,\eta(R(X,Y)Z) + A(X)\,\eta(R(Y,W)Z) + A(Y)\,\eta(R(W,X)Z) = 0. \tag{32}$$

With the help of (21), (32) reduces to

$$\begin{aligned}
&A(W)\left[\kappa\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\right. \\
&\left.+\mu\{g(hY,Z)\eta(X) - g(hX,Z)\eta(Y)\}\right] \\
&A(X)\left[\kappa\{g(W,Z)\eta(Y) - g(Y,Z)\eta(W)\}\right. \\
&\left.+\mu\{g(hW,Z)\eta(Y) - g(hY,Z)\eta(W)\}\right] \\
&A(Y)\left[\kappa\{g(X,Z)\eta(W) - g(W,Z)\eta(X)\}\right. \\
&\left.+\mu\{g(hX,Z)\eta(W) - g(hW,Z)\eta(X)\}\right] \\
&= 0.
\end{aligned} \tag{33}$$

Putting $Y = Z = e_i$ in (33) and taking summation over $1 \leq i \leq 2n+1$, we find

$$\begin{aligned}
&(2n-1)\kappa\left[A(W)\eta(X) - A(X)\eta(W)\right] \\
&+\mu\left[A(hX)\eta(W) - A(hW)\eta(X)\right] = 0.
\end{aligned} \tag{34}$$

Substituting $X$ by $\xi$ in (34), we have

$$(2n-1)\kappa\left[A(W) - A(\xi)\eta(W)\right] - \mu A(hW) = 0 \tag{35}$$

Replacing $W$ by $hW$ in (35) and using (9), we get

$$(2n-1)\,\kappa A\,(hW) = \mu\kappa\,[-A\,(W) + \eta\,(W)\,A\,(\xi)]\,. \tag{36}$$

From (35) and (36), we obtain

$$A\,(W) = A\,(\xi)\,\eta\,(W) = \eta\,(\rho)\,\eta\,(W)\,,$$

provided that

$$(2n-1)^2\,\kappa + \mu^2 \neq 0,$$

where $A\,(\xi) = g\,(\xi,\rho)$. This completes the proof. □

## 5. 3-dimensional locally $\varphi$-recurrent almost cosymplectic $(\kappa,\mu)$-spaces

On any 3-dimensional Riemannian manifold, we have

$$\begin{aligned} R\,(X,Y)\,Z =& g\,(Y,Z)\,QX - g\,(X,Z)\,QY + S\,(Y,Z)\,X \\ &- S\,(X,Z)\,Y - \frac{r}{2}\,[g\,(Y,Z)\,X - g\,(X,Z)\,Y]\,, \end{aligned} \tag{37}$$

for any vector fields $X, Y, Z$, where $Q$ is the Ricci operator, that is, $g\,(QX,Y) = S\,(X,Y)$ and $r$ is the scalar curvature of the manifold. Moreover, using (15), we have

$$QX = -\mu\lambda X, \tag{38}$$

where $\lambda = \sqrt{-\kappa}$, $\kappa \leq 0$. Therefore, it follows from (38) that

$$S\,(X,Y) = -\mu\lambda\,g\,(X,Y)\,. \tag{39}$$

Thus from (37), (38) and (39), we get

$$\begin{aligned} R\,(X,Y)\,Z =& 2\mu\lambda\,[g\,(Y,Z)\,X - g\,(X,Z)\,Y] \\ &- \frac{r}{2}\,[g\,(Y,Z)\,X - g\,(X,Z)\,Y]\,, \end{aligned} \tag{40}$$

Taking the covariant differentiation to the both sides of the equation (40), we derive

$$(\nabla_W R)\,(X,Y)\,Z = \frac{dr\,(W)}{2}\,[g\,(X,Z)\,Y - g\,(Y,Z)\,X]\,. \tag{41}$$

Applying $\varphi^2$ to the both sides of (41) and using (1), we find that

$$\varphi^2\,(\nabla_W R)\,(X,Y)\,Z = \frac{dr\,(W)}{2}\,\left[g\,(X,Z)\,\varphi^2 Y - g\,(Y,Z)\,\varphi^2 X\right]. \tag{42}$$

By (22) the equation (42) reduces to

$$A\,(W)\,R\,(X,Y,Z) = \frac{dr\,(W)}{2}\,[g\,(Y,Z)\,X - g\,(X,Z)\,Y + g\,(X,Z)\,\eta\,(Y)\,\xi - g\,(Y,Z)\,\eta\,(X)\,\xi]\,.$$

Notice that we may assume that all vector fields $X, Y, Z, W$ are orthogonal to $\xi$, thus we derive

$$A(W)R(X,Y,Z) = \frac{dr(W)}{2}\left[g(Y,Z)X - g(X,Z)Y\right].$$

Putting $W = \{e_i\}$, where $\{e_i\}$, $i = 1, 2, 3$ is an orthonormal basis of the tangent space at any point of the manifold and taking summation over $i$, $1 \leq i \leq 3$, we obtain

$$R(X,Y,Z) = \lambda\left[g(Y,Z)X - g(X,Z)Y\right],$$

where $\lambda = \dfrac{dr(e_i)}{2A(e_i)}$ is scalar, since $A$ is a non-zero 1-form. Then by Schur's theorem $\lambda$ will be a constant on the manifold. Therefore, $M^3$ is a constant curvature $\lambda$. Thus we get the following theorem:

**Theorem 5.1.** *A 3-dimensional connected locally $\varphi$-recurrent almost cosymplectic $(\kappa, \mu)$-space is the space of constant curvature.*

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