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NEWTON INEQUALITIES FOR p-HARMONIC CONVEX FUNCTIONS

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Abstract. In this paper, we establish some new Newton's type integral inequalities for p-harmonic convex functions. Some special cases are also discussed as applications of our main results. Results obtained in this paper may be starting point for further research.

1. Introduction

Convexity theory is an effective and powerful technique for studying a wide class of problems which arise in various branches of pure and applied sciences. It is well known that inequalities have played a fundamental role in the development of almost all the fields of pure and applied sciences, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19]. It is worth mentioning that the weighted arithmetic mean is used to define the convex set. Related to the arithmetic mean, we have harmonic mean which is used to define the harmonic convex set and it has applications in electrical circuit theory and other branches of sciences. It is known that the total resistance of a set of parallel resistors is obtained by adding up the reciprocal of the individual resistance value and then considering the reciprocal of their total. For example, if R_1 and R_2 are the resistance of two parallel resistors, then the total resistance $R = \frac{R_1R_2}{R_1+R_2}$ is simply the half of the harmonic mean. The harmonic convex set was introduced in 2013 by Shi et. al. [18]. A significant class of convex functions, called harmonic convex, was introduced by Anderson et al.[1] and Iscan [6], independently. See also [6, 7] and the references therein. Noor et. al.[11] introduced the class of relative harmonic convex functions with respect

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to an arbitrary non-negative function h. This class is more general and contains several known and new classes of harmonic convex functions as a special case. Another class of convex functions is considered by Noor et. al. [13], which is called p-harmonic convex function and derived several Hermite-Hadamard type inequalities.

Gao et al. [3] and Noor et al. [12] have obtained Newton's type inequalities for the class of functions, whose second derivatives in absolute value are convex functions. In this paper, we derive a new integral identity for differentiable p-harmonic convex functions. This auxiliary result is used to obtain some new Newton's type inequalities for p-harmonic convex functions. Some special cases are also discussed. Using the technique of this paper, one can derive Two-point Trapezoidal type inequalities for p-harmonic convex functions. It is an interesting problem to obtain Newton type inequalities for coordinated p-harmonic convex functions. The ideas and technique of this paper may inspire further research in this field.

2. Preliminaries

In this section we recall some known concept.

Definition 2.1 ([16]). *A set $I \subset \mathbb{R}^n$ is said to be a* **convex set**, *if*

$$(1-t)x + ty \in I, \qquad \forall x, y \in I,\ t \in [0,1].$$

Definition 2.2 ([16]). *A function $f : I \to \mathbb{R}$ is said to be a* **convex function**, *if*

$$f((1-t)x + ty) \le (1-t)f(x) + tf(y), \qquad \forall x, y \in I,\ t \in [0,1].$$

Note that, if $t = \frac{1}{2}$, then we have

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2}, \qquad \forall x, y \in I,$$

which is known as Jensen or mid-convex function.

Noor et al [13, 14] introduced and considered some new concepts of p-harmonic convex sets and p-harmonic convex functions, which are unifying ones.

Definition 2.3. *A set $H_p \subseteq \mathbb{R}^n \setminus \{0\}$ is said to be* **p-harmonic convex set**, *if*

$$\left[\frac{x^p y^p}{tx^p + (1-t)y^p}\right]^{\frac{1}{p}} \in I, \quad \forall x, y \in H_p,\ t \in [0,1], \quad p \ne 0.$$

Definition 2.4. *A function* $f : H_p \to \mathbb{R}$ *is said to be* p**-harmonic convex function**, *if*

$$f\left(\left[\frac{x^p y^p}{tx^p + (1-t)y^p}\right]^{\frac{1}{p}}\right) \leq (1-t)f(x)+tf(y), \forall x, y \in H_p,\ t \in [0,1]. \tag{1}$$

Note that for $t = \frac{1}{2}$ in (1), we have

$$f\left(\left[\frac{2x^p y^p}{x^p + y^p}\right]^{\frac{1}{p}}\right) \leq \frac{f(x)+f(y)}{2}, \quad \forall x, y \in H_p.$$

The function f is called **Jensen** p**-harmonic convex function**.

It is clear that, if $p = -1$ and $p = 1$, then the definitions of p-harmonic convex sets and p-harmonic convex functions reduce to definition of classical convex sets, harmonic convex sets and harmonic convex convex functions, respectively.

Noor et. al [13, 14] have obtained the Hermite-Hadamard inequality for p-harmonic convex function as follows.

Theorem 2.5. *Let* $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *be a harmonic* p*-convex function on the interval* $[a, b]$*. Then for any* $\lambda \in [0, 1]$*, we have*

$$\begin{aligned} f\left(\left[\frac{2a^p b^p}{a^p + b^p}\right]^{\frac{1}{p}}\right) &\leq (1-\lambda)f\left(\left[\frac{2a^p b^p}{(1-\lambda)a^p + (\lambda+1)b^p}\right]^{\frac{1}{p}}\right) \\ &\quad + \lambda f\left(\left[\frac{2a^p b^p}{(2-\lambda)a^p + \lambda b^p}\right]^{\frac{1}{p}}\right) \\ &\leq \frac{pa^p b^p}{b^p - a^p}\int_a^b \frac{f(x)}{x^{1+p}}\mathrm{d}x \\ &\leq \frac{1}{2}\left[f\left(\left[\frac{a^p b^p}{(1-\lambda)a^p + \lambda b^p}\right]^{\frac{1}{p}}\right) + (1-\lambda)f(a) + \lambda f(b)\right] \\ &\leq \frac{f(a)+f(b)}{2}. \end{aligned} \tag{2}$$

3. Main Results

First of all, we prove an auxiliary Lemma which plays an important role in order to prove our main results.

Lemma 3.1. *Let $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a, b]$, then*

$$\frac{1}{8}\left[f(a)+3f\left(\left[\frac{3a^pb^p}{a^p+2b^p}\right]^{\frac{1}{p}}\right)+3f\left(\left[\frac{3a^pb^p}{2a^p+b^p}\right]^{\frac{1}{p}}\right)+f(b)\right] - \frac{pa^pb^p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1+p}}\mathrm{d}x = \frac{ab(b^p-a^p)}{p}\int_0^1 \frac{\mu(t)}{A_t^{p+1}} f'\left(\frac{ab}{A_t}\right)\mathrm{d}t$$

where $A_t = [ta^p + (1-t)b^p]^{\frac{1}{p}}$ and

$$\mu(t) = \begin{cases} t-\frac{1}{8}, & t \in [0, \frac{1}{3}) \\ t-\frac{1}{2}, & t \in [\frac{1}{3}, \frac{2}{3}) \\ t-\frac{7}{8}, & t \in [\frac{2}{3}, 1]. \end{cases} \tag{3}$$

Proof. Let

$$\begin{aligned} I &= \frac{b^p-a^p}{pa^pb^p}\int_0^1 \mu(t)\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}+1} f'\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t \\ &= \frac{b^p-a^p}{pa^pb^p}\int_0^{\frac{1}{3}} \left(t-\frac{1}{8}\right)\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}+1} f'\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t \\ &\quad + \frac{b^p-a^p}{pa^pb^p}\int_{\frac{1}{3}}^{\frac{2}{3}} \left(t-\frac{1}{2}\right)\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}+1} f'\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t \\ &\quad + \frac{b^p-a^p}{pa^pb^p}\int_{\frac{2}{3}}^{1} \left(t-\frac{7}{8}\right)\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}+1} f'\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Thus

$$\begin{aligned} I_1 &= \frac{b^p-a^p}{pa^pb^p}\int_0^{\frac{1}{3}} \left(t-\frac{1}{8}\right)\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}+1} f'\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t \\ &= \left|(t-\frac{1}{8})f\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\right|_0^{\frac{1}{3}} - \int_0^{\frac{1}{3}} f\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t \\ &= \frac{1}{8}f(a) + \frac{5}{24}f\left(\left[\frac{3a^pb^p}{a^p+2b^p}\right]^{\frac{1}{p}}\right) - \int_0^{\frac{1}{3}} f\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t \end{aligned}$$

Similarly, one can obtain

$$I_2=\frac{b^p-a^p}{pa^pb^p}\int_{\frac{1}{3}}^{\frac{2}{3}}\left(t-\frac{1}{2}\right)\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}+1}f'\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t$$
$$=\frac{1}{6}f\left(\left[\frac{3a^pb^p}{2a^p+b^p}\right]^{\frac{1}{p}}\right)+\frac{1}{6}f\left(\left[\frac{3a^pb^p}{a^p+2b^p}\right]^{\frac{1}{p}}\right)-\int_{\frac{1}{3}}^{\frac{2}{3}}f\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t$$

and

$$I_3=\frac{b^p-a^p}{pa^pb^p}\int_{\frac{2}{3}}^{1}\left(t-\frac{7}{8}\right)\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}+1}f'\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t$$
$$=\frac{1}{8}f(b)+\frac{5}{24}f\left(\left[\frac{3a^pb^p}{2a^p+b^p}\right]^{\frac{1}{p}}\right)-\int_{\frac{2}{3}}^{1}f\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t.$$

Adding I_1, I_2 and I_3, and making the change of variable $x=[\frac{a^pb^p}{ta^p+(1-t)b^p}]^{\frac{1}{p}}$, we have

$$I=I_1+I_2+I_3$$
$$=\frac{1}{8}\left[f(a)+3f\left(\left[\frac{3a^pb^p}{a^p+2b^p}\right]^{\frac{1}{p}}\right)\right.$$
$$\left.+3f\left(\left[\frac{3a^pb^p}{2a^p+b^p}\right]^{\frac{1}{p}}\right)+f(b)\right]-\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1+p}}\mathrm{d}x.$$

□

Theorem 3.2. *Let $f:I=[a,b]\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a differentiable function on the interior I^o of I. If $f'\in L[a,b]$ and $|f'|^q$ is p-harmonic convex function on I, $q\geq 1$, then*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\left[\frac{3a^pb^p}{a^p+2b^p}\right]^{\frac{1}{p}}\right)+3f\left(\left[\frac{3a^pb^p}{2a^p+b^p}\right]^{\frac{1}{p}}\right)+f(b)\right]-\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1+p}}\mathrm{d}x\right|$$
$$\leq\frac{ab(b^p-a^p)}{p}\{(\varphi_1(p;a,b))^{1-\frac{1}{q}}[\varphi_4(p;a,b)|f'(a)|^q+\varphi_5(p;a,b)|f'(b)|^q]^{\frac{1}{q}}$$
$$+(\varphi_2(p;a,b))^{1-\frac{1}{q}}[\varphi_6(p;a,b)|f'(a)|^q+\varphi_7(p;a,b)|f'(b)|^q]^{\frac{1}{q}}$$
$$+(\varphi_3(p;a,b))^{1-\frac{1}{q}}[\varphi_8(p;a,b)|f'(a)|^q+\varphi_9(p;a,b)|f'(b)|^q]^{\frac{1}{q}}\},$$

where

$$\varphi_1(p;a,b)=\int_0^{\frac{1}{3}}\frac{\left|t-\frac{1}{8}\right|}{A_t^{1+p}}\mathrm{d}t,$$
$$\varphi_2(p;a,b)=\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{\left|t-\frac{1}{2}\right|}{A_t^{1+p}}\mathrm{d}t,$$

$$\varphi_3(p;a,b)=\int_{\frac{2}{3}}^{1}\frac{\left|t-\frac{7}{8}\right|}{A_t^{1+p}}\mathrm{d}t,$$

(4)
$$\varphi_4(p;a,b)=\int_{0}^{\frac{1}{3}}\frac{\left|t-\frac{1}{8}\right|(1-t)}{A_t^{1+p}}\mathrm{d}t,$$

(5)
$$\varphi_5(p;a,b)=\int_{0}^{\frac{1}{3}}\frac{\left|t-\frac{1}{8}\right|t}{A_t^{1+p}}\mathrm{d}t,$$

(6)
$$\varphi_6(p;a,b)=\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{\left|t-\frac{1}{2}\right|(1-t)}{A_t^{1+p}}\mathrm{d}t,$$

(7)
$$\varphi_7(p;a,b)=\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{\left|t-\frac{1}{2}\right|t}{A_t^{1+p}}\mathrm{d}t,$$

(8)
$$\varphi_8(p;a,b)=\int_{\frac{2}{3}}^{1}\frac{\left|t-\frac{7}{8}\right|(1-t)}{A_t^{1+p}}\mathrm{d}t,$$

(9)
$$\varphi_9(p;a,b)=\int_{\frac{2}{3}}^{1}\frac{\left|t-\frac{7}{8}\right|t}{A_t^{1+p}}\mathrm{d}t.$$

Proof. Using Lemma 3.1, power mean inequality and p-harmonic convexity of the $|f'|^q$, we have

$$\begin{aligned}
&\left|\frac{1}{8}\left[f(a)+3f\left(\left[\frac{3a^pb^p}{a^p+2b^p}\right]^{\frac{1}{p}}\right)+3f\left(\left[\frac{3a^pb^p}{2a^p+b^p}\right]^{\frac{1}{p}}\right)+f(b)\right]-\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1+p}}\mathrm{d}x\right|\\
&\le\frac{ab(b^p-a^p)}{p}\left[\int_0^{\frac{1}{3}}\frac{|t-\frac{1}{8}|}{A_t^{1+p}}\left|f'\left(\frac{ab}{A_t}\right)\right|\mathrm{d}t+\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{|t-\frac{1}{2}|}{A_t^{1+p}}\left|f'\left(\frac{ab}{A_t}\right)\right|\mathrm{d}t+\int_{\frac{2}{3}}^{1}\frac{|t-\frac{7}{8}|}{A_t^{1+p}}\left|f'\left(\frac{ab}{A_t}\right)\right|\mathrm{d}t\right]\\
&\le\frac{ab(b^p-a^p)}{p}\left[\left(\int_0^{\frac{1}{3}}\frac{|t-\frac{1}{8}|}{A_t^{1+p}}\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{3}}\frac{|t-\frac{1}{8}|}{A_t^{1+p}}\left|f'\left(\frac{ab}{A_t}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right.\\
&\quad+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{|t-\frac{1}{2}|}{A_t^{1+p}}\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{|t-\frac{1}{2}|}{A_t^{1+p}}\left|f'\left(\frac{ab}{A_t}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\int_{\frac{2}{3}}^{1}\frac{|t-\frac{7}{8}|}{A_t^{1+p}}\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_{\frac{2}{3}}^{1}\frac{|t-\frac{7}{8}|}{A_t^{1+p}}\left|f'\left(\frac{ab}{A_t}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right]\\
&\le\frac{ab(b^p-a^p)}{p}\left[\left(\int_0^{\frac{1}{3}}\frac{|t-\frac{1}{8}|}{A_t^{1+p}}\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{3}}\frac{|t-\frac{1}{8}|\left[(1-t)|f'(a)|^q+t|f'(b)|^q\right]}{A_t^{1+p}}\mathrm{d}t\right)^{\frac{1}{q}}\right.\\
&\quad+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{|t-\frac{1}{2}|}{A_t^{1+p}}\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{|t-\frac{1}{2}|\left[(1-t)|f'(a)|^q+t|f'(b)|^q\right]}{A_t^{1+p}}\mathrm{d}t\right)^{\frac{1}{q}}\\
&\quad\left.+\left(\int_{\frac{2}{3}}^{1}\frac{|t-\frac{7}{8}|}{A_t^{1+p}}\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_{\frac{2}{3}}^{1}\frac{|t-\frac{7}{8}|\left[(1-t)|f'(a)|^q+t|f'(b)|^q\right]}{A_t^{1+p}}\mathrm{d}t\right)^{\frac{1}{q}}\right]
\end{aligned}$$

$$= \frac{ab(b^p-a^p)}{p}\Bigg[\Big(\int_0^{\frac{1}{3}}\frac{|t-\frac{1}{8}|}{A_t^{1+p}}\mathrm{d}t\Big)^{1-\frac{1}{q}}\Big(\int_0^{\frac{1}{3}}\frac{|t-\frac{1}{8}|(1-t)}{A_t^{1+p}}|f'(a)|^q\mathrm{d}t+\int_0^{\frac{1}{3}}\frac{|t-\frac{1}{8}|t}{A_t^{1+p}}|f'(b)|^q\mathrm{d}t\Big)^{\frac{1}{q}}$$
$$+\Big(\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{|t-\frac{1}{2}|}{A_t^{1+p}}\mathrm{d}t\Big)^{1-\frac{1}{q}}\Big(\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{|t-\frac{1}{2}|(1-t)}{A_t^{1+p}}|f'(a)|^q\mathrm{d}t+\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{|t-\frac{1}{2}|t}{A_t^{1+p}}|f'(b)|^q\mathrm{d}t\Big)^{\frac{1}{q}}$$
$$+\Big(\int_{\frac{2}{3}}^{1}\frac{|t-\frac{7}{8}|}{A_t^{1+p}}\mathrm{d}t\Big)^{1-\frac{1}{q}}\Big(\int_{\frac{2}{3}}^{1}\frac{|t-\frac{7}{8}|(1-t)}{A_t^{1+p}}|f'(a)|^q\mathrm{d}t+\int_{\frac{2}{3}}^{1}\frac{|t-\frac{7}{8}|t}{A_t^{1+p}}|f'(b)|^q\mathrm{d}t\Big)^{\frac{1}{q}}\Bigg]$$
$$= \frac{ab(b^p-a^p)}{p}\{(\varphi_1(p;a,b))^{1-\frac{1}{q}}[\varphi_4(p;a,b)|f'(a)|^q+\varphi_5(p;a,b)|f'(b)|^q]^{\frac{1}{q}}$$
$$+(\varphi_2(p;a,b))^{1-\frac{1}{q}}[\varphi_6(p;a,b)|f'(a)|^q+\varphi_7(p;a,b)|f'(b)|^q]^{\frac{1}{q}}$$
$$+(\varphi_3(p;a,b))^{1-\frac{1}{q}}[\varphi_8(p;a,b)|f'(a)|^q+\varphi_9(p;a,b)|f'(b)|^q]^{\frac{1}{q}}\},$$

is the required result. □

Corollary 3.3. *Under the assumptions of Theorem* 3.2 *with* $p=-1$, *we have the following new result.*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|$$
$$\le (b-a)\Bigg\{\left(\frac{17}{576}\right)\left[\frac{973|f'(a)|^q+251|f'(b)|^q}{1224}\right]^{\frac{1}{q}}$$
$$+\left(\frac{1}{36}\right)\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}}+\left(\frac{17}{576}\right)\left[\frac{251|f'(a)|^q+973|f'(b)|^q}{1224}\right]^{\frac{1}{q}}\Bigg\}.$$

Theorem 3.4. *Let* $f: I=[a,b]\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ *be a differentiable function on the interior* I^o *of* I. *If* $f'\in L[a,b]$ *and* $|f'|^q$ *is p-harmonic convex function on I,* $r,q>1$ *and* $\frac{1}{r}+\frac{1}{q}=1$, *then*

$$\left|\frac{1}{8}\left[f(a)+3f\Big(\Big[\frac{3a^pb^p}{a^p+2b^p}\Big]^{\frac{1}{p}}\Big)+3f\Big(\Big[\frac{3a^pb^p}{2a^p+b^p}\Big]^{\frac{1}{p}}\Big)+f(b)\right]-\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1+p}}\mathrm{d}x\right|$$
$$(10)\quad \le \frac{ab(b^p-a^p)}{p}\Bigg[(\varphi_{10}(p;a,b))^{\frac{1}{r}}\Big(\frac{|f'(a)|^q+|f'([\frac{3a^pb^p}{a^p+2b^p}]^{\frac{1}{p}})|^q}{6}\Big)^{\frac{1}{q}}$$
$$+(\varphi_{11}(p;a,b))^{\frac{1}{r}}\Big(\frac{|f'([\frac{3a^pb^p}{a^p+2b^p}]^{\frac{1}{p}})|^q+|f'([\frac{3a^pb^p}{2a^p+b^p}]^{\frac{1}{p}})|^q}{6}\Big)^{\frac{1}{q}}$$
$$+(\varphi_{12}(p;a,b))^{\frac{1}{r}}\Big(\frac{|f'([\frac{3a^pb^p}{2a^p+b^p}]^{\frac{1}{p}})|^q+|f'(b)|^q}{6}\Big)^{\frac{1}{q}}\Bigg],$$

where

$$\varphi_{10}(p;a,b)=\int_0^{\frac{1}{3}}\frac{|t-\frac{1}{8}|^r}{A_t^{(1+p)r}}\mathrm{d}t,$$

$$\varphi_{11}(p;a,b)=\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{|t-\frac{1}{2}|^r}{A_t^{(1+p)r}}\mathrm{d}t,$$

$$\varphi_{12}(p;a,b)=\int_{\frac{2}{3}}^{1}\frac{|t-\frac{7}{8}|^r}{A_t^{(1+p)r}}\mathrm{d}t.$$

Proof. Using Lemma 3.1 and Holder's integral inequality, we have

$$\begin{aligned}
&\left|\frac{1}{8}\left[f(a)+3f\left(\left[\frac{3a^pb^p}{a^p+2b^p}\right]^{\frac{1}{p}}\right)+3f\left(\left[\frac{3a^pb^p}{2a^p+b^p}\right]^{\frac{1}{p}}\right)+f(b)\right]\right.\\
&\qquad\left.-\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1+p}}\mathrm{d}x\right|\\
&\le\frac{ab(b^p-a^p)}{p}\left[\int_0^{\frac{1}{3}}\left|\frac{t-\frac{1}{8}}{A_t^{1+p}}\right|\left|f'\left(\frac{ab}{A_t}\right)\right|\mathrm{d}t\right.\\
&\qquad+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{t-\frac{1}{2}}{A_t^{1+p}}\right|\left|f'\left(\frac{ab}{A_t}\right)\right|\mathrm{d}t\\
&\qquad\left.+\int_{\frac{2}{3}}^{1}\left|\frac{t-\frac{7}{8}}{A_t^{1+p}}\right|\left|f'\left(\frac{ab}{A_t}\right)\right|\mathrm{d}t\right]\\
&\le\frac{ab(b^p-a^p)}{p}\left[\left(\int_0^{\frac{1}{3}}\left|\frac{t-\frac{1}{8}}{A_t^{1+p}}\right|^r\mathrm{d}t\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{3}}\left|f'\left(\frac{ab}{A_t}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right.\\
&\qquad+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{t-\frac{1}{2}}{A_t^{1+p}}\right|^r\mathrm{d}t\right)^{\frac{1}{r}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|f'\left(\frac{ab}{A_t}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\\
&\qquad\left.+\left(\int_{\frac{2}{3}}^{1}\left|\frac{t-\frac{7}{8}}{A_t^{1+p}}\right|^r\mathrm{d}t\right)^{\frac{1}{r}}\left(\int_{\frac{2}{3}}^{1}\left|f'\left(\frac{ab}{A_t}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right]\\
&=\frac{b^p-a^p}{p}\left[\left(\int_0^{\frac{1}{3}}\frac{|t-\frac{1}{8}|^r}{A_t^{(1+p)r}}\mathrm{d}t\right)^{\frac{1}{r}}\left(\frac{pa^pb^p}{b^p-a^p}\int_a^{[\frac{3a^pb^p}{a^p+2b^p}]^{\frac{1}{p}}}\frac{|f'(x)|^q}{x^{1+p}}\mathrm{d}x\right)^{\frac{1}{q}}\right.\\
&\qquad+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{|t-\frac{1}{2}|^r}{A_t^{(1+p)r}}\mathrm{d}t\right)^{\frac{1}{r}}\left(\frac{pa^pb^p}{b^p-a^p}\int_{[\frac{3a^pb^p}{a^p+2b^p}]^{\frac{1}{p}}}^{[\frac{3a^pb^p}{2a^p+b^p}]^{\frac{1}{p}}}\frac{|f'(x)|^q}{x^{1+p}}\mathrm{d}x\right)^{\frac{1}{q}}\\
&\qquad\left.+\left(\int_{\frac{2}{3}}^{1}\frac{|t-\frac{7}{8}|^r}{A_t^{(1+p)r}}\mathrm{d}t\right)^{\frac{1}{r}}\left(\frac{pa^pb^p}{b^p-a^p}\int_{[\frac{3a^pb^p}{2a^p+b^p}]^{\frac{1}{p}}}^{b}\frac{|f'(x)|^q}{x^{1+p}}\mathrm{d}x\right)^{\frac{1}{q}}\right]
\end{aligned}\tag{11}$$

Using the p-harmonic convexity of $|f'|^q$, we obtain the following inequalities from inequality (2)

$$\frac{pa^pb^p}{b^p-a^p}\int_a^{[\frac{3a^pb^p}{a^p+2b^p}]^{\frac{1}{p}}}\frac{|f'(x)|^q}{x^{1+p}}\mathrm{d}x\le\frac{|f'(a)|^q+|f'([\frac{3a^pb^p}{a^p+2b^p}]^{\frac{1}{p}})|^q}{6},\tag{12}$$

$$\frac{pa^pb^p}{b^p-a^p}\int_{[\frac{3a^pb^p}{a^p+2b^p}]^{\frac{1}{p}}}^{[\frac{3a^pb^p}{2a^p+b^p}]^{\frac{1}{p}}}\frac{|f'(x)|^q}{x^{1+p}}\mathrm{d}x\le\frac{|f'([\frac{3a^pb^p}{a^p+2b^p}]^{\frac{1}{p}})|^q+|f'([\frac{3a^pb^p}{2a^p+b^p}]^{\frac{1}{p}})|^q}{6},\tag{13}$$

and

$$(14)\quad \frac{pa^pb^p}{b^p-a^p}\int_{[\frac{3a^pb^p}{2a^p+b^p}]^{\frac{1}{p}}}^{b}\frac{|f'(x)|^q}{x^{1+p}}\mathrm{d}x \le \frac{|f'\big([\frac{3a^pb^p}{2a^p+b^p}]^{\frac{1}{p}}\big)|^q+|f'(b)|^q}{6}.$$

A combination of (11)-(14) gives the required inequality (10). □

Corollary 3.5. *Under the assumptions of Theorem* 3.4 *with* $p=-1$, *we obtain the following new result.*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|$$
$$\le (b-a)\Bigg[\left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{\frac{1}{r}}\left(\frac{|f'(a)|^q+|f'\big(\frac{2a+b}{3}\big)|^q}{6}\right)^{\frac{1}{q}}$$
$$+\left(\frac{2}{6^{r+1}(r+1)}\right)^{\frac{1}{r}}\left(\frac{|f'\big(\frac{2a+b}{3}\big)|^q+|f'\big(\frac{a+2b}{3}\big)|^q}{6}\right)^{\frac{1}{q}}$$
$$+\left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{\frac{1}{r}}\left(\frac{|f'\big(\frac{a+2b}{3}\big)|^q+|f'(b)|^q}{6}\right)^{\frac{1}{q}}\Bigg].$$

Theorem 3.6. *Let* $f: I=[a,b]\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ *be a differentiable function on the interior* I^o *of* I. *If* $f'\in L[a,b]$ *and* $|f'|^q$ *is p-harmonic convex function on* I, $r,q>1$ *and* $\frac{1}{r}+\frac{1}{q}=1$, *then*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\left[\frac{3a^pb^p}{a^p+2b^p}\right]^{\frac{1}{p}}\right)+3f\left(\left[\frac{3a^pb^p}{2a^p+b^p}\right]^{\frac{1}{p}}\right)+f(b)\right]\right.$$
$$\left.-\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1+p}}\mathrm{d}x\right|$$
$$\le\frac{ab(b^p-a^p)}{p}\Bigg[\left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{\frac{1}{r}}\big(\varphi_{13}(q;a,b)|f'(a)|^q$$
$$+\varphi_{14}(q;a,b)|f'(b)|^q\big)^{\frac{1}{q}}+\left(\frac{2}{6^{r+1}(r+1)}\right)^{\frac{1}{r}}\big(\varphi_{15}(q;a,b)|f'(a)|^q$$
$$+\varphi_{16}(q;a,b)|f'(b)|^q\big)^{\frac{1}{q}}+\left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{\frac{1}{r}}\big(\varphi_{17}(q;a,b)|f'(a)|^q$$
$$+\varphi_{18}(q;a,b)|f'(b)|^q\big)^{\frac{1}{q}}\Bigg],$$

where

$$\varphi_{13}(q;a,b)=\int_0^{\frac{1}{3}}\frac{1-t}{A_t^{(1+p)q}},\quad \varphi_{14}(q;a,b)=\int_0^{\frac{1}{3}}\frac{t}{A_t^{(1+p)q}}$$
$$\varphi_{15}(q;a,b)=\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{1-t}{A_t^{(1+p)q}},\quad \varphi_{16}(q;a,b)=\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{t}{A_t^{(1+p)q}}$$
$$\varphi_{17}(q;a,b)=\int_{\frac{2}{3}}^{1}\frac{1-t}{A_t^{(1+p)q}},\quad \varphi_{18}(q;a,b)=\int_{\frac{2}{3}}^{1}\frac{t}{A_t^{(1+p)q}}.$$

Proof. Using Lemma 3.1, Holder's inequality and p-harmonic convexity of the function $|f'|^q$, we have

$$\left|\frac{1}{8}\left[f(a)+3f\left(\left[\frac{3a^pb^p}{a^p+2b^p}\right]^{\frac{1}{p}}\right)+3f\left(\left[\frac{3a^pb^p}{2a^p+b^p}\right]^{\frac{1}{p}}\right)+f(b)\right]-\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1+p}}\mathrm{d}x\right|$$
$$\le\frac{ab(b^p-a^p)}{p}\left[\int_0^{\frac{1}{3}}|t-\frac{1}{8}|\left|\frac{1}{A_t^{1+p}}f'\left(\frac{ab}{A_t}\right)\right|\mathrm{d}t+\int_{\frac{1}{3}}^{\frac{2}{3}}|t-\frac{1}{2}|\left|\frac{1}{A_t^{1+p}}f'\left(\frac{ab}{A_t}\right)\right|\mathrm{d}t\right.$$
$$\left.+\int_{\frac{2}{3}}^{1}|t-\frac{7}{8}|\left|\frac{1}{A_t^{1+p}}f'\left(\frac{ab}{A_t}\right)\right|\mathrm{d}t\right]$$
$$\le\frac{ab(b^p-a^p)}{p}\left[\left(\int_0^{\frac{1}{3}}|t-\frac{1}{8}|^r\mathrm{d}t\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{3}}\frac{1}{A_t^{(1+p)q}}\left|f'\left(\frac{ab}{A_t}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}|t-\frac{1}{2}|^r\mathrm{d}t\right)^{\frac{1}{r}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{1}{A_t^{(1+p)q}}\left|f'\left(\frac{ab}{A_t}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_{\frac{2}{3}}^{1}|t-\frac{7}{8}|^r\mathrm{d}t\right)^{\frac{1}{r}}\left(\int_{\frac{2}{3}}^{1}\frac{1}{A_t^{(1+p)q}}\left|f'\left(\frac{ab}{A_t}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right]$$
$$\le\frac{ab(b^p-a^p)}{p}\left[\left(\int_0^{\frac{1}{3}}|t-\frac{1}{8}|^r\mathrm{d}t\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{3}}\frac{1}{A_t^{(1+p)q}}\left[(1-t)|f'(a)|^q+t|f'(b)|^q\right]\mathrm{d}t\right)^{\frac{1}{q}}\right.$$
$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}|t-\frac{1}{2}|^r\mathrm{d}t\right)^{\frac{1}{r}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{1}{A_t^{(1+p)q}}\left[(1-t)|f'(a)|^q+t|f'(b)|^q\right]\mathrm{d}t\right)^{\frac{1}{q}}$$
$$\left.+\left(\int_{\frac{2}{3}}^{1}|t-\frac{7}{8}|^r\mathrm{d}t\right)^{\frac{1}{r}}\left(\int_{\frac{2}{3}}^{1}\frac{1}{A_t^{(1+p)q}}\left[(1-t)|f'(a)|^q+t|f'(b)|^q\right]\mathrm{d}t\right)^{\frac{1}{q}}\right]$$
$$=\frac{ab(b^p-a^p)}{p}\left[\left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{3}}\frac{(1-t)}{A_t^{(1+p)q}}|f'(a)|^q\mathrm{d}t+\int_0^{\frac{1}{3}}\frac{t}{A_t^{(1+p)q}}|f'(b)|^q\mathrm{d}t\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{2}{6^{r+1}(r+1)}\right)^{\frac{1}{r}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{(1-t)}{A_t^{(1+p)q}}|f'(a)|^q\mathrm{d}t+\int_{\frac{1}{3}}^{\frac{2}{3}}\frac{t}{A_t^{(1+p)q}}|f'(b)|^q\mathrm{d}t\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{\frac{1}{r}}\left(\int_{\frac{2}{3}}^{1}\frac{(1-t)}{A_t^{(1+p)q}}|f'(a)|^q\mathrm{d}t+\int_{\frac{2}{3}}^{1}\frac{t}{A_t^{(1+p)q}}|f'(b)|^q\mathrm{d}t\right)^{\frac{1}{q}}\right]$$

$$
\begin{aligned}
&= \frac{ab(b^p - a^p)}{p}\Bigg[\left(\frac{3^{r+1}+5^{r+1}}{24^r(r+1)}\right)^{\frac{1}{r}}\left(\varphi_{13}(q;a,b)|f'(a)|^q + \varphi_{14}(q;a,b)|f'(b)|^q\right)^{\frac{1}{q}}\\
&+ \left(\frac{1}{6^{r+1}(r+1)}\right)^{\frac{1}{r}}\left(\varphi_{15}(q;a,b)|f'(a)|^q + \varphi_{16}(q;a,b)|f'(b)|^q\right)^{\frac{1}{q}}\\
&+ \left(\frac{3^{r+1}+5^{r+1}}{24^r(r+1)}\right)^{\frac{1}{r}}\left(\varphi_{17}(q;a,b)|f'(a)|^q + \varphi_{18}(q;a,b)|f'(b)|^q\right)^{\frac{1}{q}}\Bigg].
\end{aligned}
$$

□

Corollary 3.7. *Under the assumptions of Theorem 3.6 with $p = -1$, we have the following new result.*

$$
\begin{aligned}
&\left|\frac{1}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\\
&\le (b-a)\Bigg[\left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{\frac{1}{r}}\left(\frac{5|f'(a)|^q + |f'(b)|^q}{18}\right)^{\frac{1}{q}}\\
&+ \left(\frac{1}{6^{r+1}(r+1)}\right)^{\frac{1}{r}}\left(\frac{|f'(a)|^q + |f'(b)|^q}{6}\right)^{\frac{1}{q}}\\
&+ \left(\frac{3^{r+1}+5^{r+1}}{24^{r+1}(r+1)}\right)^{\frac{1}{r}}\left(\frac{|f'(a)|^q + 5|f'(b)|^q}{18}\right)^{\frac{1}{q}}\Bigg].
\end{aligned}
$$

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