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ON PRIME SUBMODULES OF AN INJECTIVE MODULE OVER A NOETHERIAN RING

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Abstract. In this paper, we characterize the prime submodules of an injective module over a Noetherian ring.

1. Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary. A proper submodule P of an R-module M is called *prime*, if $rm \in P$ for some $r \in R$ and $m \in M$ implies $m \in P$ or $r \in (P : M)$, where $(P : M) = \{r \in R \mid rM \subseteq P\}$. If P is a prime submodule of an R-module M then $(P : M)$ is a prime ideal of R. The set of all prime submodules of an R-module M is denoted by $Spec(M)$. Prime submodules of a module over a commutative ring have been studied by many authors, see [5, 6, 9]. Also prime submodules of a finitely generated free module over a PID were studied in [2, 3]. The authors in [2], described prime submodules of a finitely generated free module over a UFD and characterized the prime submodules of a free module of finite rank over a PID. The authors in [7, 8], extended some results obtained in [2] to a Dedekind and a valuation domain. In this paper, we characterize the prime submodules of an injective module over a Noetherian ring.

2. Main Results

An R-module M is injective if for every R-module monomorphism $f : N \longrightarrow N'$ and for every R-module homomorphism $g : N \longrightarrow M$, there exists an R-module homomorphism $h : N' \longrightarrow M$ such that

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$hf = g$. We know that every injective submodule N of an R-module M is a direct summand of M. Let $N \subseteq M$ be R-modules. We say that M is an essential extension of N, if for any nonzero R-submodue U of M one has $U \cap N \neq 0$. An essential extension M of N is called *proper* if $N \neq M$. By [1, Proposition 3.2.2], an R-module M is injective if and only if it has no proper essential extension. Let M be an R-module. An injective module E is called an *injective envelope* of M, if E is an essential extension of M and denoted by $E(M)$. We know that any module M can be embedded into an injective module and injective envelope of M is the minimal embedding. In this case, the corresponding injective module is unique up to isomorphism.

The following lemma is an special case of Lemma 9.8 in [4].

Lemma 2.1. *Let R be an integral domain with quotient field K. Then $E(R) \simeq K$.*

Proof. At first we show that K is an essential extension of R. If $0 \neq x = \frac{a}{b} \in K$, then $a = bx \in Rx \cap R$ and so K is an essential extension of R. It is sufficient to show that K is an injective R-module. Let I be a nonzero ideal of R and $f : I \longrightarrow K$ be an R-module homomorphism. Let a be a nonzero element of I and $f(a) = \frac{c}{d}$. We define $\Phi : R \longrightarrow K$, by $\Phi(x) = \frac{xc}{ad}$, where $x \in R$. Clearly, Φ is an R-module homomorphism. Suppose that $a' \in I$ and $f(a') = \frac{c'}{d'}$. We have $f(aa') = af(a') = a'f(a)$ and hence $\frac{ac'}{d'} = \frac{a'c}{d}$. Therefore, $\Phi(a') = \frac{a'c}{ad} = \frac{ac'}{ad'} = \frac{c'}{d'} = f(a')$. Now by Baer's Criterion, K is an injective R-module and so $E(R) \simeq K$. □

An element x of an R-module M is called *torsion*, if it has a nonzero annihilator in R. Let $t(M)$ be the set of all torsion elements of M. It is clear that if R is an integral domain, then $t(M)$ is a submodule of M. We say that $t(M)$ is the torsion submodule of M. If $t(M) = M$, then M is called a *torsion module* and if $t(M) = 0$, then M is called a *torsion-free module*. An R-module M is *divisible* if for every $0 \neq r \in R$, $rM = M$. It is easy to see that every injective module over an integral domain R is divisible. If M is a divisible R-module, then for every proper submodule N of M, $(N : M) = 0$.

Lemma 2.2. *Let R be a ring and M be a divisible R-module with $P \in Spec(M)$. We have*

i) *If $t(M) \neq M$, then $t(M) \subseteq P$.*
ii) *P is a divisible R-module.*
iii) *For every $x \in M - P$, $ann(x) = 0$.*

Proof. i) Let $x \in t(M)$. Then there exists $0 \neq r \in R$ such that $rx = 0 \in P$. Since $(P : M) = 0$, $x \in P$. So $t(M) \subseteq P$.

ii) Let $0 \neq r \in R$ and $x \in P$. There exists $y \in M$ such that $ry = x$. Since $(P : M) = 0$, $y \in P$ and we have $x \in rP$. So $rP = P$ and hence P is a divisible R-module.

iii) Let $r \in R$, $x \in M - P$ and $rx = 0$. Since $rx \in P$ and $x \notin P$, $r \in (P : M) = 0$ and hence $ann(x) = 0$. □

Corollary 2.3. *Let R be an integral domain and M be a divisible R-module. Then M is torsion if and only if $Spec(M) = \emptyset$.*

Proof. If $t(M) \neq M$ then by [5, Result 3], $t(M)$ is a prime submodule of M. Now the proof follows by Lemma 2.2(i). □

Lemma 2.4. *Let R be an integral domain with quotient field K. If M is a torsion-free divisible R-module, then M is isomorphic to a direct sum of copies of K.*

Proof. We prove that M is a K-module. Let r be a nonzero element of R and $x \in M$. Since M is torsion-free and $rM = M$, there exists a unique $y \in M$ such that $ry = x$. Then the R-module homomorphism $f_r : M \longrightarrow M$ defined by $f_r(x) = y$ is well-defined. Now we define $f : K \times M \longrightarrow M$, by $f\left(\frac{r}{s}, x\right) = f_s(rx)$. So M is a free K-module and hence M is isomorphic to a direct sum of copies of K. □

Recall that a prime ideal p of a ring R is an *associated prime* of an R-module M, if $p = ann(x)$ for some nonzero element $x \in M$. The set of all associated primes of an R-module M is denoted by $Ass(M)$.

Lemma 2.5. *Let R be a Noetherian domain and p be a nonzero prime ideal of R. Then $Spec\left(E\left(\frac{R}{p}\right)\right) = \emptyset$.*

Proof. Let $E = E\left(\frac{R}{p}\right)$ and $P \in Spec(E)$. Suppose that $x \in E - P$. By Lemma 2.2(iii), $ann(x) = 0$ and hence $\{0\} \in Ass(E)$. But by [1, Lemma 3.2.7], $Ass(E) = \{p\}$, which is a contradiction. So $Spec(E) = \emptyset$. □

For a Noetherian ring R, by [1, Theorem 3.2.8], an injective module can be uniquely written as a direct sum of indecomposable injective modules such that the indecomposable injective R-modules are just the injective envelopes of the cyclic R-modules $\frac{R}{p}$, where $p \in Spec(R)$. The next Theorem characterizes the prime submodules of an injective module over a Noetherian domain.

Theorem 2.6. *Let R be a Noetherian domain with quotient field K. Suppose that M is an injective R-module. Then*

i) $M = t(M) \oplus N$, *where* $N \simeq \oplus_{i\in I} K$, *for some index set* I.
ii) $Spec(M) = \emptyset$ *or* $Spec(M) = \{t(M) \oplus D \mid D \lneqq N,\ D \simeq \oplus_{j\in J} K,$ *for some index set* $J\}$.

Proof. i) For every $p \in Spec(R)$, let $M(p) = \oplus E\left(\frac{R}{p}\right)$ such that the number of indecomposable summands in the decomposition of $M(p)$ equals $dim_{k(p)}\ Hom_{R_p}(k(p), M_p)$, where $k(p) = \frac{R_p}{pR_p}$. By Lemma 2.1, we have $M((0)) \simeq \oplus_{x\in X} K$, for some index set X. Let $A = \bigoplus_{0\neq p\in Spec(R)} M(p)$, $B = \oplus_{x\in X} K$ and $U = A \oplus B$. By [1, Theorem 3.2.8], we have $M \simeq U$. Now we prove that $t(U) = A \oplus \{0\}$. Let $0 \neq p \in Spec(R)$. By Lemma 2.5, $Spec\left(E\left(\frac{R}{p}\right)\right) = \emptyset$ and by Corollary 2.3, $E\left(\frac{R}{p}\right)$ is a torsion R-module. So $M(p)$ is a torsion R-module. Therefore $A \oplus \{0\} \subseteq t(U)$. On the other hand, let $y = (x_1, x_2) \in t(U)$, where $x_1 \in A$ and $x_2 \in B$. Then there exists $0 \neq r \in R$ such that $ry = 0$ and hence $rx_2 = 0$. Since B is a torsion-free R-module, we have $x_2 = 0$ and hence $y \in A \oplus \{0\}$. Therefore $t(U) = A \oplus \{0\}$. Thus $U \simeq t(U) \oplus B$. Since M is an injective R-module, U is an injective R-module and hence $t(U)$ is an injective R-module. On the other hand, $t(M) \simeq t(U)$ and hence $t(M)$ is an injective submodule of M. Then there exists a submodule N of M such that $M = t(M) \oplus N$. Therefore N is a torsion-free divisible R-module and by Lemma 2.4, $N \simeq \oplus_{i\in I} K$, for some index set I.

ii) Suppose that $M = t(M)$. Since M is an injective module over an integral domain, then M is divisible and hence by Corollary 2.3, $Spec(M) = \emptyset$. Now suppose that, $M \neq t(M)$. Let $\Omega = \{t(M) \oplus D \mid D \lneqq N,\ D \simeq \oplus_{j\in J} K$, for some index set $J\}$. We show that $Spec(M) = \Omega$. Assume that $P \in \Omega$. Then there exist an index set J and proper submodule D of N, such that $P = t(M) \oplus D$ and $D \simeq \oplus_{j\in J} K$. Now we show that $P = t(M) \oplus D \in Spec(M)$. Suppose that $0 \neq r \in R$ and $x = x_1 + x_2 \in t(M) \oplus N$ such that $rx \in t(M) \oplus D$. Thus $rx_2 \in D$. Since D is a prime K-submodule of N, we have $x_2 \in D$ and hence $x = x_1 + x_2 \in P$. We conclude that $P \in Spec(M)$ and hence $\Omega \subseteq Spec(M)$. Conversely, suppose that $P \in Spec(M)$. By Lemma 2.2(i), $t(M) \subseteq P$. Let $D = \{x_2 \in N \mid \exists x_1 \in t(M)$ such that $x_1 + x_2 \in P\}$. Clearly D is a proper submodule of N and $t(M) \cap D = \{0\}$. We show that $P = t(M) \oplus D$. By definition of D, we have $P \subseteq t(M) \oplus D$. Conversely, let $x = x_1 + x_2 \in t(M) \oplus D$. Then there exists $y \in t(M)$ such that

$y + x_2 \in P$. We have $(x_1 + x_2) - (y + x_2) = x_1 - y \in t(M) \subseteq P$ and hence $x = x_1 + x_2 \in P$. So $P = t(M) \oplus D$. Now we prove that $D \in Spec(N)$. Let $0 \neq r \in R$, $x \in N$ and $rx \in D$. Then there exists $y \in t(M)$ such that $y + rx \in P$. By Corollary 2.3, $t(M) \in Spec(M)$ and by Lemma 2.2(ii), $t(M)$ is divisible. So $rt(M) = t(M)$ and hence there exists $z \in t(M)$ such that $rz = y$. Thus $r(z + x) \in P$ and so $z + x \in P$. Therefore $x \in D$ and $D \in Spec(N)$. Now by Lemma 2.2(ii), D is a proper torsion-free divisible submodule of N and by Lemma 2.4, $D \simeq \oplus_{j\in J} K$, for some index set J. □

Example 2.7. *Let M be a divisible abelian group. We know that M is a direct sum of copies of $\mathbb{Z}(p^\infty)$, for various primes p and copies of the rational numbers $\mathbb{Q}$. So we have three cases:*

Case (i) $M = \oplus_{p\in\Gamma}\mathbb{Z}(p^\infty)$, where Γ is a subset of prime numbers. Then $Spec(M) = \emptyset$.

Case (ii) $M = \oplus_{i\in I}\mathbb{Q}$, for some index set I. Then $Spec(M) = \{D \mid D = 0$ or $D \simeq \oplus_{j\in J}\mathbb{Q}$, for some index set $J\}$.

Case (iii) $M = (\oplus_{p\in\Gamma}\mathbb{Z}(p^\infty)) \oplus (\oplus_{i\in I}\mathbb{Q})$, where Γ is a subset of prime numbers and I is an index set. Then $Spec(M) = \{(\oplus_{p\in\Gamma}\mathbb{Z}(p^\infty)) \oplus D \mid D \simeq \oplus_{j\in J}\mathbb{Q}$, for some index set $J\}$.

In particular, let $M = \mathbb{Z}(2^\infty) \oplus \mathbb{Z}(3^\infty) \oplus \mathbb{Q} \oplus \mathbb{Q}$ and Ω be the set of all $N(a, b)$ such that $N(a, b) = \{(x, y) \mid x, y \in \mathbb{Q}$ and $ax + by = 0\}$, where $a, b \in \mathbb{Q}$ and $(a, b) \neq (0, 0)$. We have Ω is the set of all non-trivial $\mathbb{Q}$-subspaces of $\mathbb{Q}$-vector space $\mathbb{Q} \oplus \mathbb{Q}$. On the other hand, $t(M) = \mathbb{Z}(2^\infty) \oplus \mathbb{Z}(3^\infty)$and hence by Theorem 2.6,

$$Spec(M) = \{\mathbb{Z}(2^\infty) \oplus \mathbb{Z}(3^\infty) \oplus D \mid D \in \Omega\} \cup \{\mathbb{Z}(2^\infty) \oplus \mathbb{Z}(3^\infty)\}.$$

Theorem 2.8. *Let R be a ring but not a domain.*

i) *If M is a divisible R-module, then $Spec(M) = \emptyset$.*
ii) *Let R be a Noetherian ring and M be an injective R-module. Then for every $p \in Spec(R)$, $Spec\left(E\left(\frac{R}{p}\right)\right) = \emptyset$ if and only if $Spec(M) = \emptyset$.*

Proof. i) Let $P \in Spec(M)$. Then $(P : M) = 0 \in Spec(R)$ and hence R is an integral domain, which is a contradiction. Hence $Spec(M) = \emptyset$.

ii) For every $p \in Spec(R)$, let $M(p) = \bigoplus E\left(\frac{R}{p}\right)$ where the number of indecomposable summands in the decomposition of $M(p)$ equals $dim_{k(p)}Hom_{R_p}(k(p), M_p)$, where $k(p) = \frac{R_p}{pR_p}$. Let $U = \bigoplus_{p\in Spec(R)} M(p)$. By [1, Theorem 3.2.8], we have $M \simeq U$. So $Spec(M) = \emptyset$ if and only if $Spec(U) = \emptyset$. Now we show that for every $p \in Spec(R)$, $Spec\left(E\left(\frac{R}{p}\right)\right) = \emptyset$ if and only if $Spec(U) = \emptyset$. Assume that for every $p \in Spec(R)$, $Spec\left(E\left(\frac{R}{p}\right)\right) = \emptyset$. We prove that $Spec(U) = \emptyset$. Let $P \in Spec(U)$. Put $Spec(R) = \{p_s | s \in S\}$, for some index set S. Assume that for every $s' \in S$, $A_{s'} = \oplus_{s \in S} B_s$ such that $B_{s'} = M(p_{s'})$ and for every $s \neq s'$, $B_s = 0$. Clearly $U = \oplus_{s \in S} A_s$. If for every $s \in S$, $A_s \subseteq P$ then $P = U$, which is a contradiction. Hence there exists $k \in S$ such that $A_k \nsubseteq P$. Now we show that $A_k \cap P \in Spec(A_k)$. Clearly $A_k \cap P \neq A_k$. Let $x \in A_k$, $r \in R$ and $rx \in A_k \cap P$ but $x \notin A_k \cap P$. Then $x \in U$, $rx \in P$ but $x \notin P$. So $r \in (P : U)$ and hence $rU \subseteq P$. Therefore $rA_k = r(U \cap A_k) \subseteq P \cap A_k$. Thus $P \cap A_k \in Spec(A_k)$. We have $A_k \simeq M(p_k)$. Then there exists $Q \in Spec(M(p_k))$ such that $Q \simeq P \cap A_k$. Now $M(p_k) = \oplus_{i \in I} E\left(\dfrac{R}{p_k}\right)$, for some index set I. For every $j \in I$, let $C_j = \oplus_{i \in I} D_i$ such that $D_j = E\left(\frac{R}{p_k}\right)$ and for every $i \neq j$, $D_i = 0$. Clearly $M(p_k) = \oplus_{i \in I} C_i$. If for every $i \in I$, $C_i \subseteq Q$, then $M(p_k) = Q$, which is a contradiction. Thus there exist $l \in I$ such that $C_l \nsubseteq Q$. Hence $C_l \cap Q \in Spec(C_l)$. Since $C_l \simeq E\left(\frac{R}{p_k}\right)$, then $Spec\left(E\left(\frac{R}{p_k}\right)\right) \neq \emptyset$, which is a contradiction. Therefore $Spec(U) = \emptyset$. Conversely, suppose that $Spec(U) = \emptyset$. We show that for every $p \in Spec(R)$, $Spec\left(E\left(\frac{R}{p}\right)\right) = \emptyset$. Let $q \in Spec(R)$ and $P \in Spec\left(E\left(\frac{R}{q}\right)\right)$. By definition, $M(q) = \oplus_{i \in I} E\left(\dfrac{R}{q}\right)$, for some index set I. Let $Q_j = \oplus_{i \in I} N_i$ such that $N_j = P$ and for every $i \in I$, $i \neq j$, $N_i = E\left(\frac{R}{q}\right)$. It is easy to see that $Q_j \in Spec(M(q))$. By the above notation, there exists $s' \in S$ such that $A_{s'} \simeq M(q)$. So there exists $W \in Spec(A_{s'})$ such that $W \simeq Q_j$. Let $V = \oplus_{s \in S} E_s$ such that $E_{s'} = W$ and for every $s \in S$, $s \neq s'$, $E_s = A_s$. It is easy to see that $V \in Spec(U)$, which is a contradiction. The proof is complete. □

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