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CENTROAFFINE GEOMETRY OF RULED SURFACES AND CENTERED CYCLIC SURFACES IN $\mathbb{R}^4$

Yun Yang and Yanhua Yu

Abstract. In this paper, we get several centroaffine invariant properties for a ruled surface in $\mathbb{R}^4$ with centroaffine theories of codimension two. Then by solving certain partial differential equations and studying a centroaffine surface with some centroaffine invariant properties in $\mathbb{R}^4$, we obtain such a surface is centroaffinely equivalent to a ruled surface or one of the flat centered cyclic surfaces. Furthermore, some centroaffine invariant properties for centered cyclic surfaces are considered.

1. Introduction

The main purpose of affine differential geometry is to study the properties of submanifolds M^n of $\mathbb{R}^m$ that are invariant under the group of all affine transformations [3, 9]. The classical theory for affine hypersurfaces was developed by Blaschke and his school [1]. Similarly, the centroaffine differential geometry is the study of the properties of submanifolds that are invariant under the centroaffine transformation group, which is the subgroup of the affine transformation group that keeps the origin invariant.

In centroaffine differential geometry, the theory of hypersurfaces has a long history. The notion of centroaffine minimal hypersurfaces was introduced by Wang [11] as extremals for the area integral of the centroaffine metric. Liu [4] got the classification of surfaces in $\mathbb{R}^3$ which are both centroaffine-minimal and equiaffine-minimal, for hypersurfaces, see [7]. See also [13, 14, 17] for the classification results about centroaffine translation surfaces and centroaffine ruled surfaces in $\mathbb{R}^3$. On the other hand, there is relatively little literature available in the field of centroaffine immersions with higher codimensions. For a centroaffine immersion into an affine space, the position vector yields its first canonical transversal vector field. A standard method of choosing second one

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was proposed in 1950 by Lopšic (see Walter [10]). Reorganizing the geometry of equi-centroaffine immersions of codimension two, Nomizu and Sasaki took the prenormalized Blaschke normal field as the second canonical transversal vector field [8], and by this structure Furuhata proved that an equi-centroaffine immersion is minimal if and only if the trace of the affine shape operator with respect to the prenormalized Blaschke normal field vanishes identically [2]. On the other hand, Liu gave another structure to study centroaffine immersions of codimension two [5,6]. He developed the centroaffine metric g of a centroaffine immersion with codimension two in a new way and then chose $\Delta_g x$ as the second transversal vector field, where Δ_g denotes the Laplacian of g. According to this structure and using the efficacious moving frame method, the authors of this paper and Liu compared these different normalizations and defined the minimal centroaffine immersions of codimension two [12,15,16].

A ruled surface is a surface generated by a family of straight lines, which leads to a wide range of practical applications. For example, the ruled surface is a typical modeling surface in computer aided geometric design. Ruled surfaces are also widely found in architecture. A cyclic surface is a surface S foliated by pieces of circles and at the same time there is a one-parameter family of planes which meet S in these pieces of circles. In this paper, we consider the ruled surfaces and the centered cyclic surfaces with codimension two in $\mathbb{R}^4$. For a ruled surface, we get that (1) its centroaffine metric is indefinite, (2) its second fundamental form is degenerate, (3) its Pick invariant is vanishing, (4) the eigenvalues of the shape operator are equal, (5) $\rho = g^{ij}\rho_i\rho_j = 0$. By studying a centroaffine surface with these invariant properties in $\mathbb{R}^4$, we obtain the surface is centroaffinely equivalent to a ruled surface or a flat centered cyclic surface. Furthermore, we study centered cyclic surfaces in $\mathbb{R}^4$.

The rest of this paper is organized as follows. In Section 2 we recall the basic machinery for centroaffine immersions of codimension 2 with normalization $\{x, \Delta_g x\}$. In Section 3 we get some centroaffine invariants for the cenroaffine ruled surface in $\mathbb{R}^4$. In Section 4 we study the centroaffine surface with the above five properties, and obtain that the surface centroaffinely equivalent to a ruled surface or a flat centered cyclic surface. In Section 5 we further consider the centroaffine geometry of centered cyclic surfaces in $\mathbb{R}^4$.

2. Centroaffine immersions in $\mathbb{R}^{n+2}$

Let $x : M = M^n \to \mathbb{R}^{n+2}$ $(n \geq 2)$ be an oriented immersed submanifold such that $x(p) \notin \mathrm{d}x(T_pM)$ for all $p \in M$, and let $x(M)$ be not contained in a hyperplane containing the origin of $\mathbb{R}^{n+2}$. Our convention for the range of indices is the following

$$1 \leq i, j, k, \ldots \leq n,$$
$$n + 1 \leq \alpha, \beta, \gamma, \ldots \leq n + 2,$$
$$1 \leq A, B, C, \ldots \leq n + 2,$$

and we shall follow the usual Einstein summation convention.

For any local oriented basis $\sigma = \{E_1, E_2, \ldots, E_n\}$ of TM with dual basis $\{\theta^1, \theta^2, \ldots, \theta^n\}$ we define

$$G := [E_1(x), \ldots, E_n(x), x, \mathrm{d}^2 x] = G_{ij}\theta^i \otimes \theta^j, \tag{2.1}$$

where $[\cdot]$ is the standard determinant in $\mathbb{R}^{n+2}$, and $G_{ij} := [E_1(x), \ldots, E_n(x), x, E_iE_j(x)]$. G is a symmetric 2-form and we assume that G is nondegenerate.

Then we define

$$g := g_{ij}\theta^i \otimes \theta^j, \quad g_{ij} = |\det(G_{pq})|^{-\frac{1}{n+2}} G_{ij}. \tag{2.2}$$

It is easy to verify g is independent of the choice of the basis σ and thus a globally defined symmetric 2-form. From Eq. (2.1) we know that the conformal class of g is a centroaffine invariant [5, 6].

Definition 2.1. g is called a centroaffine metric of a centroaffine immersion $x : M \to \mathbb{R}^{n+2}$.

Remark. x is called a nondegenerate centroaffine submanifold if g is nondegenerate, and x is definite or indefinite if g is definite or indefinite, respectively.

Definition 2.2. Let Δ_g denote the Laplacian of g. $\{x, \frac{\Delta_g x}{n}\}$ is called the centroaffine normalization of a centroaffine immersion $x : M \to \mathbb{R}^{n+2}$.

The structure equations can be given by [12]

$$\frac{\partial^2 x}{\partial u^i \partial u^j} = \Gamma^k_{ij} x_k + h_{ij} x + g_{ij} \frac{\Delta_g x}{n}, \tag{2.3}$$

$$\left(\frac{\Delta_g x}{n}\right)_i = -S^k_i x_k + \rho_i x, \tag{2.4}$$

where $x_k = \frac{\partial x}{\partial u^k}$ and $(\frac{\Delta_g x}{n})_i = \frac{\partial(\frac{\Delta_g x}{n})}{\partial u^i}$. Let $\nabla = \{\Gamma^k_{ij}\}, \tilde{\nabla} = \{\tilde{\Gamma}^k_{ij}\}$ be the induced connection and the Levi-Civita connection of centroaffine metric g, respectively, and then we define the Fubini-Pick form by

$$C^k_{ij} := \Gamma^k_{ij} - \tilde{\Gamma}^k_{ij}, \tag{2.5}$$

$$C_{ijk} := g_{il} C^l_{jk}. \tag{2.6}$$

The Pick invariant is defined by

$$J = \frac{1}{n(n-1)} g^{ij} C^l_{ik} C^k_{lj}. \tag{2.7}$$

In [12] we obtained that C^k_{ij} is symmetric for i and j and C_{ijk} is symmetric for i, j and k. Furthermore

$$g^{ij} C^k_{ij} = 0, \quad g^{ij} h_{ij} = 0, \quad C^i_{ij} = 0. \tag{2.8}$$

Introducing now $\tilde{h}_{ij} = -g_{il} S^l_j$ and $\tilde{H} = (\tilde{h}_{ij})$,

$$H = \frac{\mathrm{tr}_g \tilde{H}}{n} = -\frac{\mathrm{tr} S}{n} \tag{2.9}$$

is called the centroaffine mean curvature of a centroaffine immersion $x : M^n \to \mathbb{R}^{n+2}$, and the centroaffine immersion x is centroaffine minimal if and only if $H = 0$.

Following the standard routine for geometry of submanifolds, we may derive the equation of Gauss, the equations of Codazzi and the equations of Ricci:

$$R^l_{kij} = \frac{\partial \Gamma^l_{kj}}{\partial u^i} - \frac{\partial \Gamma^l_{ki}}{\partial u^j} + \Gamma^h_{kj}\Gamma^l_{hi} - \Gamma^h_{ki}\Gamma^l_{hj} = h_{ki}\delta^l_j - h_{kj}\delta^l_i + g_{kj}S^l_i - g_{ki}S^l_j, \tag{2.10}$$

$$\Gamma^l_{ki}h_{lj} + \frac{\partial h_{ki}}{\partial u^j} + g_{ki}\rho_j = \Gamma^l_{kj}h_{li} + \frac{\partial h_{kj}}{\partial u^i} + g_{kj}\rho_i, \tag{2.11}$$

$$\Gamma^l_{ki}g_{lj} + \frac{\partial g_{ki}}{\partial u^j} = \Gamma^l_{kj}g_{li} + \frac{\partial g_{kj}}{\partial u^i}, \tag{2.12}$$

$$\frac{\partial S^i_k}{\partial u^j} + S^i_l\Gamma^k_{jl} + \rho_j\delta^i_k = \frac{\partial S^j_k}{\partial u^i} + S^j_l\Gamma^k_{il} + \rho_i\delta^j_k, \tag{2.13}$$

$$S^k_i h_{jk} + \frac{\partial \rho_j}{\partial u^i} = S^k_j h_{ik} + \frac{\partial \rho_i}{\partial u^j}, \tag{2.14}$$

$$\tilde{h}_{ij} = \tilde{h}_{ji}. \tag{2.15}$$

By studying the basic centroaffine theories we have obtained the following result [15].

Theorem 2.3. *Let $x : M \to \mathbb{R}^{n+2}$ be a centroaffine immersion of codimension two, then normalized scalar curvature χ of the metric g, the Pick invariant J and the centroaffine mean curvature H satisfy the following relation*

$$\chi = J - H. \tag{2.16}$$

By a direct computation, it is also easy to verify that:

Lemma 2.4. *Let $x : M \to \mathbb{R}^{n+2}$ be a centroaffine immersion of codimension two. Then under the parameter transformation $\bar{u}^i = \bar{u}(u^1, u^2, \dots, u^n), (i = 1, 2, \dots, n)$, $\Delta_{\bar{g}}x = \Delta_g x$, and under the centroaffine transformation $\bar{x} = Ax$, $\Delta_{\bar{g}}\bar{x} = |A|^{-\frac{2}{n+2}} A\Delta_g x$.*

Following the structure equations (2.3) and (2.4) we have

$$h_{ij} = \frac{[x_1, x_2, \dots, x_n, \frac{\partial^2 x}{\partial u^i \partial u^j}, \Delta_g x]}{[x_1, x_2, \dots, x_n, x, \Delta_g x]}, \tag{2.17}$$

$$C^k_{ij} = \frac{[x_1, x_2, \dots, x_{k-1}, \frac{\partial^2 x}{\partial u^i \partial u^j}, x_{k+1}, \dots, x_n, x, \Delta_g x]}{[x_1, x_2, \dots, x_n, x, \Delta_g x]} - \tilde{\Gamma}^k_{ij}, \tag{2.18}$$

$$\tilde{h}_{ij} = -\frac{[x_1, x_2, \dots, x_{k-1}, (\frac{\Delta_g x}{n})_i, x_{k+1}, \dots, x_n, x, \Delta_g x]}{[x_1, x_2, \dots, x_n, x, \Delta_g x]} g_{kj}, \tag{2.19}$$

$$\rho_i g_{jk} = \frac{[x_1, x_2, \dots, x_n, (\frac{\Delta_g x}{n})_i, \Delta_g x]}{[x_1, x_2, \dots, x_n, x, \Delta_g x]} g_{jk}. \tag{2.20}$$

Therefore, it is straightforward to check that:

Proposition 2.5. *If two centroaffine immersion $x, \bar{x} : M \to \mathbb{R}^{n+2}$ are centroaffine equivalent, then*

$$h_{ij} = \bar{h}_{ij}, \quad C^k_{ij} = \bar{C}^k_{ij}, \quad \tilde{h}_{ij} = \bar{\tilde{h}}_{ij}, \quad \rho_i g_{jk} = \bar{\rho}_i \bar{g}_{jk}. \tag{2.21}$$

The following definition has been given in [15, 16].

Definition 2.6.

$$L = h_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j$$

is call the second fundamental form of a centroaffine immersion $x : M \to \mathbb{R}^{n+2}$.

3. Centroaffine ruled surfaces in $\mathbb{R}^4$

Let $x : M^2 \to \mathbb{R}^4$ be a non-degenerate centroaffine ruled surface given by $x(u, v) = a(u) + vb(u)$, where $a(u)$ and $b(u)$ are 1-variable vector fields, and in the following, we assume that the vector fields $a(u)$ and $b(u)$ are linearly independent. For the centroaffine ruled surface x, the position vector, denoted also by x, is always transversal to the tangent space $x_*(TM)$ at each point of M, that is, the vector fields $a' + vb', b, a + vb$ are linearly independent.

For a ruled surface $x(u, v) = a(u) + vb(u)$, by Eq. (2.1), we can directly obtain the symmetric 2-form $G_{ij} (i, j = 1, 2)$, that is,

$$G_{11} = [b', b, a, b'']v^2 + \{[b', b, a, a''] + [a', b, a, b'']\}v + [a', b, a, a''], \tag{3.1}$$

$$G_{12} = [a', b, a, b'], \tag{3.2}$$

$$G_{22} = 0. \tag{3.3}$$

Certainly, through a parameter transformation $\bar{u} = \bar{u}(u)$, we can assume that $[a', b, a, b'] = 1$, that is, $G_{12} = 1$. Now taking the notations $f_1(u) = [b', b, a, b''], f_2(u) = 2[a', b, a, b''] = -2[a'', b, a, b']$ and $f_3(u) = [a', b, a, a'']$, the symmetric 2-form G_{ij} can be written as

$$G_{11} = f_1(u)v^2 + f_2(u)v + f_3(u), \tag{3.4}$$

$$G_{12} = 1, \tag{3.5}$$

$$G_{22} = 0. \tag{3.6}$$

According to the definition of the centroaffine metric, namely, Eq. (2.2), we have

$$g_{11} = f_1(u)v^2 + f_2(u)v + f_3(u), \tag{3.7}$$

$$g_{12} = 1, \tag{3.8}$$

$$g_{22} = 0. \tag{3.9}$$

Using centroaffine metric g, we can get its Levi-Civita connection

$$\tilde{\Gamma}^1_{11} = -\frac{1}{2}(2f_1 v + f_2), \tag{3.10}$$

$$\tilde{\Gamma}^2_{11} = \frac{1}{2}[2f_1 v^3 + (f_1' + 3f_1 f_2)v^2 + (f_2' + 2f_1 f_3 + f_2^2)v + f_3' + f_2 f_3], \tag{3.11}$$

$$\tilde{\Gamma}_{12}^1 = 0, \quad \tilde{\Gamma}_{12}^2 = \frac{1}{2}(2f_1 v + f_2), \tag{3.12}$$

$$\tilde{\Gamma}_{22}^1 = \tilde{\Gamma}_{22}^2 = 0. \tag{3.13}$$

Thus, it is easy to show that

$$\Delta_g x = 2b' - (2f_1 v + f_2)b, \tag{3.14}$$

which is the second transversal vector field for centroaffine immersion with codimention two. Correspondingly, the structure equation can be expressed by

$$\begin{aligned} x_{uu} &= a'' + vb'' \\ &= \Gamma_{11}^1(a' + vb') + \Gamma_{11}^2 b + h_{11}(a + vb) \\ &\quad + (f_1 v^2 + f_2 v + f_3)\frac{\Delta_g x}{2}, \end{aligned} \tag{3.15}$$

$$x_{uv} = b' = \Gamma_{12}^1(a' + vb') + \Gamma_{12}^2 b + h_{12}(a + vb) + \frac{\Delta_g x}{2}, \tag{3.16}$$

$$x_{vv} = 0, \tag{3.17}$$

$$(\frac{\Delta_g x}{2})_u = -f_1(a' + vb') + \alpha(u,v)b + \frac{\partial h_{11}}{\partial v}(a + vb), \tag{3.18}$$

$$(\frac{\Delta_g x}{2})_v = -f_1 b, \tag{3.19}$$

where $\alpha(u,v) = \frac{\partial \Gamma_{11}^2}{\partial v} + h_{11} - \frac{1}{2}(2f_1 v + f_2)^2 - f_1(f_1 v^2 + f_2 v + f_3) - \frac{1}{2}(2f_1' v + f_2')$ and $\Gamma_{11}^1 = -\frac{1}{2}(2f_1 v + f_2)$.

By the integrability conditions (2.10)-(2.15), we conclude that

$$h_{12} = 0, \quad h_{22} = 0, \tag{3.20}$$

$$\Gamma_{12}^1 = 0, \quad \Gamma_{12}^2 = \frac{1}{2}(2f_1 v + f_2), \quad \Gamma_{22}^1 = \Gamma_{22}^2 = 0, \tag{3.21}$$

$$\frac{\partial^2 h_{11}}{\partial v^2} = 0, \quad \frac{\partial^2 \Gamma_{11}^2}{\partial v^2} + 2\frac{\partial h_{11}}{\partial v} - 3f_1(2f_1 v + f_2) = 0. \tag{3.22}$$

Using Eqs. (2.5), (3.10)-(3.13) and (3.21), we see that

$$C_{11}^1 = C_{12}^1 = C_{12}^2 = C_{22}^1 = C_{22}^2 = 0. \tag{3.23}$$

Applying the above results to Eq. (2.7), we have

$$J = 0. \tag{3.24}$$

Hence, the combining of Eqs. (2.9), (2.16), (3.18) and (3.19) shows that

$$K = -H = f_1, \tag{3.25}$$

where K is centroaffine Gauss curvature, and H is centroaffine mean curvature. Now, by Eqs. (3.18) and (3.19), the shape operator matrix can be written in the form

$$(S_j^i) = \begin{pmatrix} f_1 & -\alpha(u,v) \\ 0 & f_1 \end{pmatrix}. \tag{3.26}$$

This shows that the eigenvalues λ_1, λ_2 of this matrix satisfy that

$$\lambda_1 = \lambda_2 = f_1 = K = -H. \tag{3.27}$$

On the other hand, Eqs. (3.18) and (3.19) generate

$$\rho_1 = \frac{\partial h_{11}}{\partial v}, \quad \rho_2 = 0, \tag{3.28}$$

which derive that

$$\rho = g^{ij}\rho_i\rho_j = 0. \tag{3.29}$$

To summarize by Eqs. (3.7)-(3.9), (3.20), (3.24), (3.27) and (3.29), we have:

Proposition 3.1. *Let $x : M^2 \to \mathbb{R}^4$ be a non-degenerate centroaffine ruled surface, then x has the following properties*:

(1) *the centroaffine metric is indefinite,*
(2) *the second fundamental form is degenerate,*
(3) *the Pick invariant is vanishing,*
(4) *the eigenvalues of the shape operator are equal,*
(5) $\rho = g^{ij}\rho_i\rho_j = 0$.

Remark 3.2. Making use of the basic theories of centroaffine immersions in $\mathbb{R}^{n+2}$, we can easily verify that the above five properties are invariant under the centroaffine transformation.

In the following we give two examples for ruled surfaces.

Example 1. Let

$$x(u, v) = (\cos u, \sin u, 0, 0)^{\mathrm{Tran}} + v(0, 0, \cos u, \sin u)^{\mathrm{Tran}}.$$

By a direct computation, we can obtain

$$f_1(u) = f_2(u) = f_3(u) = 0$$

and

$$g_{11} = g_{22} = 0, \quad g_{12} = 1.$$

According to Eq. (3.27), it is easy to see

$$K = H = 0.$$

Obviously, this surface is a centroaffine flat and centroaffine minimal ruled surface in $\mathbb{R}^4$.

Example 2. For the ruled surface

$$x(u, v) = (e^u, e^{-u}, 0, 0)^{\mathrm{Tran}} + v(e^u, -e^{-u}, u + \frac{1}{4}, u - \frac{1}{4})^{\mathrm{Tran}},$$

it is not difficult to get that

$$f_1(u) = -1, \quad f_2(u) = f_3(u) = 0$$

and

$$g_{11} = -v^2, \quad g_{22} = 0, \quad g_{12} = 1.$$

By Eq. (3.27), we see that

$$K = -1, \quad H = 1.$$

Exactly, the surface is a ruled surface with constant centroaffine mean curvature and constant Gauss curvature in $\mathbb{R}^4$.

4. Indefinite centroaffine surfaces in $\mathbb{R}^4$

Let $x : M^2 \to \mathbb{R}^4$ be an indefinite centroaffine surface, by choosing the asymptotic local basis $\sigma = \{E_1, E_2\}$ of TM such that

$$g = e^w(\mathrm{d}u \otimes \mathrm{d}v + \mathrm{d}v \otimes \mathrm{d}u). \tag{4.1}$$

Then from [5] we can obtain the following equations directly. Under the asymptotic local coordinates (u, v), the structure equations can be expressed as

$$x_{uu} = w_u x_u + e^{-w}\varphi x_v + \psi x, \tag{4.2}$$

$$x_{uv} = e^w \frac{\Delta_g x}{2}, \tag{4.3}$$

$$x_{vv} = e^{-w}\lambda x_u + w_v x_v + \theta x, \tag{4.4}$$

$$\begin{aligned}(\frac{\Delta_g x}{2})_u = {} & e^{-w}(w_{uv} + e^{-2w}\lambda\varphi)x_u + e^{-w}(e^{-w}\varphi_v + \psi)x_v \\ & + e^{-w}(e^{-w}\varphi\theta + \psi_v)x,\end{aligned} \tag{4.5}$$

$$\begin{aligned}(\frac{\Delta_g x}{2})_v = {} & e^{-w}(e^{-w}\lambda_u + \theta)x_u + e^{-w}(w_{uv} + e^{-2w}\lambda\varphi)x_v \\ & + e^{-w}(e^{-w}\lambda\psi + \theta_u)x.\end{aligned} \tag{4.6}$$

By a direct computation, the integrability conditions (2.10)-(2.15) turn out to be

$$(e^{-w}w_{uv} + e^{-3w}\lambda\varphi)_v + e^{-3w}\lambda\varphi_v - e^{-w}(e^{-w}\lambda_u)_u - 2e^{-w}\theta_u = 0, \tag{4.7}$$

$$(e^{-w}w_{uv} + e^{-3w}\lambda\varphi)_u + e^{-3w}\lambda_u\varphi - e^{-w}(e^{-w}\varphi_v)_v - 2e^{-w}\psi_v = 0, \tag{4.8}$$

$$(e^{-2w}\varphi\theta + e^{-w}\psi_v)_v + e^{-2w}\theta\varphi_v = (e^{-2w}\lambda\psi + e^{-w}\theta_u)_u + e^{-2w}\psi\lambda_u. \tag{4.9}$$

Note that the following results have been obtained in [5].

$$C^1_{11} = C^1_{12} = C^2_{12} = C^2_{22} = 0, \tag{4.10}$$

$$C^2_{11} = e^{-w}\varphi, \quad C^1_{22} = e^{-w}\lambda, \tag{4.11}$$

$$h_{11} = \psi, \quad h_{12} = 0, \quad h_{22} = \theta. \tag{4.12}$$

The expressions of the Gauss curvature and the Pick invariant also can be found in [5].

$$K = -e^{-w}w_{uv}, \quad J = e^{-3w}\lambda\varphi. \tag{4.13}$$

We now prove the following results:

Proposition 4.1. *If x is an indefinite centroaffine surface in $\mathbb{R}^4$ with vanishing Pick invariant and constant Gauss curvature, then $\rho = 0$.*

Proof. Now observe Eq. (4.13), it is easy to see $J = 0$ yields

$$\lambda\varphi = 0. \tag{4.14}$$

Without loss of generality, we can assume

$$\lambda = 0. \tag{4.15}$$

Note that since the Gauss curvature is constant, $\lambda = 0$, Eqs. (4.7)-(4.9) and (4.13) generate

$$\theta_u = 0. \tag{4.16}$$

On the other hand, from the structure equations (4.5) and (4.6) we obtain

$$\rho_1 = e^{-w}(e^{-w}\varphi\theta + \psi_v), \tag{4.17}$$

$$\rho_2 = e^{-w}(e^{-w}\lambda\psi + \theta_u). \tag{4.18}$$

Obviously, it follows from Eqs. (4.15) and (4.16) that

$$\rho_2 = 0.$$

Therefore, a direct calculation shows that

$$\rho = g^{ij}\rho_i\rho_j = 0.$$

This completes the proof. □

In the following, we begin to consider a centroaffine surface x which has the properties (1)-(5) in Proposition 3.1.

Firstly, from Eq. (4.13), it is obvious that $J = 0$ is equivalent to

$$\lambda\varphi = 0. \tag{4.19}$$

Similarly, without loss of generality, we assume

$$\lambda = 0. \tag{4.20}$$

Note that since the second fundamental form is degenerate, by Eq. (4.12), it follows that

$$\psi\theta = 0, \tag{4.21}$$

which includes the following two different cases:

Case 1. $\theta = 0$.

Solving the structure equation (4.4) by $\lambda = 0$ and $\theta = 0$, we have that

$$x = \int e^w \mathrm{d}v A(u) + B(u), \tag{4.22}$$

where $A(u)$ and $B(u)$ are 1-parameter vector fields. Putting $\bar{v} = \int e^w \mathrm{d}v, \bar{u} = u$, it is not hard to check x is a ruled surface.

Case 2. If $\theta \neq 0$, then $\psi = 0$.

According to the property that the eigenvalues of the shape operator are same, that is, from Eqs. (4.5) and (4.6), the matrix

$$(S^i_j) = \begin{pmatrix} e^{-w}(w_{uv} + e^{-2w}\lambda\varphi) & e^{-w}(e^{-w}\varphi_v + \psi) \\ e^{-w}(e^{-w}\lambda_u + \theta) & e^{-w}(w_{uv} + e^{-2w}\lambda\varphi) \end{pmatrix} \tag{4.23}$$

has same eigenvalues. Then we get by $\lambda = 0, \psi = 0$ and $\theta \neq 0$

$$\varphi_v = 0. \tag{4.24}$$

Furthermore, the structure equations (4.5) and (4.6) show that

$$\rho_1 = e^{-w}(e^{-w}\varphi\theta + \psi_v), \tag{4.25}$$

$$\rho_2 = e^{-w}(e^{-w}\lambda\psi + \theta_u). \tag{4.26}$$

So the property $\rho = g^{ij}\rho_i\rho_j = 0$ yields by $\lambda = 0, \psi = 0$ and $\theta \neq 0$

$$\varphi\theta\theta_u = 0. \tag{4.27}$$

Notice that $\varphi = 0, \psi = 0$ and Eq. (4.2) generate the same results as Case 1. Thus, for $\varphi \neq 0$, Eq. (4.27) implies

$$\theta_u = 0. \tag{4.28}$$

Hence, according to Eqs. (4.7)-(4.9) we get

$$(e^{-w}w_{uv})_u = (e^{-w}w_{uv})_v = 0, \tag{4.29}$$

$$(e^{-2w}\varphi\theta)_v = 0. \tag{4.30}$$

In particular, Eqs. (4.24), (4.30) and $\varphi\theta \neq 0$ yield

$$2w_v\theta = \theta_v. \tag{4.31}$$

Next, using (4.28) and $\varphi\theta \neq 0$, and differentiating both side of (4.31) with respect to u, we deduce that

$$w_{uv} = 0, \tag{4.32}$$

which implies the Gauss curvature is zero, that is, x is flat. By the flatness of the surface, we can choose a local basis such that $w \equiv 0$. Thus, the combining of Eqs. (4.24), (4.28) and (4.30) yields that θ must be a nonzero constant, and the structure equations can be changed to

$$x_{uu} = \varphi(u)x_v, \tag{4.33}$$

$$x_{vv} = \theta x. \tag{4.34}$$

Since θ is a nonzero constant, by a parameter transformation, we can assume $\theta = \pm 1$. Solving this system of partial differential equation, we can verify the surface is centroaffinely equivalent to one of the following surfaces, which also appear in [15]:

- $x = e^v V_1(u) + e^{-v}V_2(u)$, where $V_1(u)$ and $V_2(u)$ are 1-parameter vector fields such that $V_1''(u) = \varphi(u)V_1(u)$ and $V_2''(u) = -\varphi(u)V_2(u)$,
- $x = \cos v V_1(u) + \sin(-v)V_2(u)$, where $V_1(u)$ and $V_2(u)$ are 1-parameter vector fields such that $V_1''(u) = \varphi(u)V_2(u)$ and $V_2''(u) = -\varphi(u)V_1(u)$.

Summarizing the arguments above, we have the following proposition.

Proposition 4.2. *If $x : M^2 \to \mathbb{R}^4$ is a centroaffine surface satisfying the properties* (1)-(5) *in Proposition 3.1, then x is centroaffinely equivalent to*

(I) *ruled surface $x = a(u) + vb(u)$,*

(II) *$x = e^v V_1(u) + e^{-v} V_2(u)$, where $V_1(u)$ and $V_2(u)$ are 1-parameter vector fields such that $V_1''(u) = \varphi(u)V_1(u)$ and $V_2''(u) = -\varphi(u)V_2(u)$,*

(III) *$x = \cos v V_1(u) + \sin(-v)V_2(u)$, where $V_1(u)$ and $V_2(u)$ are 1-parameter vector fields such that $V_1''(u) = \varphi(u)V_2(u)$ and $V_2''(u) = -\varphi(u)V_1(u)$.*

If we look carefully at the derivation process,we see that the last two surfaces in Proposition 4.2 are centroaffine flat. So it follows:

Corollary 4.3. *Let $x : M^2 \to \mathbb{R}^4$ be a centroaffine surface satisfying the properties* (1)-(5) *in Proposition 3.1. If x is not a ruled surface, x is centroaffine flat.*

Now, from Proposition 4.1 and Proposition 4.2 we directly get:

Corollary 4.4. *Let $x : M^2 \to \mathbb{R}^4$ be a centroaffine surface with the constant Gauss curvature in $\mathbb{R}^4$, which satisfies the properties* (1)-(4) *in Proposition 3.1. Then if x is not a ruled surface, x must be centroaffine flat.*

5. Centered cyclic surfaces in $\mathbb{R}^4$

Let Σ be a surface of $\mathbb{R}^4$ foliated by circles with common center located at the origin of $\mathbb{R}^4$, Locally, Σ may be parameterized by the following immersion

$$x : I \times \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^4,$$
$$(u, v) \to r(u)\left(e_1(u)\cos v + e_2(u)\sin v\right),$$

where $r(u)$ is a positive function and $(e_1(u), e_2(u))$ is an orthonormal basis of the plane containing the circle. And then, under the centroaffine transformation, this surface can be expressed by

$$a(u)\cos v + b(u)\sin v,$$

where $a(u)$ and $b(u)$ are arbitrary 1-parameter vector fields with respect to u, and they should be linearly independent.

To extend this concept to hyperbolas, the surface will be changed to

$$a(u)\cosh v + b(u)\sinh v.$$

Obviously, this surface also can be written as

$$a(u)e^v + b(u)e^{-v}.$$

For convenience's sake, we uniformly call these two kinds of surfaces $a(u)\cos v + b(u)\sin v$ and $a(u)e^v + b(u)e^{-v}$ *centered cyclic surfaces.* In the following two subsections, we will consider these two kinds of surface separately.

5.1. $x = e^v a(u) + e^{-v} b(u)$

Let $x : M^2 \to \mathbb{R}^4$ be a non-degenerate centroaffine surface given by $x = e^v a(u) + e^{-v} b(u)$, where $a(u)$ and $b(u)$ are 1-variable vector fields, and in the following, we assume that the vector fields $a(u)$ and $b(u)$ are linearly independent. For the centroaffine surface x, the position vector, denoted also by x, is always transversal to the tangent space $x_*(TM)$ at each point of M, that is, the vector fields $e^v a' + e^{-v} b', e^v a - e^{-v} b, e^v a + e^{-v} b$ are linearly independent.

In terms of Eq. (2.1), we obtain

$$G_{11} = 2e^{2v}[a', a, b, a''] + 2e^{-2v}[b', a, b, b''] + 2\left([a', a, b, b''] + [b', a, b, a'']\right), \tag{5.1}$$

$$G_{12} = 4[b', a, b, a'], \tag{5.2}$$

$$G_{22} = 0. \tag{5.3}$$

As before, by a parameter transformation $\bar{u} = \bar{u}(u)$, we can assume that $[a', b, a, b'] = 1$, that is, $G_{12} = 1$. Here using the notations $\bar{f}_1(u) = 2[a', a, b, a'']$, $\bar{f}_2(u) = 2[b', a, b, b''] = -2[a'', b, a, b']$ and $\bar{f}_3(u) = 2([a', a, b, b''] + [b', a, b, a'']) = 4[a', a, b, b''] = 4[b', a, b, a'']$, we have

$$G_{11} = \bar{f}_1(u)e^{2v} + \bar{f}_2(u)e^{-2v} + \bar{f}_3(u), \tag{5.4}$$

$$G_{12} = 1, \tag{5.5}$$

$$G_{22} = 0. \tag{5.6}$$

Then by Eq. (2.2) it follows that

$$g_{11} = \bar{f}_1(u)e^{2v} + \bar{f}_2(u)e^{-2v} + \bar{f}_3(u), \tag{5.7}$$

$$g_{12} = 1, \tag{5.8}$$

$$g_{22} = 0. \tag{5.9}$$

Hence, the Levi-Civita connection of g can be written as

$$\tilde{\Gamma}^1_{11} = \bar{f}_2 e^{-2v} - \bar{f}_1 e^{2v}, \tag{5.10}$$

$$\tilde{\Gamma}^2_{11} = \bar{f}_1^2 e^{4v} + \bar{f}_2^2 e^{-4v} + (\bar{f}_1\bar{f}_3 + \frac{\bar{f}_1'}{2})e^{2v} + (\frac{\bar{f}_2'}{2} - \bar{f}_2\bar{f}_3)e^{-2v} + \frac{\bar{f}_3'}{2}, \tag{5.11}$$

$$\tilde{\Gamma}^1_{12} = 0, \quad \tilde{\Gamma}^2_{12} = \bar{f}_1 e^{2v} - \bar{f}_2 e^{-2v}, \tag{5.12}$$

$$\tilde{\Gamma}^1_{22} = \tilde{\Gamma}^2_{22} = 0. \tag{5.13}$$

Thus, we have

$$\frac{\Delta_g x}{2} = x_{uv} - (\bar{f}_1 e^{2v} - \bar{f}_2 e^{-2v})x_v - \frac{\bar{f}_1 e^{2v} + \bar{f}_2 e^{-2v} + \bar{f}_3}{2}x. \tag{5.14}$$

Totally, the structure equations can be given by

$$x_{uu} = \Gamma^1_{11}(e^v a' + e^{-v} b') + \Gamma^2_{11}(e^v a - e^{-v} b) + h_{11}(e^v a + e^{-v} b) + (\bar{f}_1 e^{2v} + \bar{f}_2 e^{-2v} + \bar{f}_3)\frac{\Delta_g x}{2}, \tag{5.15}$$

$$x_{uv} = \Gamma_{12}^1(e^v a' + e^{-v} b') + \Gamma_{12}^2(e^v a - e^{-v} b) + h_{12}(e^v a + e^{-v} b) + \frac{\Delta_g x}{2}, \tag{5.16}$$

$$x_{vv} = x, \tag{5.17}$$

$$(\frac{\Delta_g x}{2})_u = -\left[\frac{3}{2}(\bar f_1 e^{2v} + \bar f_2 e^{-2v}) - \frac{\bar f_3}{2}\right] x_u + \alpha(u,v) x_v + \beta(u,v) x, \tag{5.18}$$

$$(\frac{\Delta_g x}{2})_v = x_u - \left[\frac{5}{2}(\bar f_1 e^{2v} + \bar f_2 e^{-2v}) + \frac{\bar f_3}{2}\right] x_v - 2(\bar f_1 e^{2v} + \bar f_2 e^{-2v}) x, \tag{5.19}$$

where $\Gamma_{11}^1 = \bar f_2 e^{-2v} - \bar f_1 e^{2v}$ and

$$\alpha(u,v) = \frac{\partial \Gamma_{11}^2}{\partial v} + h_{11} - g_{11}[\frac{5}{2}(\bar f_1 e^{2v} + \bar f_2 e^{-2v}) + \frac{\bar f_3}{2}] - (\bar f_1' e^{2v} - \bar f_2' e^{-2v}) - 2(\bar f_1 e^{2v} - \bar f_2 e^{-2v})^2, \tag{5.20}$$

$$\beta(u,v) = \Gamma_{11}^2 + \frac{\partial h_{11}}{\partial v} + g_{11}(3\bar f_1 e^{2v} + \bar f_2 e^{-2v}) - \frac{\bar f_1' e^{2v} + \bar f_2' e^{-2v} + \bar f_3'}{2}. \tag{5.21}$$

By the integrability conditions (2.10)-(2.15), it is easy to verify

$$h_{12} = \frac{\bar f_1 e^{2v} + \bar f_2 e^{-2v} + \bar f_3}{2}, \quad h_{22} = 1, \tag{5.22}$$

$$\Gamma_{12}^1 = 0, \quad \Gamma_{12}^2 = \bar f_1 e^{2v} - \bar f_2 e^{-2v}, \quad \Gamma_{22}^1 = \Gamma_{22}^2 = 0, \tag{5.23}$$

$$\frac{\partial^2 \Gamma_{11}^2}{\partial v^2} + 2\frac{\partial h_{11}}{\partial v} = 18(\bar f_1^2 e^{4v} - \bar f_2^2 e^{-4v}) + \bar f_3(8\bar f_1 e^{2v} - 4\bar f_2 e^{-2v}) + 2(\bar f_1 e^{2v} + \bar f_2 e^{-2v})^2, \tag{5.24}$$

$$2\frac{\partial \Gamma_{11}^2}{\partial v} + \frac{\partial^2 h_{11}}{\partial v^2} = 8\bar f_1 e^{2v}(\bar f_1 e^{2v} + \bar f_2 e^{-2v} + \bar f_3) + 8(\bar f_1^2 e^{4v} - \bar f_2^2 e^{-4v}). \tag{5.25}$$

Now it is direct to give the Fubini-Pick form

$$C_{11}^1 = C_{12}^1 = C_{12}^2 = C_{22}^1 = C_{22}^2 = 0 \tag{5.26}$$

and the Pick invariant

$$J = 0. \tag{5.27}$$

The above results can be described as follows:

Proposition 5.1. *The centroaffine surface $x = e^v a(u) + e^{-v} b(u)$ in $\mathbb{R}^4$ is indefinite and its Pick invariant $J = 0$.*

Then, obviously, the Gauss curvature can be obtain by centroaffine metric directly

$$K = 2(\bar f_1 e^{2v} + \bar f_2 e^{-2v}). \tag{5.28}$$

This yields the following proposition.

Proposition 5.2. *The Gauss curvature K is constant if and only if $K = 0$.*

Proof. We may easily verify that $\bar{f}_1 e^{2v} + \bar{f}_2 e^{-2v}$ is constant if and only if $\bar{f}_1 = \bar{f}_2 = 0$, which implies $K = 0$. □

Remark 5.3. Due to Theorem 2.3 and Proposition 5.1, we have $H = -K$. From Proposition 5.2 it is obvious that the mean curvature H is constant if and only if x is minimal.

According to Propositions 4.1 and 5.1 we have:

Lemma 5.4. *If $K = 0$, then centroaffine invariant $\rho = 0$.*

Then for a centroaffine surface $x(u, v) = e^v a(u) + e^{-v} b(u)$, we can prove the following proposition.

Proposition 5.5. *If x is flat, the property* (2) *and the property* (4) *in Proposition 3.1 are equivalent.*

Proof. If x is flat, from Proposition 5.2, we know $\bar{f}_1 = \bar{f}_2 = 0$. Eqs. (5.24) and (5.25) give

$$\frac{\partial^2 \Gamma_{11}^2}{\partial v^2} + 2\frac{\partial h_{11}}{\partial v} = 0, \tag{5.29}$$

$$2\frac{\partial \Gamma_{11}^2}{\partial v} + \frac{\partial^2 h_{11}}{\partial v^2} = 0. \tag{5.30}$$

Solving this partial differential equation system, we have

$$h_{11} = -\frac{1}{2}e^{-2v} g_1(u) - \frac{1}{2}e^{2v} g_2(u) + g_3(u), \tag{5.31}$$

$$\Gamma_{11}^2 = -\frac{1}{2}e^{-2v} g_1(u) + \frac{1}{2}e^{2v} g_2(u) + g_4(u), \tag{5.32}$$

where $g_i(u), i = 1, 2, 3, 4$ are arbitrary 1-parameter functions with respect to u.

Firstly, if the second fundamental form is degenerate, that is, $h_{11}h_{22} - h_{12}h_{21} = 0$, Eq. (5.22) and $\bar{f}_1 = \bar{f}_2 = 0$ yield

$$h_{11} = \frac{\bar{f}_3^2}{4}. \tag{5.33}$$

The combining of Eqs. (5.31) and (5.33) generates

$$g_1(u) = g_2(u) = 0, \quad g_3(u) = \frac{\bar{f}_3^2}{4}. \tag{5.34}$$

Then in view of Eqs. (5.18), (5.19), (5.20), (5.32) and (5.34), we get the shape operator matrix

$$(S_j^i) = \begin{pmatrix} \frac{\bar{f}_3}{2} & -\frac{\bar{f}_3^2}{4} \\ 1 & -\frac{\bar{f}_3}{2} \end{pmatrix}. \tag{5.35}$$

The eigenvalues λ_1, λ_2 of this matrix satisfy that $\lambda_1 + \lambda_2 = 0, \lambda_1\lambda_2 = \det(S_j^i) = 0$, so it follows that $\lambda_1 = \lambda_2 = 0$. This verified the property (4) in Proposition 3.1.

Conversely, from (5.18), (5.19) and $\bar{f}_1 = \bar{f}_2 = 0$, we get the shape operator matrix

$$(S^i_j) = \begin{pmatrix} \dfrac{\bar{f}_3}{2} & \alpha(u,v) \\ 1 & -\dfrac{\bar{f}_3}{2} \end{pmatrix}. \tag{5.36}$$

Notice that the trace of this matrix is zero. So the eigenvalues λ_1, λ_2 of the shape operator are same implies $\lambda_1 = \lambda_2 = 0$, that is, $\det(S^i_j) = 0$. Then we have

$$\alpha(u,v) = -\frac{\bar{f}_3^2}{4}. \tag{5.37}$$

The combining of Eqs. (5.20), (5.31), (5.32) and (5.37) gives

$$g_1(u) = g_2(u) = 0, \quad h_{11} = g_3(u) = \frac{\bar{f}_3^2}{4}. \tag{5.38}$$

Obviously, $h_{11}h_{22} - h_{12}h_{21} = 0$, which means the second fundamental form is degenerate. □

Finally, from Propositions 4.2, 5.1 and Corollary 4.3, it is easy to verify:

Corollary 5.6. *If the centroaffine surface $x = e^v a(u) + e^{-v} b(u)$ in $\mathbb{R}^4$ satisfies properties* (2), (4) *and* (5) *in Proposition 3.1, then x is flat.*

5.2. $x = \cos va(u) + \sin vb(u)$

Let $x : M^2 \to \mathbb{R}^4$ be a non-degenerate centroaffine surface given by $x = \cos va(u) + \sin vb(u)$, where $a(u)$ and $b(u)$ are 1-variable vector fields, and in the following, we assume that the vector fields $a(u)$ and $b(u)$ are linearly independent. For the centroaffine surface x, the position vector, denoted also by x, is always transversal to the tangent space $x_*(TM)$ at each point of M, that is, the vector fields $\cos va' + \sin vb'$, $-\sin va + \cos vb$, $\cos va + \sin vb$ are linearly independent.

From Eq. (2.1), it is easy to check that

$$\begin{aligned} G_{11} = \ & \cos 2v \frac{[a', b, a, a''] - [b', b, a, b'']}{2} \\ & + \sin 2v \frac{[a', b, a, b''] + [b', b, a, a'']}{2} \\ & + \frac{[a', b, a, a''] + [b', b, a, b'']}{2}, \end{aligned} \tag{5.39}$$

$$G_{12} = [a', b, a, b'], \tag{5.40}$$

$$G_{22} = 0. \tag{5.41}$$

By a parameter transformation $\bar{u} = \bar{u}(u)$, we can also assume that $[a', b, a, b'] = 1$, that is, $G_{12} = 1$. Now taking the notations

$$\tilde{f}_1 = \frac{[a', b, a, a''] - [b', b, a, b'']}{2}, \tag{5.42}$$

$$\tilde{f}_2 = \frac{[a', b, a, b''] + [b', b, a, a'']}{2}, \tag{5.43}$$

$$\tilde{f}_3 = \frac{[a', b, a, a''] + [b', b, a, b'']}{2}, \tag{5.44}$$

we have

$$G_{11} = \tilde{f}_1(u)\cos 2v + \tilde{f}_2(u)\sin 2v + \tilde{f}_3(u), \tag{5.45}$$

$$G_{12} = 1, \tag{5.46}$$

$$G_{22} = 0. \tag{5.47}$$

Hence, by Eq. (2.2),

$$g_{11} = \tilde{f}_1(u)\cos 2v + \tilde{f}_2(u)\sin 2v + \tilde{f}_3(u), \tag{5.48}$$

$$g_{12} = 1, \tag{5.49}$$

$$g_{22} = 0. \tag{5.50}$$

By a direct computation, the Levi-Civita connection of g can be given by

$$\tilde{\Gamma}^1_{11} = \tilde{f}_1 \sin 2v - \tilde{f}_2 \cos 2v, \tag{5.51}$$

$$\begin{aligned}\tilde{\Gamma}^2_{11} = {} & \frac{\tilde{f}_1' \cos 2v + \tilde{f}_2' \sin 2v + \tilde{f}_3'}{2} \\ & - (\tilde{f}_1 - \tilde{f}_2 \cos 2v)(\tilde{f}_1 \cos 2v + \tilde{f}_2 \sin 2v + \tilde{f}_3),\end{aligned} \tag{5.52}$$

$$\tilde{\Gamma}^1_{12} = 0, \quad \tilde{\Gamma}^2_{12} = \tilde{f}_2 \cos 2v - \tilde{f}_1 \sin 2v, \tag{5.53}$$

$$\tilde{\Gamma}^1_{22} = \tilde{\Gamma}^2_{22} = 0. \tag{5.54}$$

It is easy to see

$$\frac{\Delta_g x}{2} = x_{uv} - (\tilde{f}_2 \cos 2v - \tilde{f}_1 \sin 2v)x_v + \frac{\tilde{f}_1 \cos 2v + \tilde{f}_2 \sin 2v + \tilde{f}_3}{2}x. \tag{5.55}$$

Then the structure equation can be written as

$$x_{uu} = \Gamma^1_{11}x_u + \Gamma^2_{11}x_v + h_{11}x + g_{11}\frac{\Delta_g x}{2}, \tag{5.56}$$

$$x_{uv} = \Gamma^1_{12}x_u + \Gamma^2_{12}x_v + h_{12}x + \frac{\Delta_g x}{2}, \tag{5.57}$$

$$x_{vv} = -x, \tag{5.58}$$

$$(\frac{\Delta_g x}{2})_u = \left[\frac{3}{2}(\tilde{f}_1 \cos 2v + \tilde{f}_2 \sin 2v) - \frac{\tilde{f}_3}{2}\right] x_u + \alpha(u,v)x_v + \beta(u,v)x, \tag{5.59}$$

$$\begin{aligned}(\frac{\Delta_g x}{2})_v = {} & -x_u + \left[\frac{5}{2}(\tilde{f}_1 \cos 2v + \tilde{f}_2 \sin 2v) + \frac{\bar{f}_3}{2}\right] x_v \\ & + 2(\tilde{f}_2 \cos 2v - \tilde{f}_1 \sin 2v)x,\end{aligned} \tag{5.60}$$

where $\Gamma^1_{11} = \tilde{f}_1 \sin 2v - \tilde{f}_2 \cos 2v$ and

$$\alpha(u,v) = \Gamma^1_{11}(\bar{f}_1 e^{2v} + \bar{f}_2 e^{-2v}) + \frac{\partial \Gamma^2_{11}}{\partial v} + h_{11} - g_{11}[\frac{5}{2}(\bar{f}_1 e^{2v} + \bar{f}_2 e^{-2v}) + \frac{\bar{f}_3}{2}], \tag{5.61}$$

$$\beta(u,v) = \Gamma^1_{11}\frac{g_{11}}{2} + \Gamma^2_{11} + \frac{\partial h_{11}}{\partial v} - 2g_{11}(\bar{f}_1 e^{2v} + \bar{f}_2 e^{-2v}). \tag{5.62}$$

By the integrability conditions (2.10)-(2.15), it follows that

$$h_{12} = -\frac{\tilde{f}_1 \cos 2v + \tilde{f}_2 \sin 2v + \tilde{f}_3}{2}, \quad h_{22} = -1, \tag{5.63}$$

$$\Gamma^1_{12} = 0, \quad \Gamma^2_{12} = \tilde{f}_2 \cos 2v - \tilde{f}_1 \sin 2v, \quad \Gamma^1_{22} = \Gamma^2_{22} = 0, \tag{5.64}$$

$$\frac{\partial^2 \Gamma^2_{11}}{\partial v^2} + 2\frac{\partial h_{11}}{\partial v} = 18(\bar{f}_1^2 e^{4v} - \bar{f}_2^2 e^{-4v}) + \bar{f}_3(8\bar{f}_1 e^{2v} - 4\bar{f}_2 e^{-2v}) + 2(\bar{f}_1 e^{2v} + \bar{f}_2 e^{-2v})^2, \tag{5.65}$$

$$2\frac{\partial \Gamma^2_{11}}{\partial v} + \frac{\partial^2 h_{11}}{\partial v^2} = 8\bar{f}_2 e^{-2v}(\bar{f}_1)e^{2v} + \bar{f}_2 e^{-2v} + \bar{f}_3) + 8(\bar{f}_1^2 e^{4v} - \bar{f}_2^2 e^{-4v}). \tag{5.66}$$

Hence, we get the Fubini-Pick form

$$C^1_{11} = C^1_{12} = C^2_{12} = C^1_{22} = C^2_{22} = 0. \tag{5.67}$$

So it is apparent that the Pick invariant

$$J = 0. \tag{5.68}$$

We also get the Gauss curvature

$$K = -2(\tilde{f}_1 \cos 2v + \tilde{f}_2 \sin 2v). \tag{5.69}$$

In fact, using the same methods as before, we have the same results for surface $x(u,v) = \cos va(u) + \sin vb(u)$.

Proposition 5.7. *The centroaffine surface* $x(u,v) = \cos va(u) + \sin vb(u)$ *is indefinite and its Pick-invariant* $J = 0$.

Proposition 5.8. *If Gauss curvature* K *is constant, then* $K = 0$.

Corollary 5.9. *If* $K = 0$*, then centroaffine invariant* $\rho = 0$.

Proposition 5.10. *If* x *is flat, the properties* (2) *and* (4) *in Proposition 3.1 are equivalent.*

Corollary 5.11. *If* x *satisfies properties* (2), (4) *and* (5) *in Proposition 3.1, then* x *is flat.*

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Yun Yang
Department of Mathematics
Northeastern University
Shenyang, Liaoning, 110004, P. R. China
Email address: `freeuse_st@126.com`

Yanhua Yu
Department of Mathematics
Northeastern University
Shenyang, Liaoning, 110004, P. R. China
Email address: `yyh_start@126.com`