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A ONE-PARAMETER FAMILY OF TOTALLY UMBILICAL HYPERSPHERES IN THE NEARLY KÄHLER 6-SPHERE

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Abstract. We discuss two kinds of almost contact metric structures on a one-parameter family of totally umbilical hyperspheres in the nearly Kähler unit 6-sphere S^6.

1. Introduction

An odd dimensional smooth manifold M with a quadruple (ϕ, ξ, η, g) of a $(1,1)$-tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g satisfying the following conditions is called an almost contact metric manifold;

$$\begin{gathered} \phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1 \\ \phi\xi = 0, \quad \eta \circ \phi = 0 \end{gathered} \tag{1}$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2}$$

for any $X, Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the Lie algebra of all smooth vector fields on M. Further, an almost contact metric manifold (M, ϕ, ξ, η, g) is called a contact metric manifold if it satisfies the following condition;

$$d\eta(X, Y) = g(X, \phi Y) \tag{3}$$

for any $X, Y \in \mathfrak{X}(M)$. An almost contact metric manifold $M = (M, \phi, \xi, \eta, g)$ is called a *quasi contact metric manifold* if the corresponding almost Hermitian cone $\bar{M} = (\bar{M}, \bar{J}, \bar{g})$ is a quasi Kähler manifold [4,6,7]. In [6], the quasi contact metric manifold is proved to be a generalization of a contact metric manifold. Further, the authors raised the following question based on the discussion.

Question A. *Does there exist a* $(2n+1)(\geq 5)$*-dimensional quasi contact metric manifold which is not a contact metric manifold?*

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Concerning the above Question A, authors discussed oriented hypersurfaces in a quasi Kähler manifold which are quasi contact metric manifolds with respect to the naturally induced almost contact metric structure, and obtained the following results in [1].

Theorem B. *Let $\bar{M} = (\bar{M}, J, \bar{g})$ be a nearly Kähler manifold and M be a hypersurface of $\bar{M}$ oriented by a unit normal vector field ν. Then $M = (M, \phi, \xi, \eta, g)$ is a quasi contact metric manifold with respect to the naturally induced almost contact metric structure (ϕ, ξ, η, g) if and only if it satisfies the equality*

$$g((A\phi + \phi A)X, Y) = -2g(\phi X, Y)$$

for any $X, Y \in \mathfrak{X}(M)$, where A is the shape operator with respect to the unit normal vector field ν, and hence, M is a contact metric manifold.

Theorem C. *There does not exist oriented totally umbilical hypersurface in the nearly Kähler unit 6-sphere which is a quasi contact metric manifold with respect to the naturally induced almost contact metric structure.*

In the present paper, we provide explicit examples of totally umbilical hyperspheres in the nearly Kähler unit 6-sphere which support Theorem B or Theorem C. We also discuss the properties from the view point of almost contact metric geometry.

2. Preliminaries

First, we shall recall fundamental the nearly Kähler structure on a unit 6-sphere S^6. Let $\mathfrak{C}$ be the Cayley algebra $\mathfrak{C} = \{x = x_0 + \sum_{i=1}^{7} x_i e_i \mid x_0, x_i \in \mathbb{R}, e_i^2 = -1\ (1 \leq i \leq 7)\}$, and $\mathfrak{C}_+ = \{x = \sum_{i=1}^{7} x_i e_i \in \mathfrak{C} \mid x_i \in \mathbb{R}\ (1 \leq i \leq 7)\}$ both set of all pure imaginary Cayley numbers. Here, the multiplication operation on $\mathfrak{C}$ is defined by the figure below;

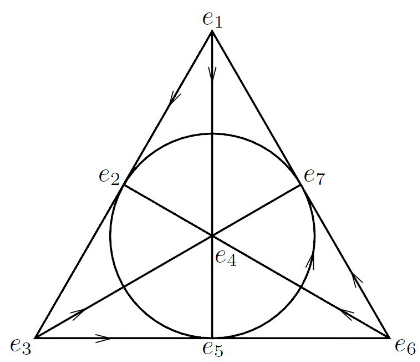

FIGURE 1

We denote by $\langle , \rangle$ the canonical inner product on $\mathfrak{C}$ and let $|x| = \sqrt{\langle x, x\rangle}$ (the length of $x \in \mathfrak{C}$). Then, $(\mathfrak{C}, \langle , \rangle)$ (resp. $(\mathfrak{C}_+, \langle , \rangle)$) can be identified with 8-dimensional Euclidean space $\mathbb{E}^8$ (resp. 7-dimensional Euclidean space $\mathbb{E}^7$) in the natural way. We also define cross product $x \times y$ for $x, y \in \mathfrak{C}_+$ by $x \times y = xy + \langle x, y\rangle 1 (\in \mathfrak{C}_+)$. Here, we identify $e_i \in \mathfrak{C}_+$ $(1 \leq i \leq 7)$ with the coordinate vector field $\frac{\partial}{\partial x_i}$ (denoted by ∂_i briefly) in our arguments and adopt them alternatively in the forthcoming arguments. We denote by D the Levi-Civita connection on $\mathbb{E}^7$ with respect to the Riemannian metric induced from the inner product $\langle , \rangle$. Let S^6 be a unit 6-sphere in $\mathbb{E}^7 (\simeq \mathfrak{C}_+)$ centered at the origin o. Then, S^6 is expressed as $S^6 = \{x \in \mathfrak{C}_+ \mid |x| = 1\}$.

For any point $x \in S^6$, we denote by N_x the outward oriented unit normal vector with initial point x, $N_x = \overrightarrow{ox}$. In this paper, we identify $N_x (x \in S^6)$ with the position vector $x (\in \mathfrak{C}_+)$. The unit normal vector N is also written as $N = \sum_{i=1}^{7} x_i \partial_i$ in terms of the coordinate vector fields $\partial_i (1 \leq i \leq 7)$. Here we note that the tangent space $T_x S^6$ can be regarded as the subspace $\{y \in \mathfrak{C}_+ \mid \langle y, x\rangle = 0\}$ of $\mathfrak{C}_+$. Now, we define $(1,1)$-tensor field J on S^6 by

$$J_x y = N_x \times y = x \times y (= xy), \quad y \in T_x S^6. \tag{4}$$

Then, we may easily check that J is an almost complex structure on S^6 and $(J, \bar{g})$ is a nearly Kähler structure on S^6, namely, $(\bar{\nabla}_X J)Y = -(\bar{\nabla}_Y J)X$ holds for any vector fields X, Y tangent to S^6, where $\bar{g}$ and $\bar{\nabla}$ are the Riemannian metric on S^6 induced from the inner product $\langle , \rangle$ on $\mathfrak{C}_+$ and $\bar{\nabla}$ is the Levi-Civita connection of $\bar{g}$, respectively. We shall call the nearly Kähler structure $(J, \bar{g})$ given above on S^6 the standard one.

3. One parameter family of totally umbilical hyperspheres in S^6

First, for each real number r $(-1 < r < 1)$, we define hypersurface M_r by

$$\begin{aligned} M_r &= S^6 \cap \{x = \sum_{i=1}^{6} x_i e_i + r e_7 \in \mathfrak{C}_+ | \ x_i \in \mathbb{R} \ (1 \leq i \leq 6)\} \\ &= \{x = \sum_{i=1}^{6} x_i e_i + r e_7 \in \mathfrak{C}_+ | \ \sum_{i=1}^{6} x_i^2 = 1 - r^2\}. \end{aligned}$$

We observe that M_r is diffeomorphic to a 5-sphere S^5.

Now, let x be any point of M_r and γ_x be the smooth curve in M_r through $x = \gamma_x(\theta)$ $(0 < \theta < \pi)$ defined by

$$\gamma_x(t) = (\cos t) e_7 + (\frac{1}{\sqrt{1 - r^2}} \sin t) \sum_{i=1}^{6} x_i e_i \ (0 \leq t \leq \pi), \tag{5}$$

where $\cos\theta = r, \ \sin\theta = \sqrt{1 - r^2}$. We here define a vector field ν on M_r by

$$\nu_x = \frac{d}{dt}|_{t=\theta} \gamma_x(t) \tag{6}$$

$$= -(\sin\theta)e_7 + (\frac{1}{\sqrt{1-r^2}}\cos\theta)\sum_{i=1}^{6} x_i e_i$$

$$= \frac{r}{\sqrt{1-r^2}}\sum_{i=1}^{6} x_i e_i - \sqrt{1-r^2}e_7.$$

Thus, from (6), the following equalities hold for any $x \in M_r$;

$$\begin{aligned} \bar{g}(\nu_x, \nu_x) &= \langle \nu_x, \nu_x \rangle = 1, \\ \langle \nu_x, N_x \rangle = \langle \nu_x, x \rangle &= \frac{r}{\sqrt{1-r^2}}(1-r^2) - r\sqrt{1-r^2} = 0. \end{aligned} \tag{7}$$

On the other hand, for any $x = \sum_{i=1}^{6} x_i e_i + re_7 \in M_r$, we may find an integer a $(1 \leq a \leq 6)$ such that $x_a \neq 0$ and fix it. Now, we shall define a smooth curve $\alpha_{a,b}(s)$ $(1 \leq b \leq 6,\ b \neq a)(-\pi < s < \pi)$ through the point $x = \alpha_{a,b}(0)$ by

$$\begin{aligned} \alpha_{a,b}(s) = (&\sqrt{x_a^2 + x_b^2}\cos(s + \theta_{a,b}))e_a + (\sqrt{x_a^2 + x_b^2}\sin(s + \theta_{a,b}))e_b \\ &+ \sum_{1 \leq i \leq 6, i \neq a,b} x_i e_i + re_7, \end{aligned} \tag{8}$$

where $\cos\theta_{a,b} = \frac{x_a}{\sqrt{x_a^2+x_b^2}}$, $\sin\theta_{a,b} = \frac{x_b}{\sqrt{x_a^2+x_b^2}}$ $(0 \leq \theta_{a,b} < 2\pi)$. Then, from (8), we have

$$\begin{aligned} \frac{d}{ds}|_{s=0}\alpha_{a,b}(s) &= -(\sqrt{x_a^2 + x_b^2}\sin\theta_{a,b})e_a + (\sqrt{x_a^2 + x_b^2}\cos\theta_{a,b})e_b \\ &= -x_b e_a + x_a e_b \ (= -x_b\partial_a + x_a\partial_b) \end{aligned} \tag{9}$$

at x. We here set

$$X_{a,b} = -x_b e_a + x_a e_b \ (= -x_b\partial_a + x_a\partial_b). \tag{10}$$

From (9) and (10), it follows that

$$T_x M_r = \mathrm{span}_{\mathbb{R}}\{X_{a,b}\ (b \neq a,\ 1 \leq b \leq 6)\} \tag{11}$$

and

$$\bar{g}(X_{a,b}, \nu_x) = 0, \tag{12}$$

at $x \in M_r$. Thus, from (11) and (12), we can see that ν_x is a unit normal vector at any $x \in M_r$ in S^6, namely the vector field ν is a unit normal vector field on M_r in S^6.

Now, since S^6 is a totally umbilical hypersurface in $\mathbb{E}^7$ $(\simeq \mathbb{C}_+)$ with respect to the unit normal vector field N, the corresponding shape operator $\bar{A}$ is given by $\bar{A} = -I$. Thus, taking account of the Gauss formula, we have

$$D_{X_{a,b}}\nu = \bar{\nabla}_{X_{a,b}}\nu. \tag{13}$$

From (6), the unit normal vector field ν can be expressed by

$$\nu = \frac{r}{\sqrt{1-r^2}} \sum_{i=1}^{6} x_i \partial_i - \sqrt{1-r^2}\partial_7. \tag{14}$$

Thus, from (10), (13) and (14), we have

$$D_{X_{a,b}}\nu = \frac{r}{\sqrt{1-r^2}}(-x_b\partial_a + x_a\partial_b) = \frac{r}{\sqrt{1-r^2}} X_{a,b} \tag{15}$$

for any $X_{a,b}$ at any point $x \in M_r$. Therefore, from (15), we see that (M_r, g) is a totally umbilical hypersurface of $(S^6, \bar{g})$ with the shape operator $A = -\frac{r}{\sqrt{1-r^2}} I$ with respect to the unit normal vector field ν on (M_r, g) in $(S^6, \bar{g})$.

4. Almost contact metric structures on (M_r, g)

In this section, we define two kinds almost contact metric structures on (M_r, g) and discuss their respective geometric properties. First, let ξ be the unit vector field on M_r defined by

$$\xi = -J\nu = -N \times \nu. \tag{16}$$

Then, from (16), it follows that the vector field ξ is orthogonal to both of the vector fields N and ν along M_r. Further, from Fig. 1, (6) and (16), we have

$$\begin{aligned} \xi &= -(\sum_{i=1}^{6} x_i e_i + r e_7) \times (\frac{r}{\sqrt{1-r^2}} \sum_{j=1}^{6} x_j e_j - \sqrt{1-r^2} e_7) \\ &= \sqrt{1-r^2}(\sum_{i=1}^{6} x_i e_i) \times e_7 - \frac{r^2}{\sqrt{1-r^2}} e_7 \times (\sum_{j=1}^{6} x_j e_j) \\ &= \frac{1}{\sqrt{1-r^2}}(x_6 e_1 + x_5 e_2 + x_4 e_3 - x_3 e_4 - x_2 e_5 - x_1 e_6). \end{aligned} \tag{17}$$

From (17), ξ is also rewritten as

$$\xi = \frac{1}{\sqrt{1-r^2}}(x_6\partial_1 + x_5\partial_2 + x_4\partial_3 - x_3\partial_4 - x_2\partial_5 - x_1\partial_6). \tag{18}$$

Thus, the 1-form η dual to the vector field ξ is given by

$$\eta = \frac{1}{\sqrt{1-r^2}}(x_6 dx_1 + x_5 dx_2 + x_4 dx_3 - x_3 dx_4 - x_2 dx_5 - x_1 dx_6). \tag{19}$$

From (19), we also have

$$d\eta = -\frac{2}{\sqrt{1-r^2}}(dx_1 \wedge dx_6 + dx_2 \wedge dx_5 + dx_3 \wedge dx_4). \tag{20}$$

From (19) and (20), we have further

$$\begin{aligned}\eta \wedge (d\eta)^2 = -\frac{8}{(\sqrt{1-r^2})^3}\{&-x_1 dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5 \wedge dx_6 \\ &+x_2 dx_1 \wedge dx_3 \wedge dx_4 \wedge dx_5 \wedge dx_6 \\ &-x_3 dx_1 \wedge dx_2 \wedge dx_4 \wedge dx_5 \wedge dx_6 \\ &+x_4 dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_5 \wedge dx_6 \\ &-x_5 dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_6 \\ &+x_6 dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5\} \neq 0.\end{aligned} \tag{21}$$

Therefore, η is a contact form on M_r. Now, we shall show $\nabla_\xi \xi = 0$. From (6), we have

$$D_\xi \xi = -\frac{1}{1-r^2}\sum_{i=1}^{6} x_i \partial_i. \tag{22}$$

Taking account of the Gauss formula for $(S^6, \bar{g})$ and $(\mathbb{E}^7, \langle , \rangle)$, we have

$$D_\xi \xi = \bar{\nabla}_\xi \xi - N = \bar{\nabla}_\xi \xi - \sum_{i=1}^{6} x_i \partial_i - r\partial_7. \tag{23}$$

On the other hand, since (M_r, g) is a totally umbilical hypersurface of $(S^6, \bar{g})$ with the shape operator $A = -\frac{r}{\sqrt{1-r^2}} I$ with respect to the unit normal vector field ν, from (14), taking account of the Gauss formula, we get

$$\begin{aligned}\bar{\nabla}_\xi \xi &= \nabla_\xi \xi - \frac{r}{\sqrt{1-r^2}}\nu \\ &= \nabla_\xi \xi - \frac{r}{\sqrt{1-r^2}}\Big(\frac{r}{\sqrt{1-r^2}}\sum_{i=1}^{6} x_i \partial_i - \sqrt{1-r^2}\partial_7\Big) \\ &= \nabla_\xi \xi - \frac{r^2}{1-r^2}\sum_{i=1}^{6} x_i \partial_i + r\partial_7.\end{aligned} \tag{24}$$

Then, from (22)$\sim$(24), we have

$$-\frac{1}{1-r^2}\sum_{i=1}^{6} x_i \partial_i = \nabla_\xi \xi - \Big(1 + \frac{r^2}{1-r^2}\Big)\sum_{i=1}^{6} x_i \partial_i,$$

and hence

$$\nabla_\xi \xi = 0. \tag{25}$$

From (25), it follows that each integral curve of the vector field ξ is a geodesic of (M_r, g). Thus, taking account of the definition of the vector field ξ in (16), we see that (M_r, g, ξ) is a Hopf hypersurface in $(S^6, J, \bar{g})$. Further, since (M_r, g) is a totally umbilical hypersurface in $(S^6, \bar{g})$ with the shape operator $A = -\frac{r}{\sqrt{1-r^2}} I$, from the Gauss equation for (M_r, g), we see that the curvature tensor R of (M_r, g) is given

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y + \frac{r^2}{1-r^2}(g(Y,Z)X - g(X,Z)Y) = \frac{1}{1-r^2}(g(Y,Z)X - g(X,Z)Y) \tag{26}$$

for any $X, Y, Z \in T_x M_r$. From (26), it follows that (M_r, g) is a hypersurface of $(S^6, \bar{g})$ of constant sectional curvature $\frac{1}{1-r^2}$. We define $(1,1)$-tensor field ϕ on M_r by

$$\phi X = JX - \eta(X)\nu \tag{27}$$

for any $X \in T_x M_r$. Then, from (16), (19) and (27), we see that (ϕ, ξ, η, g) is the naturally induced almost contact metric structure on M_r. Now, choose $x = \sum_{i=1}^{6} x_i e_i + re_7 \in M_r$ arbitrary. Without loss of essentiality, we may suppose $x_1 \neq 0$, for example. Then, from (4), (10) and (27), taking account of Fig. 1, we have

$$X_{1,2} = -x_2\partial_1 + x_1\partial_2, \quad X_{1,3} = -x_3\partial_1 + x_1\partial_3, \tag{28}$$

and

$$\begin{aligned}
\phi X_{1,3} &= JX_{1,3} - \eta(X_{1,3})\nu \\
&= (x_1x_2 - \frac{r}{1-r^2}(x_1^2x_4 - x_1x_3x_6))\partial_1 \\
&\quad + (-(x_1^2 + x_3^2) - \frac{r}{1-r^2}(x_1x_2x_4 - x_2x_3x_6))\partial_2 \\
&\quad + (x_2x_3 - \frac{r}{1-r^2}(x_1x_3x_4 - x_3^2x_6))\partial_3 \\
&\quad + ((-x_3x_5 + rx_1) - \frac{r}{1-r^2}(x_1x_4^2 - x_3x_4x_6))\partial_4 \\
&\quad + (x_1x_6 + x_3x_4 - \frac{r}{1-r^2}(x_1x_4x_5 - x_3x_5x_6))\partial_5 \\
&\quad + (-(rx_3 + x_1x_5) - \frac{r}{1-r^2}(x_1x_4x_6 - x_3x_6^2))\partial_6.
\end{aligned} \tag{29}$$

Thus, from (28) and (29), we have

$$g(X_{1,2}, \phi X_{1,3}) = -x_1(x_1^2 + x_2^2 + x_3^2)(\neq 0). \tag{30}$$

On the other hand, from (20) and (28), we have

$$\begin{aligned}
d\eta(X_{1,2}, X_{1,3}) = &-\frac{2}{\sqrt{1-r^2}}(dx_1 \wedge dx_6 + dx_2 \wedge dx_5 + dx_3 \wedge dx_4) \\
&(-x_2\partial_1 + x_1\partial_2, -x_3\partial_1 + x_1\partial_3) \\
= &\ 0.
\end{aligned} \tag{31}$$

Thus, from (30) and (31), we have

$$d\eta(X_{1,2}, X_{1,3}) \neq g(X_{1,2}, \phi X_{1,3}). \tag{32}$$

We may also derive the similar conclusion as (32) for the other cases $x_b \neq 0$ $(3 \leq b \leq 6)$. Thus, from (32), the almost contact metric manifold $(M_r, \phi, \xi, \eta, g)$ is not a contact metric manifold for any r $(-1 < r < 1)$. This supports Theorem C, since the quasi contact metric structure is a generalization of a contact metric structure.

Now, let τ be the scalar curvature of (M_r, g) and define the smooth functions f on M_r and the mean curvature α respectively by

$$f = g(A\xi, \xi) \tag{33}$$

and

$$\alpha = \frac{1}{5} trA. \tag{34}$$

Then, from (26), since $A = -\frac{r}{\sqrt{1-r^2}} I$ and $g(\xi, \xi) = 1$, we have

$$\tau = \frac{20}{1-r^2}, \quad f = -\frac{r}{\sqrt{1-r^2}}, \quad \alpha = -\frac{r}{\sqrt{1-r^2}}. \tag{35}$$

Then, from (35), we may check that the hypersurface $(M_r, \phi, \xi, \eta, g)$ satisfies the following equality;

$$\tau = 20 + 5\alpha(5\alpha - f) \tag{36}$$

for any real number r $(-1 < r < 1)$. Here, we note that $(M_r, \phi, \xi, \eta, g)$ is totally geodesic in (S^6, g) if and only if $r = 0$. Now, let M be an orientable compact and connected hypersurface of the nearly Kähler unit 6-sphere $(S^6, J, \bar{g})$ endowed with the naturally induced almost contact metric structure (ϕ, ξ, η, g) and define the functions f and α on M by (33) and (34) respectively, in terms of the hypersurface M. Under the above setting, in [5], the authors asserted that if the scalar curvature τ of M satisfies the inequality $\tau \geq 20 + 5\alpha(5\alpha - f)$, then (M, ϕ, ξ, η, g) is a totally geodesic hypersphere S^5 of $(S^6, J, \bar{g})$ ([5], Theorem 1.1). However, by taking account of the equality (36), we may check that their assertion is not appropriate, since every hyperspheres (M_r, ξ, g) $(-1 < r < 1, r \neq 0)$ belonging to our one parameter family of totally umbilical hypersurfaces of the nearly Kähler unit 6-sphere $(S^6, J, \bar{g})$ is not totally geodesic for any $r(-1 < r < 1, r \neq 0)$.

Next, we define another almost contact metric structure on the hypersurface (M_r, g) and discuss on the geometric properties. Let ϕ' be the $(1,1)$-tensor field on M_r defined by

$$\phi'\xi = 0 \tag{37}$$

and

$$\phi' X = -\nu \times X = -\nu X \tag{38}$$

for any $X \in \mathfrak{X}(M_r)$ with $X \perp \xi$. Then, from (16), (37) and (38), we may check

$$\begin{aligned} \langle \phi' X, N \rangle &= -\langle \nu X, N \rangle = -\langle \nu, \nu \rangle \langle \nu X, N \rangle \\ &= -\langle \nu(\nu X), \nu N \rangle = -\langle \nu^2 X, \nu N \rangle \\ &= \langle X, \nu N \rangle = -\langle X, N\nu \rangle \\ &= \langle X, \xi \rangle = 0, \end{aligned} \tag{39}$$

$$\begin{aligned} \langle \phi' X, \nu \rangle &= \langle -\nu X, \nu \rangle \\ &= -\langle \nu, \nu \rangle \langle \nu X, \nu \rangle \\ &= -\langle \nu^2 X, \nu^2 \rangle = \langle X, 1 \rangle \\ &= 0, \end{aligned} \tag{40}$$

$$\begin{aligned} \langle \phi' X, \xi \rangle &= \langle -\nu X, -N\nu \rangle \\ &= \langle -\nu X, \nu N \rangle \\ &= -\langle \nu, \nu \rangle \langle X, N \rangle \\ &= 0. \end{aligned} \tag{41}$$

Thus, from (38) $\sim$ (41), we have finally

$$\phi'^2 X = \phi'(\phi' X) = \nu(\nu X) = \nu^2 X = -X. \tag{42}$$

Taking account of (37) $\sim$ (42), we see that (ϕ', ξ, η, g) is an almost contact metric structure on M_r. Further, we may note that the almost contact metric manifold $(M_0, \phi', \xi, \eta, g)$ coincides with the almost contact metric manifold introduced in ([3], p. 64) which is different from the naturally induced cone from the nearly Kähler structure $(J, \bar{g})$ on S^6 with respect to the unit normal vector field ν. Later, we shall show that (ϕ', ξ, η, g) is a contact metric structure on the hypersurface M_0.

For any $X \in \mathfrak{X}(M_r)$, we set

$$Y = X - \eta(X)\xi. \tag{43}$$

Then, $Y \in \mathfrak{X}(M_r)$ and $Y \perp \xi$. From (16), (37) and (38), we have

$$\begin{aligned} \phi' Y &= -\nu \times Y = -\nu \times (X - \eta(X)\xi) \\ &= -\nu \times X + \eta(X)(\nu \times \xi) = -\nu X + \eta(X)\nu\xi \\ &= -\nu X - \eta(X)\nu(N\nu) = -\nu X + \eta(X)\nu^2 N \\ &= -\nu X - \eta(X) N \end{aligned} \tag{44}$$

for any $X \in \mathfrak{X}(M_r)$. Comparing (27) and (44), we see that $\phi X \neq \phi' X$ for $X \in \mathfrak{X}(M_r)$ with $X \perp \xi$. It is known that the almost contact metric structure (ϕ', ξ, η, g) on the hypersphere M_0 in the nearly Kähler 6-sphere S^6 is a contact metric structure by ([3], p. 64). Here, we shall provide an exact proof for this fact, now, we choose a point $x = \sum_{i=1}^{6} x_i e_i \in M_0$ arbitrary and fix it. Here, for our purpose without also discuss in the case where $x_i \neq 0$, now, we set

$$Y_{1,b} = X_{1,b} - \eta(X_{1,b})\xi \quad (1 < b \leq 6) \tag{45}$$

for any $X \in \mathfrak{X}(M_r)$. Then, from (45), taking account of (10), (17) with $r = 0$, (18) and Fig. 1, we have

$$\begin{aligned} Y_{1,2} = {} & (-x_2 + x_2x_6^2 - x_1x_5x_6)\partial_1 + (x_1 + x_2x_5x_6 - x_1x_5^2)\partial_2 \\ & + (x_2x_4x_6 - x_1x_4x_5)\partial_3 + (-x_2x_3x_6 + x_1x_3x_5)\partial_4 \\ & + (-x_2^2x_6 + x_1x_2x_5)\partial_5 + (-x_1x_2x_6 + x_1^2x_5)\partial_6, \end{aligned} \tag{46}$$

$$\begin{aligned} Y_{1,3} = {} & (-x_3 + x_3x_6^2 - x_1x_4x_6)\partial_1 + (x_3x_5x_6 - x_1x_4x_5)\partial_2 \\ & + (x_1 + x_3x_4x_6 - x_1x_4^2)\partial_3 + (-x_3^2x_6 + x_1x_3x_4)\partial_4 \\ & + (-x_2x_3x_6 + x_1x_2x_4)\partial_5 + (-x_1x_3x_6 + x_1^2x_4)\partial_6, \end{aligned}$$

$$\begin{aligned} Y_{1,4} = {} & (-x_4 + x_4x_6^2 + x_1x_3x_6)\partial_1 + (x_4x_5x_6 + x_1x_3x_5)\partial_2 \\ & + (x_1x_3x_4 + x_4^2x_6)\partial_3 + (x_1 - x_3x_4x_6 - x_1x_3^2)\partial_4 \\ & + (-x_2x_4x_6 - x_1x_2x_3)\partial_5 + (-x_1x_4x_6 - x_1^2x_3)\partial_6, \end{aligned}$$

$$\begin{aligned} Y_{1,5} = {} & (-x_5 + x_5x_6^2 + x_1x_2x_6)\partial_1 + (x_5^2x_6 + x_1x_2x_5)\partial_2 \\ & + (x_4x_5x_6 + x_1x_2x_4)\partial_3 + (-x_3x_5x_6 - x_1x_2x_3)\partial_4 \\ & + (x_1 - x_2x_5x_6 - x_1x_2^2)\partial_5 + (-x_1x_5x_6 - x_1^2x_2)\partial_6, \end{aligned}$$

$$\begin{aligned} Y_{1,6} = {} & (-x_6 + x_1^2x_6 + x_6^3)\partial_1 + (x_1^2x_5 + x_6^2x_5)\partial_2 \\ & + (x_1^2x_4 + x_6^2x_4)\partial_3 + (-x_1^2x_3 - x_6^2x_3)\partial_4 \\ & + (-x_1^2x_2 - x_6^2x_2)\partial_5 + (x_1 - x_1^3 - x_1x_6^2)\partial_6. \end{aligned}$$

Thus, from (14) with $r = 0$, (44) and (46), taking account of Fig. 1, we have

$$\begin{aligned} \phi' Y_{1,3} = {} & (x_1x_3x_6 - x_1^2x_4)\partial_1 + (x_2x_3x_6 - x_1x_2x_4)\partial_2 \\ & + (x_3^2x_6 - x_1x_3x_4)\partial_3 + (x_1 + x_3x_4x_6 - x_1x_4^2)\partial_4 \\ & + (x_3x_5x_6 - x_1x_4x_5)\partial_5 + (-x_3 + x_3x_6^2 - x_1x_4x_6)\partial_6, \end{aligned} \tag{47}$$

$$\begin{aligned} \phi' Y_{1,4} = {} & (x_1x_4x_6 + x_1^2x_3)\partial_1 + (x_2x_4x_6 + x_1x_2x_3)\partial_2 \\ & + (-x_1 + x_3x_4x_6 + x_1x_3^2)\partial_3 + (x_4^2x_6 + x_1x_3x_4)\partial_4 \\ & + (x_4x_5x_6 + x_1x_3x_5)\partial_5 + (-x_4 + x_4x_6^2 + x_1x_3x_6)\partial_6, \end{aligned}$$

$$\begin{aligned} \phi' Y_{1,5} = {} & (x_1x_5x_6 + x_1^2x_2)\partial_1 + (-x_1 + x_2x_5x_6 + x_1x_2^2)\partial_2 \\ & + (x_3x_5x_6 + x_1x_2x_3)\partial_3 + (x_4x_5x_6 + x_1x_2x_4)\partial_4 \\ & + (x_5^2x_6 + x_1x_2x_5)\partial_5 + (-x_5 + x_5x_6^2 + x_1x_2x_6)\partial_6, \end{aligned}$$

$$\begin{aligned}\phi' Y_{1,6} = (-x_1 + x_1^3 + x_6^2 x_1)\partial_1 + (x_1^2 x_2 + x_6^2 x_2)\partial_2 \\ + (x_1^2 x_3 + x_6^2 x_3)\partial_3 + (x_1^2 x_4 + x_6^2 x_4)\partial_4 \\ + (x_1^2 x_5 + x_6^2 x_5)\partial_5 + (-x_6 + x_1^2 x_6 + x_6^3)\partial_6.\end{aligned}$$

Thus, from (17), (20) and (46), we have

$$\begin{aligned}&d\eta(Y_{1,2}, Y_{1,3}) = 0, \quad d\eta(Y_{1,2}, Y_{1,4}) = 0, \\ &d\eta(Y_{1,2}, Y_{1,5}) = -x_1^2, \quad d\eta(Y_{1,2}, Y_{1,6}) = x_1 x_2, \quad d\eta(Y_{1,b}, \xi) = 0\end{aligned} \tag{48}$$

for any b $(1 < b \leq 6)$. Similarly, from (37), (46) and (47), we have

$$\begin{aligned}&g(Y_{1,2}, \phi' Y_{1,3}) = 0, \quad g(Y_{1,2}, \phi' Y_{1,4}) = 0, \\ &g(Y_{1,2}, \phi' Y_{1,5}) = -x_1^2, \quad g(Y_{1,2}, \phi' Y_{1,6}) = x_1 x_2, \quad g(Y_{1,b}, \phi' \xi) = 0\end{aligned} \tag{49}$$

for any b $(1 < b \leq 6)$. Therefore, from (3), (48) and (49), we can see that $(M_0, \phi', \xi, \eta, g)$ is a contact metric manifold. This support both Theorem B and Theorem C, since the contact metric structure (ϕ', ξ, η, g) is different from the naturally induced one (ϕ, ξ, η, g) on M_0. On the other hand, from (26) with $r = 0$, $(M_0, \phi', \xi, \eta, g)$ is a space of constant sectional curvature 1. Therefore, from the fact ([3], Theorem 7.3), we see finally that $(M_0, \phi', \xi, \eta, g)$ is a Sasakian manifold. Further, taking account of (5), we may check that, for each $r(-1 < r < 1)$, the map $F_r : M_r \to M_0$ defined by $F_r(\sum_{i=1}^6 x_i e_i + r e_7) = \frac{1}{\sqrt{1-r^2}}(\sum_{i=1}^6 x_i e_i)$, $(\sum_{i=1}^6 {x_i}^2 = 1 - r^2)$ on M_0 is a diffeomorphism from M_r to M_0. Thus, the pullback of the Sasakian structure $({\phi'}_0, \xi_0, \eta_0, g_0)$ on M_0 to M_r by the diffeomorphism F_r is also a Sasakian structure on M_r of constant sectional curvature 1 for each $r(-1 < r < 1)$. We here note that the pullback Sasakian structure is given by $\bar{\phi} = ({F_r}^{-1})_* \circ \phi_0 \circ (F_r)_*$, $\bar{\xi} = ({F_r}^{-1})_* \xi_0 = \xi$, $\bar{\eta} = {F_r}^*(\eta_0) = \eta$, $\bar{g} = {F_r}^*(g_0) = g$. On the other hand, by modifying the above arguments suitably, we may also check that $(M_r, \phi', \xi, \eta, g)$ is not a contact metric manifold for any r with $(-1 < r < 1, r \neq 0)$.

Remark 1. Let M be a hypersurface in the nearly Kähler unit 6-sphere $(S^6, J, \bar{g})$ oriented by unit normal vector field ν and (ϕ, ξ, η, g) be the corresponding naturally induced almost contact metric structure on M. Now, let G be the $(1,2)$-tensor field on $(S^6, J, \bar{g})$ given by $G(\bar{X}, \bar{Y}) = (\bar{\nabla}_{\bar{X}} J)(\bar{Y})$ for any $\bar{X}, \bar{Y} \in \mathfrak{X}(S^6)$, and ψ be the $(1,1)$-tensor field on M defined by $\psi X = G(X, \nu)$ for any $X \in \mathfrak{X}(M)$ [5]. Here, specifying (M, ν) as the hypersurface (M_0, ν) introduced in §3, we can show that $\psi = \phi'$ holds for M_0 by making use of the discussions in [5, 8].

Remark 2. J. Berndt, J. Bolton and L. M. Woodward have proved that a Hopf hypersurface in the nearly Kähler 6-sphere S^6 is either an open part of (i) a geodesic hypersphere of S^6 or (ii) a tube around an almost complex curve in S^6([2], Theorem 2). Taking account of this result, it seems also meaningful to discuss the Hopf hypersurfaces of type (ii) in S^6 from the geometry of almost contact metric structures viewpoint.

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