J. Korean Math. Soc. **55** (2018), No. 4, pp. 939–962
https://doi.org/10.4134/JKMS.j170540
pISSN: 0304-9914 / eISSN: 2234-3008

BOUNDEDNESS OF THE COMMUTATOR OF THE INTRINSIC SQUARE FUNCTION IN VARIABLE EXPONENT SPACES

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Abstract. In this paper, we show that the commutator of the intrinsic square function with BMO symbols is bounded on the variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ applying a generalization of the classical Rubio de Francia extrapolation. As a consequence we further establish its boundedness on the variable exponent Morrey spaces $\mathcal{M}_{p(\cdot),u}$, Morrey-Herz spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ and Herz type Hardy spaces $H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)$, where the exponents $\alpha(\cdot)$ and $p(\cdot)$ are variable. Observe that, even when $\alpha(\cdot)\equiv\alpha$ is constant, the corresponding main results are completely new.

1. Introduction

We begin with the definition of intrinsic square function S_β from Wilson [38, Page 773]. For $0<\beta\le 1$, let $\mathcal{C}_\beta$ be the family of functions $\phi:\mathbb{R}^n\to\mathbb{R}$ such that ϕ's support is contained in $\{x\in\mathbb{R}^n:|x|\le 1\}$, $\int_{\mathbb{R}^n}\phi(x)dx=0$ and for all $x_1,x_2\in\mathbb{R}^n$,

$$|\phi(x_1)-\phi(x_2)|\le|x_1-x_2|^\beta. \tag{1}$$

For $(y,t)\in\mathbb{R}^{n+1}_+=\mathbb{R}^n\times(0,\infty)$ and $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, set

$$A_\beta(f)(y,t)=\sup_{\phi\in\mathcal{C}_\beta}|f*\phi_t(y)|=\sup_{\phi\in\mathcal{C}_\beta}\left|\int_{\mathbb{R}^n}\phi_t(y-z)f(z)dz\right|. \tag{2}$$

We define the intrinsic square function of f (of order β) by

$$S_\beta(f)(x)=\left(\int\int_{\Gamma(x)}(A_\beta(f)(y,t))^2\frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}, \tag{3}$$

where $\phi_t(x)=\frac{1}{t^n}\phi(\frac{x}{t})$ and $\Gamma(x)=\{(y,t)\in\mathbb{R}^{n+1}_+:|x-y|<t\}$.

Received August 14, 2017; Accepted February 21, 2018.
2010 *Mathematics Subject Classification.* Primary 46E30; Secondary 42B35.
Key words and phrases. the intrinsic square function, commutator, variable exponents, Morrey spaces, Herz type spaces.
The author was partly supported by National Natural Science Foundation of China (11471033).

The intrinsic square function S_β was first introduced by Wilson in [38] to solve a conjecture raised by Fefferman and Stein [11] on the boundedness of the classical Lusin area function S from the weighted Lebesgue space $L^2_{M(\nu)}(\mathbb{R}^n)$ to $L^2_\nu(\mathbb{R}^n)$, where ν is a non-negative, locally integrable function and M denotes the Hardy-Littlewood maximal operator. Lerner [23] proved sharp $L^p(\omega)$ norm inequalities for the intrinsic square function in terms of the A_p (the class of Muckenhoupt weights) characteristic of ω for all $1 < p < \infty$. Liang and Yang [24] obtained the intrinsic square function characterizations of Musielak-Orlicz Hardy spaces. More applications of such intrinsic square function were also given by Wilson [39] and Wang and Liu [36].

Let B be any ball centered at $x \in \mathbb{R}^n$ and radius of $r > 0$. A locally integrable function b is said to be a BMO function, if it satisfies

$$\|b\|_* := \sup_{x\in\mathbb{R}^n, r>0} |B|^{-1} \int_B |b(y) - b_B|\, dy < \infty,$$

where $b_B := |B|^{-1} \int_B b(t)dt$ and $\|b\|_*$ is the norm in $\mathrm{BMO}(\mathbb{R}^n)$. For $b \in \mathrm{BMO}(\mathbb{R}^n)$, $[b, S_\beta]$, the commutator of the intrinsic square function $S_\beta(f)$, is then defined by

$$[b,S_\beta](f)(x) = \left(\iint_{\Gamma(x)} \sup_{\phi\in\mathcal{C}_\beta} \left| \int_{\mathbb{R}^n} (b(x)-b(z))\phi_t(y-z)f(z)dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}}. \tag{4}$$

Recently, Wang [33] proved that if $\omega \in A_p$ for some $1 < p < \infty$, then the commutator $[b, S_\beta]$ is bounded on the weighted Lebesgue space $L^p(\omega)$, and is also bounded on the weighted Morrey space $L^{p,\kappa}(\omega)$ for $0 < \kappa < 1$. Hu and Wang [19] established its boundedness on the classical weighted Herz spaces. Guliyev et al. [13] considered the intrinsic square function and its commutator on generalized weighted Orlicz-Morrey spaces. There are many other interesting works on this operator, among them we refer to [34] and references therein.

On the other hand, following the fundamental work of Kováčik and Rákosník [22], the variable exponent Lebesgue and some other function spaces arising in analysis such as Besov spaces and Triebel-Lizorkin spaces etc. have been intensively studied in the recent years, see [4, 6, 29, 31, 35, 42, 43] and their references. These spaces are of interest in their own right, and also have applications to fluid dynamics [30], image restoration [2] and PDE with non-standard growth conditions [15].

In many applications, a crucial step has been to prove that the classical operators, such as maximal operators, singular integrals and their commutators, are bounded in variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$. As shown in [5], one of the important methods used for extending the boundedness results from the weighted Lebesgue spaces $L^p(\omega)$ to variable $L^{p(\cdot)}(\mathbb{R}^n)$ spaces is the method of Rudio de Francia extrapolation. In Section 2 we show that the commutator $[b, S_\beta]$ is bounded on variable $L^{p(\cdot)}(\mathbb{R}^n)$ spaces applying a generalization of the classical Rubio de Francia extrapolation, which generalizes the corresponding statement in [33, Theorem 3.1] to the variable exponent case.

The variable exponent Morrey spaces $\mathcal{M}_{p(\cdot),u}$ are introduced and studied in Ho [16]. Simultaneously, he has given some sufficient conditions on u for the boundedness of fractional integrals and fractional maximal operator on such spaces. In 2017, Ho [18] established the boundedness of the vector-valued intrinsic square function on variable Morrey spaces $\mathcal{M}_{p(\cdot),u}$, where $u \in \mathbb{W}_{p(\cdot)}$ (see Definition 9 below) is a Morrey weight function for $L^{p(\cdot)}(\mathbb{R}^n)$. Hence, it is natural to ask whether the variable exponent Morrey spaces estimates for the commutator $[b, S_\beta]$ are still true if $b \in \mathrm{BMO}(\mathbb{R}^n)$? The main result in Section 3 is to give an affirmative answer to this question.

The variable exponent Herz spaces $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ and Morrey-Herz spaces $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ were recently studied by Izuki [20,21], and he also obtained some basic lemmas on generalization of the BMO norms. Under natural regularity assumptions on the exponent α and p, either at the origin or at infinity, Almeida and Direhem in [1] established the boundedness of a wide class of sublinear operators including maximal, potential and Calderón-Zygmund operators on the variable exponent Herz spaces $\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)$, where the exponent α is variable as well. Lu and Zhu [27] further generalized some results in [1] and obtained the boundedness for such sublinear operators and their commutators on the Morrey-Herz spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ with variable exponent $\alpha(\cdot)$ and $p(\cdot)$. Motivated by the above results, in Section 4 we consider the boundedness properties of the commutator $[b, S_\beta]$ on the variable Morrey-Herz spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$. It is worth pointing out that even in the particular case $\alpha(\cdot) \equiv \alpha$ is constant, the main results are also new.

There are several versions of Herz-type Hardy spaces with variable exponents, see [9, 10, 37]. The variable Herz type Hardy spaces $H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)$, as well as their atomic decomposition characterizations, have been studied by Dong and Xu in [9]. Using these decompositions, they proved some boundedness results for singular integral operators on these spaces. Following the idea of [9], in Section 5 we further obtain the boundedness of the commutator $[b, S_\beta]$ from the variable Herz type Hardy spaces $H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)$ to the variable Herz spaces $\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)$.

In general, we define $B := B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$. f_B denotes the integral average of f on B, namely, $f_B = |B|^{-1} \int_B f(x)dx$. $p'(\cdot)$ means the conjugate exponent defined by $1/p(\cdot) + 1/p'(\cdot) = 1$. By $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ we denote the Schwartz class and the space of tempered distributions, respectively. For $x \in \mathbb{R}$, we denote by $[x]$ the largest integer less than or equal to x. The symbol C stands for a positive constant, which may vary from line to line. The expression $f \lesssim g$ means that $f \le Cg$, and $f \approx g$ means $f \lesssim g \lesssim f$.

2. Preliminaries and lemmas

We first recall some basic lemmas and definitions on the variable exponent Lebesgue spaces, see [3,8] for more information.

Given a measurable set $E \subset \mathbb{R}^n$, and a measurable function $p(\cdot) : E \to [1, \infty)$, let $L^{p(\cdot)}(E)$ denote the space of all measurable functions f on E such that

$$I_{p(\cdot)}(f) := \int_E |f(x)|^{p(x)} dx < \infty.$$

This is a Banach space with respect to the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(E)} = \inf\{\mu > 0 : I_{p(\cdot)}(f/\mu) \leq 1\}.$$

The space $L^{p(\cdot)}_{\text{loc}}(E)$ is defined by

$$L^{p(\cdot)}_{\text{loc}}(E) = \{f : f \in L^{p(\cdot)}(K) \text{ for all compact subsets } K \subset E\}.$$

For brevity, we denote

$$p_E^- := \text{ess inf}_{x \in E} p(x),\ p_E^+ := \text{ess sup}_{x \in E} p(x),\ p_- := p^-_{\mathbb{R}^n},\ p_+ := p^+_{\mathbb{R}^n}.$$

Using this notation, we define

$$\mathcal{P}(E) := \{p(\cdot) : E \to [1, \infty) : p_E^- > 1 \text{ and } p_E^+ < \infty\}$$

and

$$\mathcal{B}(E) := \{p(\cdot) \in \mathcal{P}(E) : M \text{ is bounded on } L^{p(\cdot)}(E)\},$$

where M is the Hardy-Littlewood maximal operator defined by

$$Mf(x) = \sup_{r>0} r^{-n} \int_{B(x,r)\cap E} |f(y)| dy.$$

When $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, the generalized Hölder inequality holds in the form

$$\int_{\mathbb{R}^n} |f(x)g(x)| dx \leq r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}, \tag{5}$$

with $r_p = 1 + 1/p_- - 1/p_+$, see [22, Theorem 2.1].

Diening's characterization of variable $L^{p(\cdot)}(\mathbb{R}^n)$ spaces on which the maximal operator is bounded has the following important consequence, see [7, Theorem 8.1].

Lemma 1. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then the following conditions are equivalent*:
(a) $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$.
(b) $p'(\cdot) \in \mathcal{B}(\mathbb{R}^n)$.
(c) $p(\cdot)/q \in \mathcal{B}(\mathbb{R}^n)$ *for some* $1 < q < p_-$.
(d) $(p(\cdot)/q)' \in \mathcal{B}(\mathbb{R}^n)$ *for some* $1 < q < p_-$.

Let ω be a weight function on $\mathbb{R}^n$, that is, ω is real-valued, non-negative and locally integrable. For $1 < p < \infty$, we say that ω is an A_p weight if

$$\sup_B \left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B \omega(x)^{1-p'}dx\right)^{p-1} < \infty.$$

The following is a direct generalization of the classical Rubio de Francia extrapolation theorem, see [5, Corollary 1.11].

Lemma 2. *Given a family $\mathcal{F}$ and an open set $\Omega \subset \mathbb{R}^n$, assume that for some $1 < p_0 < \infty$, and for every $\omega \in A_{p_0}$,*

$$\int_\Omega f(x)^{p_0}\omega(x)dx \le C_0 \int_\Omega g(x)^{p_0}\omega(x)dx, \quad (f,g) \in \mathcal{F}.$$

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ be such that there exists $1 < p_1 < p_-$ with $(p(\cdot)/p_1)' \in \mathcal{B}(\Omega)$. Then for all $(f,g) \in \mathcal{F}$ such that $f \in L^{p(\cdot)}(\Omega)$

$$\|f\|_{L^{p(\cdot)}(\Omega)} \le C\|g\|_{L^{p(\cdot)}(\Omega)}.$$

In [33, Theorem 3.1] it was shown that for all $1 < p < \infty$ and all $\omega \in A_p$,

$$\int_{\mathbb{R}^n} |[b, S_\beta]f(x)|^p \omega(x)dx \le C\int_{\mathbb{R}^n} |f(x)|^p\omega(x)dx.$$

Thus, we can apply Lemmas 1 and 2 to get the following.

Theorem 3. *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then we have for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$,*

$$\|[b, S_\beta]f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

The following Lemmas 4, 5 and 7 are due to Izuki [20] (see also Diening [7]), and Lemma 8 comes from Almeida and Drihem [1, Lemma 2.1].

Lemma 4. *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then we have*

$$|B|^{-1}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \le C.$$

Lemma 5. *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then we have for all measurable subsets $E \subset B$,*

$$\frac{\|\chi_E\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|E|}{|B|}\right)^{\delta_1}, \quad \frac{\|\chi_E\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|E|}{|B|}\right)^{\delta_2},$$

where δ_1, δ_2 are constants with $0 < \delta_1, \delta_2 < 1$.

Remark 6. We would like to stress that everywhere below, the constants δ_1 and δ_2 are always the same as in Lemma 5.

Lemma 7. *If $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $b \in \mathrm{BMO}(\mathbb{R}^n)$, $k > j$ $(k, j \in \mathbb{N})$, then we have*

$$\sup_{B\subset\mathbb{R}^n} \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\|(b - b_B)\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx \|b\|_*,$$

$$\|(b - b_{B_j})\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C(k-j)\|b\|_*\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where $B_k = B(0, 2^k)$ and χ_{B_k} is the characteristic function of B_k for $k \in \mathbb{Z}$.

A function $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is called log-Hölder continuous at the origin (or has a log decay at the origin), if there exists a constant $C_{\log} > 0$ such that

$$|\alpha(x) - \alpha(0)| \leq \frac{C_{\log}}{\log(e + 1/|x|)}, \quad x \in \mathbb{R}^n.$$

If, for some $\alpha_\infty \in \mathbb{R}$ and $C_{\log} > 0$, there holds

$$|\alpha(x) - \alpha_\infty| \leq \frac{C_{\log}}{\log(e + |x|)}, \quad x \in \mathbb{R}^n,$$

then $\alpha(\cdot)$ is called log-Hölder continuous at infinity (or has a log decay at the infinity).

Lemma 8. *Let $r_1, r_2 > 0$ and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ be log-Hölder continuous both at origin and at infinity. Then we have*

$$r_1^{\alpha(x)} \leq C r_2^{\alpha(y)} \times \begin{cases} \left(\frac{r_1}{r_2}\right)^{\alpha_+}, & 0 < r_2 \leq r_1/2, \\ 1, & r_1/2 < r_2 \leq 2r_1, \\ \left(\frac{r_1}{r_2}\right)^{\alpha_-}, & r_2 > 2r_1, \end{cases}$$

for any $x \in B(0, r_1)\backslash B(0, r_1/2)$ and $y \in B(0, r_2)\backslash B(0, r_2/2)$.

3. Boundedness on variable exponent Morrey spaces

We first recall the following definitions given by Ho in [16].

Definition 9. Let $p(\cdot) : \mathbb{R}^n \to (1, \infty)$ with $1 < p_- \leq p_+ < \infty$. A Lebesgue measurable function $u(x, r) : \mathbb{R}_+^{n+1} \to \mathbb{R}_+$ is said to be a Morrey weight function for $L^{p(\cdot)}(\mathbb{R}^n)$ if, for any $x \in \mathbb{R}^n$ and $r > 0$,

$$\sum_{j=0}^{\infty} \frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} u(x, 2^{j+1}r) < Cu(x, r). \tag{6}$$

We denote the class of Morrey weight functions by $\mathbb{W}_{p(\cdot)}$.

Definition 10. Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $u \in \mathbb{W}_{p(\cdot)}$. The variable exponent Morrey space $\mathcal{M}_{p(\cdot),u}$ is the collection of all Lebesgue measurable functions f satisfying

$$\|f\|_{\mathcal{M}_{p(\cdot),u}} = \sup_{x \in \mathbb{R}^n, R > 0} \frac{1}{u(x, R)} \|f\chi_{B(x,R)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty.$$

We note that condition (6) is introduced by Ho in [16] to generalize the well known Hardy-Littlewood-Sobolev theorem to the case of variable exponent Morrey spaces on unbounded domains, and is also used to show the Fefferman-Stein vector-valued maximal inequalities for weighted Morrey spaces, see [17].

For any $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, let $\mathcal{K}_{p(\cdot)}$ denote the supremum of those $q > 1$ such that $p(\cdot)/q \in \mathcal{B}(\mathbb{R}^n)$ and $\mathcal{E}_{p(\cdot)}$ be the conjugate of $\mathcal{K}_{p'(\cdot)}$. The following result can be seen as a special case of the general result in [17] for Banach function spaces.

Proposition 11. *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. For any $1 < q < \mathcal{K}_{p(\cdot)}$ and $1 < \tau < \mathcal{K}_{p'(\cdot)}$, there exist constants $C_1, C_2 > 0$ such that for any $x \in \mathbb{R}^n$ and $r > 0$,*

$$C_1 2^{jn(1-\frac{1}{\tau})} \leq \frac{\|\chi_{B(x,2^j r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C_2 2^{\frac{jn}{q}}, \quad \forall j \in \mathbb{N}.$$

Now, let us state the main result in this section.

Theorem 12. *Suppose $0 < \beta \leq 1$ and $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. If there exists a constant $C > 0$ such that for any $x \in \mathbb{R}^n$ and $r > 0$, u fulfills*

$$\sum_{j=0}^{\infty} (j+1) \frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} u(x, 2^{j+1}r) < C u(x,r), \tag{7}$$

then we have for all $f \in \mathcal{M}_{p(\cdot),u}$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$,

$$\|[b, S_\beta](f)\|_{\mathcal{M}_{p(\cdot),u}} \leq C \|b\|_* \|f\|_{\mathcal{M}_{p(\cdot),u}}.$$

Remark 13. Note that if $u \equiv 1$, then the variable exponent Morrey spaces $\mathcal{M}_{p(\cdot),u}$ reduce to the variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$. Therefore, the above theorem is a generalization of Theorem 3.

Remark 14. Condition (7) is satisfied by a number of Lebesgue measurable functions. For instance, if for any $0 \leq \gamma < 1/\mathcal{E}_{p(\cdot)}$, a weight function u satisfies $u(x, 2r) \leq 2^{n\gamma} u(x,r)$ for any $x \in \mathbb{R}^n$ and $r > 0$, then (7) holds. In fact, for any $\gamma < 1/\mathcal{E}_{p(\cdot)}$, there always exists a $\tau < 1/\mathcal{K}_{p'(\cdot)}$ such that $\gamma < 1 - 1/\tau < 1 - 1/\mathcal{K}_{p'(\cdot)} = 1/\mathcal{E}_{p(\cdot)}$. An application of Proposition 11 gives

$$\sum_{j=0}^{\infty} (j+1) \frac{\|\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \frac{u(x, 2^{j+1}r)}{u(x,r)} < C \sum_{j=0}^{\infty} (j+1) 2^{jn(\frac{1}{\tau}+\gamma-1)} < C.$$

To be more precise, the weight function $u(x,r) = r^{\gamma(x)}$, $0 \leq \gamma(x) \leq \gamma_+ < 1/\mathcal{E}_{p(\cdot)}$, satisfies condition (7). In addition, it is easy to check that condition (7) together with Proposition 11 yields $u(x, 2r) \leq C u(x,r)$ for any $x \in \mathbb{R}^n$ and $r > 0$.

Proof of Theorem 12. Let $f \in \mathcal{M}_{p(\cdot),u}$. For any $x_0 \in \mathbb{R}^n$ and $r > 0$, we decompose $f = g + h$, where $g = f\chi_{B(x_0,2r)}$ and $h = \sum_{j=1}^{\infty} f\chi_{B(x_0,2^{j+1}r)\setminus B(x_0,2^j r)}$. Then we have

$$\begin{aligned}
&\frac{1}{u(x_0,r)} \|\chi_{B(x_0,r)} [b, S_\beta](f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\leq \frac{1}{u(x_0,r)} \|\chi_{B(x_0,r)} [b, S_\beta](g)\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \frac{1}{u(x_0,r)} \|\chi_{B(x_0,r)} [b, S_\beta](h)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&:= U + V.
\end{aligned}$$

For U, using $u(x_0, 2r) \leq C u(x_0, r)$ and the $L^{p(\cdot)}(\mathbb{R}^n)$-boundedness of $[b, S_\beta]$, we obtain

$$U \leq C \|b\|_* \frac{1}{u(x_0, 2r)} \|f\chi_{B(x_0,2r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

$$\leq C\|b\|_* \sup_{x_0\in\mathbb{R}^n,R>0} \frac{1}{u(x_0,R)} \|f\chi_{B(x_0,R)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$
$$\leq C\|b\|_*\|f\|_{\mathcal{M}_{p(\cdot),u}}.$$

We now turn to estimate V. For any $0<\beta\leq 1$, $\phi\in\mathcal{C}_\beta$ and $(y,t)\in\Gamma(x)$, we have

$$\left|\int_{\mathbb{R}^n}(b(x)-b(z))\phi_t(y-z)h(z)dz\right| \leq Ct^{-n}\sum_{j=1}^{\infty}\int_{\widetilde{R_j}\bigcap\{z:|y-z|\leq t\}}|b(x)-b(z)||f(z)|dz, \tag{8}$$

where $\widetilde{R_j}:=B(x_0,2^{j+1}r)\setminus B(x_0,2^jr)$.

Note that if $x\in B(x_0,r)$, $(y,t)\in\Gamma(x)$ and $z\in\widetilde{R_j}\bigcap\{z:|y-z|\leq t\}$, then

$$t\geq\frac{1}{2}(|x-y|+|y-z|)\geq\frac{1}{2}|x-z|\geq\frac{1}{2}(|z-x_0|-|x-x_0|)>2^{j-2}r. \tag{9}$$

Thus, from (8), (9) and the Minkowski inequality, we get

$$\begin{aligned}
&|[b,S_\beta](h)(x)|\\
&=\left(\int\int_{\Gamma(x)}\sup_{\phi\in\mathcal{C}_\beta}\left|\int_{\mathbb{R}^n}(b(x)-b(z))\phi_t(y-z)h(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}\\
&\leq C\left(\int_{2^{j-2}r}^{\infty}\int_{|x-y|<t}\left|t^{-n}\sum_{j=1}^{\infty}\int_{\widetilde{R_j}\bigcap\{z:|y-z|\leq t\}}|b(x)-b(z)||f(z)|dz\right|^2\frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}\\
&\leq C\sum_{j=1}^{\infty}\frac{1}{(2^{j+1}r)^n}\int_{B(x_0,2^{j+1}r)}|b(x)-b(z)||f(z)|dz\\
&\leq C\sum_{j=1}^{\infty}\frac{1}{(2^{j+1}r)^n}|b(x)-b_{B(x_0,r)}|\int_{B(x_0,2^{j+1}r)}|f(z)|dz\\
&\quad+C\sum_{j=1}^{\infty}\frac{1}{(2^{j+1}r)^n}\int_{B(x_0,2^{j+1}r)}|b(z)-b_{B(x_0,r)}||f(z)|dz\\
&:=V_1+V_2.
\end{aligned}$$

For V_1, the generalized Hölder inequality implies that

$$V_1\leq C\sum_{j=1}^{\infty}\frac{1}{(2^{j+1}r)^n}|b(x)-b_{B(x_0,r)}|\|f\chi_{B(x_0,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B(x_0,2^{j+1}r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.$$

For V_2, noting that $|b_{B(x_0,2^{j+1}r)}-b_{B(x_0,r)}|\leq C(j+1)\|b\|_*$ (see [32, Page 206]), we apply Lemma 7 with $B=B(x_0,2^{j+1}r)$ and obtain

$$V_2\leq C\sum_{j=1}^{\infty}\frac{1}{(2^{j+1}r)^n}|b_{B(x_0,2^{j+1}r)}-b_{B(x_0,r)}|\int_{B(x_0,2^{j+1}r)}|f(z)|dz$$

$$
\begin{aligned}
&+ C\sum_{j=1}^{\infty}\frac{1}{(2^{j+1}r)^n}\int_{B(x_0,2^{j+1}r)}|b(z)-b_{B(x_0,2^{j+1}r)}||f(z)|dz\\
&\le C\|b\|_*\sum_{j=1}^{\infty}(j+1)\frac{1}{(2^{j+1}r)^n}\|f\chi_{B(x_0,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B(x_0,2^{j+1}r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\\
&\quad + C\sum_{j=1}^{\infty}\frac{1}{(2^{j+1}r)^n}\|f\chi_{B(x_0,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|(b-b_{B(x_0,2^{j+1}r)})\chi_{B(x_0,2^{j+1}r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\\
&\le C\|b\|_*\sum_{j=1}^{\infty}(j+1)\frac{1}{(2^{j+1}r)^n}\|f\chi_{B(x_0,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B(x_0,2^{j+1}r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

Combining the estimates of V_1 and V_2, by Lemma 4, we get

$$
\begin{aligned}
&\|\chi_{B(x_0,r)}[b,S_\beta](h)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\le C\|b\|_*\sum_{j=1}^{\infty}(j+1)\frac{1}{(2^{j+1}r)^n}\|\chi_{B(x_0,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|f\chi_{B(x_0,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\qquad \|\chi_{B(x_0,2^{j+1}r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\\
&\le C\|b\|_*\sum_{j=1}^{\infty}(j+1)\frac{\|\chi_{B(x_0,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x_0,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}u(x_0,2^{j+1}r)\\
&\quad\times\sup_{x_0\in\mathbb{R}^n,R>0}\frac{1}{u(x_0,R)}\|f\chi_{B(x_0,R)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\le C\|b\|_*\|f\|_{\mathcal{M}_{p(\cdot),u}}\sum_{j=1}^{\infty}(j+1)\frac{\|\chi_{B(x_0,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x_0,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}u(x_0,2^{j+1}r).
\end{aligned}
$$

Thus, in view of the condition (7), we arrive at the estimate

$$
\begin{aligned}
V &= \frac{1}{u(x_0,r)}\|\chi_{B(x_0,r)}[b,S_\beta](h)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\le C\|b\|_*\|f\|_{\mathcal{M}_{p(\cdot),u}}\sum_{j=1}^{\infty}(j+1)\frac{\|\chi_{B(x_0,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(x_0,2^{j+1}r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\frac{u(x_0,2^{j+1}r)}{u(x_0,r)}\\
&\le C\|b\|_*\|f\|_{\mathcal{M}_{p(\cdot),u}}.
\end{aligned}
$$

This completes the proof of Theorem 12. □

4. Boundedness on variable exponent Morrey-Herz spaces

Let $B_l = \{x\in\mathbb{R}^n : |x|\le 2^l\}$, $R_l = B_l\backslash B_{l-1}$ and $\chi_l = \chi_{R_l}$ be the characteristic function of the set R_l for $l\in\mathbb{Z}$.

Definition 15. Let $0<q\le\infty$, $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ and $\alpha(\cdot):\mathbb{R}^n\to\mathbb{R}$ with $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$. The homogeneous Herz space $\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)$ is defined as the

class of all $f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\})$ such that

$$\|f\|_{\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)} := \left(\sum_{k\in\mathbb{Z}} \|2^{k\alpha(\cdot)} f\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/q} < \infty$$

with the usual modification when $q = \infty$.

The next proposition was initially proved by Almeida and Drihem in [1, Page 785].

Proposition 16. *Let $0 < q \le \infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. If $\alpha(\cdot)$ is log-Hölder continuous both at the origin and at infinity, then*

$$\|f\|_{\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)} \approx \left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \|f\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{q}} + \left(\sum_{k=0}^{\infty} 2^{k\alpha_\infty q} \|f\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{q}}.$$

Definition 17. Let $0 \le \lambda < \infty$, $0 < q \le \infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ with $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. The homogeneous Morrey-Herz space $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ is defined as the class of all $f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\})$ such that

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} := \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} \|2^{k\alpha(\cdot)} f\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/q} < \infty,$$

with the usual modification when $q = \infty$.

Remark 18. It obviously follows that $M\dot{K}^{\alpha(\cdot),0}_{q,p(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)$. If $\alpha(\cdot) \equiv \alpha$, $p(\cdot) \equiv p$ are constant, then $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ coincides with the classical Morrey-Herz spaces $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$ defined by Lu and Xu in [25].

Similar to Proposition 16, Lu and Zhu [27] obtained the following result.

Proposition 19. *Let $0 \le \lambda < \infty$, $0 < q \le \infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. If $\alpha(\cdot)$ is log-Hölder continuous both at origin and at infinity, then*

$$\begin{aligned}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} \approx \max\Bigg\{ &\sup_{k_0<0,k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q} \|f\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/q}, \\ &\sup_{k_0\ge 0,k_0\in\mathbb{Z}} \Bigg[2^{-k_0\lambda} \left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \|f\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/q} \\ &+ 2^{-k_0\lambda} \left(\sum_{k=0}^{k_0} 2^{k\alpha_\infty q} \|f\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/q}\Bigg]\Bigg\}.\end{aligned}$$

The main results in this section can be summarized as follows.

Theorem 20. *Suppose $0 < \beta \le 1$ and $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Let $\lambda > 0$, $0 < q \le \infty$ and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ be log-Hölder continuous both at the origin and at infinity, such that*

$$\lambda - n\delta_1 < \alpha_- \le \alpha_+ < n\delta_2,$$

where $0 < \delta_1, \delta_2 < 1$ *are the constants appearing in Lemma 5. Then we have for all* $f \in M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ *and* $b \in \mathrm{BMO}(\mathbb{R}^n)$,

$$\|[b, S_\beta]f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} \le C\|b\|_*\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.$$

In fact, Theorem 20 remains valid for $\lambda = 0$, namely, in the framework of variable exponent Herz spaces. Using the same method of proving Theorem 20, we get the following.

Corollary 21. *Suppose* $0 < \beta \le 1$ *and* $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. *Let* $0 < q \le \infty$ *and* $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ *be log-Hölder continuous both at the origin and at infinity, such that*

$$-n\delta_1 < \alpha_- \le \alpha_+ < n\delta_2,$$

where $0 < \delta_1, \delta_2 < 1$ *are the constants appearing in Lemma 5. Then we have for all* $f \in \dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)$ *and* $b \in \mathrm{BMO}(\mathbb{R}^n)$,

$$\|[b, S_\beta]f\|_{\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)} \le C\|b\|_*\|f\|_{\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)}.$$

Proof of Theorem 20. Without loss of generality, we may assume that $\|b\|_* = 1$. Let $f \in M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$. We decompose

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x).$$

The Minkowski inequality implies that

$$\begin{aligned}
&\|[b, S_\beta](f)\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} \\
&= \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} \left\| 2^{k\alpha(\cdot)} |[b, S_\beta](f)|\chi_k \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\le C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} \left\| 2^{k\alpha(\cdot)} \Bigg(\sum_{j=-\infty}^{k-2} |[b, S_\beta](f_j)| \Bigg) \chi_k \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\quad + C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} \left\| 2^{k\alpha(\cdot)} \Bigg(\sum_{j=k-1}^{k+1} |[b, S_\beta](f_j)| \Bigg) \chi_k \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\quad + C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} \left\| 2^{k\alpha(\cdot)} \Bigg(\sum_{j=k+2}^{\infty} |[b, S_\beta](f_j)| \Bigg) \chi_k \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&:= E + F + G.
\end{aligned}$$

We treat F first. By Proposition 19 and the $L^{p(\cdot)}(\mathbb{R}^n)$-boundedness of the commutator $[b, S_\beta]$, we get

$$F \approx \max\Bigg\{ \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q} \left\| \Bigg(\sum_{j=k-1}^{k+1} |[b, S_\beta](f_j)| \Bigg) \chi_k \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)},$$

$$\sup_{k_0\geq 0,k_0\in\mathbb{Z}}\Bigg[2^{-k_0\lambda q}\sum_{k=-\infty}^{-1}2^{k\alpha(0)q}\Big\|\Big(\sum_{j=k-1}^{k+1}|[b,S_\beta](f_j)|\Big)\chi_k\Big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}$$

$$+2^{-k_0\lambda q}\sum_{k=0}^{k_0}2^{k\alpha_\infty q}\Big\|\Big(\sum_{j=k-1}^{k+1}|[b,S_\beta](f_j)|\Big)\chi_k\Big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Bigg]\Bigg\}$$

$$\leq C\max\Bigg\{\sup_{k_0<0,k_0\in\mathbb{Z}}2^{-k_0\lambda q}\sum_{k=-\infty}^{k_0}\Big\|2^{k\alpha(0)}|f\chi_k|\Big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)},$$

$$\sup_{k_0\geq 0,k_0\in\mathbb{Z}}\Bigg[2^{-k_0\lambda q}\sum_{k=-\infty}^{-1}\Big\|2^{k\alpha(0)}|f\chi_k|\Big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}$$

$$+2^{-k_0\lambda q}\sum_{k=0}^{k_0}\Big\|2^{k\alpha_\infty}|f\chi_k|\Big\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Bigg]\Bigg\}$$

$$\leq C\|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.$$

For E, noticing that $x\in R_k$, $(y,t)\in\Gamma(x)$ and $z\in R_j\bigcap\{z:|y-z|\leq t\}$ with $j\leq k-2$, then we have

$$t\geq\frac{1}{2}(|x-y|+|y-z|)\geq\frac{1}{2}|x-z|\geq\frac{1}{2}(|x|-|z|)\geq\frac{1}{4}|x|. \tag{10}$$

Thus, we obtain

$$\begin{aligned} &|[b,S_\beta](f_j)(x)| \\ &=\Bigg(\int\int_{\Gamma(x)}\sup_{\phi\in\mathcal{C}_\beta}\Big|\int_{\mathbb{R}^n}(b(x)-b(z))\phi_t(y-z)f_j(z)dz\Big|^2\frac{dydt}{t^{n+1}}\Bigg)^{\frac{1}{2}} \\ &\leq C\Bigg(\int_{\frac{|x|}{4}}^{\infty}\int_{|x-y|<t}\Big|t^{-n}\int_{R_j\bigcap\{z:|y-z|\leq t\}}(b(x)-b(z))f_j(z)dz\Big|^2\frac{dydt}{t^{n+1}}\Bigg)^{\frac{1}{2}} \\ &\leq C\Bigg(\int_{R_j}|b(x)-b(z)||f_j(z)|dz\Bigg)\Bigg(\int_{\frac{|x|}{4}}^{\infty}\frac{dt}{t^{2n+1}}\Bigg)^{\frac{1}{2}} \\ &\leq C2^{-kn}\int_{R_j}|b(x)-b(z)||f_j(z)|dz. \end{aligned}\tag{11}$$

This together with Lemma 8 yields

$$\begin{aligned} &2^{k\alpha(x)}\sum_{j=-\infty}^{k-2}|[b,S_\beta](f_j)(x)|\chi_k(x) \\ &\leq C\sum_{j=-\infty}^{k-2}2^{-kn}\int_{R_j}2^{k\alpha(x)}|b(x)-b(z)||f_j(z)|dz\cdot\chi_k(x) \end{aligned}\tag{12}$$

$$\le C \sum_{j=-\infty}^{k-2} 2^{(k-j)\alpha_+ - kn} \int_{R_j} 2^{j\alpha(z)} |b(x) - b(z)||f_j(z)|dz \cdot \chi_k(x)$$
$$\le C \sum_{j=-\infty}^{k-2} 2^{(k-j)\alpha_+ - kn} \bigg(|b(x) - b_{B_j}| \int_{R_j} 2^{j\alpha(z)} |f_j(z)|dz$$
$$+ \int_{R_j} 2^{j\alpha(z)} |b_{B_j} - b(z)||f_j(z)|dz \bigg) \cdot \chi_k(x).$$

Then, from (11), (5), Lemmas 4, 5 and 7, we deduce that

$$\bigg\| 2^{k\alpha(\cdot)} \sum_{j=-\infty}^{k-2} |[b, S_\beta](f_j)| \chi_k \bigg\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$
$$\le C \sum_{j=-\infty}^{k-2} 2^{(k-j)\alpha_+ - kn} \|2^{j\alpha(\cdot)} f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \bigg(\|(b - b_{B_j})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)}$$
$$+ \|(b - b_{B_j})\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \|\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \bigg)$$
$$\le C \sum_{j=-\infty}^{k-2} 2^{(k-j)\alpha_+ - kn} \|2^{j\alpha(\cdot)} f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$
$$\times \bigg((k-j) \|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \bigg)$$
$$\le C \sum_{j=-\infty}^{k-2} (k-j) 2^{(k-j)\alpha_+ - kn} \|2^{j\alpha(\cdot)} f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$
$$\le C \sum_{j=-\infty}^{k-2} (k-j) 2^{(k-j)\alpha_+} \|2^{j\alpha(\cdot)} f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}$$
$$\le C \sum_{j=-\infty}^{k-2} (k-j) 2^{(j-k)(n\delta_2 - \alpha_+)} \|2^{j\alpha(\cdot)} f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Therefore, we arrive at the estimate

$$E \approx \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q} \sum_{k=-\infty}^{k_0} \bigg\| 2^{k\alpha(\cdot)} \bigg(\sum_{j=-\infty}^{k-2} |[b, S_\beta](f_j)| \bigg) \chi_k \bigg\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}$$
$$\le C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda q} \sum_{k=-\infty}^{k_0} \bigg(\sum_{j=-\infty}^{k-2} (k-j) 2^{(j-k)(n\delta_2 - \alpha_+)} \|2^{j\alpha(\cdot)} f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \bigg)^q.$$

Now we can distinguish two cases as follows:

Case 1°: If $0 < q \le 1$, using the well-known inequality

$$\left(\sum_{j=1}^{\infty}\theta_j\right)^q \le \sum_{j=1}^{\infty}\theta_j^q,\ (\theta_j > 0,\ j = 1, 2, \ldots), \tag{13}$$

we obtain

$$\begin{aligned}
E &\le C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} \sum_{j=-\infty}^{k-2} (k-j)^q 2^{(j-k)(n\delta_2-\alpha_+)q} \|2^{j\alpha(\cdot)} f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\le C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{j=-\infty}^{k_0-2} \|2^{j\alpha(\cdot)} f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \sum_{k=j+2}^{k_0} (k-j)^q 2^{(j-k)(n\delta_2-\alpha_+)q} \\
&\le C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{j=-\infty}^{k_0-2} \|2^{j\alpha(\cdot)} f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\le C\|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Case 2°: If $1 < q < \infty$, the Hölder inequality implies that

$$\begin{aligned}
E &\le C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} \left(\sum_{j=-\infty}^{k-2} 2^{(j-k)(n\delta_2-\alpha_+)q/2} \|2^{j\alpha(\cdot)} f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right) \\
&\quad \times \left(\sum_{j=-\infty}^{k-2} (k-j)^{q'} 2^{(j-k)(n\delta_2-\alpha_+)q'/2} \right)^{q/q'} \\
&\le C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{j=-\infty}^{k_0-2} \|2^{j\alpha(\cdot)} f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \sum_{k=j+2}^{k_0} 2^{(j-k)(n\delta_2-\alpha_+)q/2} \\
&\le C\|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

We proceed now to estimate G. Once again by Proposition 19, we have

$$G \approx \max\{H, J\},$$

where

$$\begin{aligned}
H &= \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q} \left\| \left(\sum_{j=k+2}^{\infty} |[b, S_\beta](f_j)| \right) \chi_k \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}, \\
J &= \sup_{k_0\ge 0, k_0\in\mathbb{Z}} \left\{ 2^{-k_0\lambda q} \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \left\| \left(\sum_{j=k+2}^{\infty} |[b, S_\beta](f_j)| \right) \chi_k \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right. \\
&\quad \left. + 2^{-k_0\lambda q} \sum_{k=0}^{k_0} 2^{k\alpha_\infty q} \left\| \left(\sum_{j=k+2}^{\infty} |[b, S_\beta](f_j)| \right) \chi_k \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right\}.
\end{aligned}$$

For H, noticing that $x \in R_k$, $(y,t) \in \Gamma(x)$ and $z \in R_j \bigcap \{z : |y-z| \le t\}$ with $j \ge k+2$, then we have

$$t \ge \frac{1}{2}(|x-y| + |y-z|) \ge \frac{1}{2}|x-z| \ge \frac{1}{2}(|z| - |x|) \ge \frac{1}{4}|z|. \tag{14}$$

Thus, arguing as in (11), we get

$$\begin{aligned} (15)\quad & |[b,S_\beta](f_j)(x)| \\ & \le C\left(\int_{\frac{|z|}{4}}^{\infty} \int_{|x-y|<t} \left| t^{-n} \int_{R_j \bigcap \{z:|y-z|\le t\}} (b(x)-b(z)) f_j(z) dz \right|^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}} \\ & \le C\left(\int_{R_j} |b(x)-b(z)||f_j(z)|dz\right)\left(\int_{\frac{|z|}{4}}^{\infty} \frac{dt}{t^{2n+1}}\right)^{\frac{1}{2}} \\ & \le C2^{-jn} \int_{R_j} |b(x)-b(z)||f_j(z)|dz \\ & \le C2^{-jn} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \Big(|b(x) - b_{B_k}| \|\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)} + \|(b-b_{B_k})\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \Big). \end{aligned}$$

We now obtain from (15), Lemmas 4, 5 and 7 that

$$\begin{aligned} & \|[b,S_\beta](f_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ & \le C2^{-jn} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \Big(\|(b-b_{B_k})\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\ & \quad + \|(b-b_{B_k})\chi_j\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \|\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \Big) \\ & \le C(j-k)2^{-jn} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\ & \le C(j-k) \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \\ & \le C(j-k)2^{(k-j)n\delta_1} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

Consequently, we have

$$\begin{aligned} H &= \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q} \left\| \Big(\sum_{j=k+2}^{\infty} |[b,S_\beta](f_j)| \Big) \chi_k \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \\ & \le C \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q} \Big(\sum_{j=k+2}^{\infty} (j-k) 2^{(k-j)n\delta_1} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \Big)^q. \end{aligned}$$

If $0 < q \le 1$, by the inequality (13), we get

$$H \le C \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q} \sum_{j=k+2}^{k_0-1} (j-k)^q 2^{(k-j)n\delta_1 q} \|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q$$

$$+\sup_{k_0<0,k_0\in\mathbb{Z}} 2^{-k_0\lambda q}\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q}\sum_{j=k_0}^{\infty}(j-k)^q 2^{(k-j)n\delta_1 q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}$$
$$:= H_1 + H_2.$$

For H_1, noting that $n\delta_1+\alpha(0)>n\delta_1+\alpha_->0$, hence we obtain

$$H_1 \le C\sup_{k_0<0,k_0\in\mathbb{Z}} 2^{-k_0\lambda q}\sum_{j=-\infty}^{k_0-1} 2^{j\alpha(0)q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\sum_{k=-\infty}^{j-2}(j-k)^q 2^{(k-j)(n\delta_1+\alpha(0))q}$$
$$\le C\sup_{k_0<0,k_0\in\mathbb{Z}} 2^{-k_0\lambda q}\sum_{j=-\infty}^{k_0-1} 2^{j\alpha(0)q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.$$

For H_2, in view of $\alpha(0)+n\delta_1-\lambda>\alpha_-+n\delta_1-\lambda>0$, we get

$$H_2 \approx \sup_{k_0<0,k_0\in\mathbb{Z}} 2^{-k_0\lambda q}\sum_{k=-\infty}^{k_0}\sum_{j=k_0}^{\infty}(j-k)^q 2^{(k-j)(n\delta_1+\alpha(0))q}2^{j\alpha(0)q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}$$
$$\le C\sup_{k_0<0,k_0\in\mathbb{Z}} 2^{-k_0\lambda q}\sum_{k=-\infty}^{k_0}\sum_{j=k_0}^{\infty}(j-k)^q 2^{(k-j)(n\delta_1+\alpha(0))q}2^{j\lambda q}$$
$$\times 2^{-j\lambda q}\sum_{l=-\infty}^{j} 2^{l\alpha(0)q}\|f_l\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}$$
$$\le C\sup_{k_0<0,k_0\in\mathbb{Z}} 2^{-k_0\lambda q}\sum_{k=-\infty}^{k_0} 2^{k\lambda q}\sum_{j=k_0}^{\infty}(j-k)^q 2^{(k-j)(n\delta_1+\alpha(0)-\lambda)q}\|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}$$
$$\le C\sup_{k_0<0,k_0\in\mathbb{Z}} 2^{-k_0\lambda q}\Big(\sum_{k=-\infty}^{k_0} 2^{k\lambda q}\Big)\|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} \le C\|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.$$

If $1<q<\infty$, we have

$$H \le C\sup_{k_0<0,k_0\in\mathbb{Z}} 2^{-k_0\lambda q}\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q}\Big(\sum_{j=k+2}^{k_0-1}(j-k)2^{(k-j)n\delta_1}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^q$$
$$+\sup_{k_0<0,k_0\in\mathbb{Z}} 2^{-k_0\lambda q}\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q}\Big(\sum_{j=k_0}^{\infty}(j-k)2^{(k-j)n\delta_1}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^q$$
$$:= H_3 + H_4.$$

For H_3, an application of Hölder's inequality gives

$$H_3 \approx \sup_{k_0<0,k_0\in\mathbb{Z}} 2^{-k_0\lambda q}\sum_{k=-\infty}^{k_0}\Big(\sum_{j=k+2}^{k_0-1}(j-k)2^{(k-j)(n\delta_1+\alpha(0))}2^{j\alpha(0)}\|f_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^q$$
$$\le C\sup_{k_0<0,k_0\in\mathbb{Z}} 2^{-k_0\lambda q}\sum_{k=-\infty}^{k_0}\Big(\sum_{j=k+2}^{k_0-1} 2^{(k-j)(n\delta_1+\alpha(0))q/2}2^{j\alpha(0)q}\|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)$$

$$\times \Big(\sum_{j=k+2}^{k_0-1} (j-k)^{q'} 2^{(k-j)(n\delta_1+\alpha(0))q'/2} \Big)^{q/q'}$$

$$\leq C \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{j=-\infty}^{k_0-1} 2^{j\alpha(0)q} \|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \sum_{k=-\infty}^{j-2} 2^{(k-j)(n\delta_1+\alpha(0))q/2}$$

$$\leq C \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{j=-\infty}^{k_0-1} 2^{j\alpha(0)q} \|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.$$

For H_4, as argued for H_2, we conclude that

$$H_4 \leq C \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} \Big(\sum_{j=k_0}^{\infty} 2^{(k-j)(n\delta_1+\alpha(0)+\lambda)q/2} 2^{j\alpha(0)q} \|f_j\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \Big)$$

$$\times \Big(\sum_{j=k_0}^{\infty} (j-k)^{q'} 2^{(k-j)(n\delta_1+\alpha(0)-\lambda)q'/2} \Big)^{q/q'}$$

$$\leq C \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} \Big(\sum_{j=k_0}^{\infty} 2^{(k-j)(n\delta_1+\alpha(0)+\lambda)q/2} 2^{j\lambda q} 2^{-j\lambda q} \sum_{l=-\infty}^{j} 2^{l\alpha(0)q} \|f_l\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \Big)$$

$$\leq C \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \sum_{k=-\infty}^{k_0} 2^{k\lambda q} \Big(\sum_{j=k_0}^{\infty} 2^{(k-j)(n\delta_1+\alpha(0)-\lambda)q/2} \Big) \|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}$$

$$\leq C \sup_{k_0<0, k_0\in\mathbb{Z}} 2^{-k_0\lambda q} \Big(\sum_{k=-\infty}^{k_0} 2^{k\lambda q} \Big) \|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} \leq C \|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)}.$$

We omit the estimation of J since it is essentially similar to that of H. Consequently, the proof of Theorem 20 is complete. □

5. Boundedness on variable exponent Herz-type Hardy spaces

Note that in Corollary 21 for the range of α, we have the restriction $\alpha < n\delta_2$. It is of interest to ask what will happen if $\alpha \geq n\delta_2$. The main aim of this section is to further consider the mapping properties of the commutator $[b, S_\beta]$ in this situation.

Let $G_N f$ be the grand maximal function of f defined by

$$G_N(f)(x) := \sup_{\phi\in\mathcal{A}_N} |\phi^*_\nabla(f)(x)|, \quad \forall x \in \mathbb{R}^n,$$

where

$$\mathcal{A}_N = \{\phi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\alpha|\leq N, |\beta|\leq N} |x^\alpha D^\beta \phi(x)| \leq 1\}, \quad N > n+1,$$

and $\phi^*_\nabla(f)(x) = \sup_{|y-x|<t} |\phi_t * f(y)|$ with $\phi_t(x) = t^{-n}\phi(\frac{x}{t})$.

The grand maximal operator G_N was first introduced by Fefferman and Stein in [12] to study the classical Hardy spaces. For an extensive study of the Hardy spaces and the variable exponent cases, see, for example, [14, 28, 40, 41, 44, 45] and their references.

Definition 22. Let $0 < q \le \infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ and $N > n+1$. The homogeneous Herz-type Hardy spaces $H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : G_N f \in \dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n) \right\}$$

and $\|f\|_{H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)} = \|G_N f\|_{\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)}$.

Definition 23. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ be log-Hölder continuous both at the origin and at infinity. Suppose $b \in L^{p'(\cdot)}_{\text{loc}}(\mathbb{R}^n)$ and non-negative integer $s \ge [\alpha_r - n\delta_2]$, where $n\delta_2 \le \alpha_r < \infty$ and δ_2 is the constant as in Lemma 5. Here $\alpha_r = \alpha(0)$ if $0 < r < 1$ and $\alpha_r = \alpha_\infty$ if $r \ge 1$. A function $a(\cdot)$ is said to be a central $(\alpha(\cdot), p(\cdot), s; b)$-atom if

(1) $\text{supp} a \subset B(0,r) := \{x \in \mathbb{R}^n : |x| < r\}$,
(2) $\|a\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le |B(0,r)|^{-\frac{\alpha(0)}{n}}$, $0 < r < 1$,
(3) $\|a\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le |B(0,r)|^{-\frac{\alpha_\infty}{n}}$, $r \ge 1$,
(4) $\int_{\mathbb{R}^n} a(x) x^\beta dx = \int_{\mathbb{R}^n} a(x) b(x) x^\beta dx = 0$, $|\beta| \le s$.

Obviously, if $p(\cdot) \equiv p$, $\alpha(\cdot) \equiv \alpha$ are constant, then taking $\delta_2 = 1 - \frac{1}{p}$ we can get the classical case, see [26]. Similar to [9, Theorem 2.1], we have the following characterizations of the spaces $H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)$ in terms of central atomic decompositions.

Proposition 24. *Let $0 < q < \infty$, $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ be log-Hölder continuous both at the origin and at infinity. Suppose $n\delta_2 \le \alpha(0), \alpha_\infty < \infty$, where δ_2 is the constant appearing in Lemma 5. Then $f \in H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)$ if and only if*

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k, \ \textit{in the sense of } \mathcal{S}'(\mathbb{R}^n),$$

where each a_k is a central $(\alpha(\cdot), p(\cdot), s; b)$-atom with $\text{supp} a_k \subset B_k$ and $\sum_{k=-\infty}^{\infty} |\lambda_k|^q < \infty$. Moreover,

$$\|f\|_{H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)} \approx \inf \left(\sum_{k=-\infty}^{\infty} |\lambda_k|^q \right)^{\frac{1}{q}},$$

where the infimum is taken over all above decompositions of f.

The main result in this section can be stated as follows.

Theorem 25. *Suppose $0 < \beta \le 1$ and $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Let $0 < q < \infty$ and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ be log-Hölder continuous both at the origin and at infinity. If*

$n\delta_2 \le \alpha(0), \alpha_\infty < n\delta_2 + \beta$, where δ_2 is the constant appearing in Lemma 5. Then we have for all $f \in H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$,

$$\|[b,S_\beta]f\|_{\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)} \le C\|b\|_*\|f\|_{H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)}.$$

Proof of Theorem 25. Without loss of generality, we may assume that $\|b\|_* = 1$. Let $f \in H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)$, by Proposition 24, we have

$$f = \sum_{j=-\infty}^{\infty} \lambda_j a_j, \text{ in the sense of } \mathcal{S}'(\mathbb{R}^n),$$

where $\lambda_j \ge 0$ and each a_j is a central $(\alpha(\cdot), p(\cdot), 0; b)$-atom with $\mathrm{supp} a_j \subset B_j$ and

$$\|f\|_{H\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)} \approx \inf\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^q\right)^{\frac{1}{q}}.$$

Therefore, we get

$$\begin{aligned}
&\|[b,S_\beta](f)\|_{\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbb{R}^n)}\\
\approx &\left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q}\|[b,S_\beta](f)\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{q}}\\
&+\left(\sum_{k=0}^{\infty} 2^{k\alpha_\infty q}\|[b,S_\beta](f)\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{q}}\\
\le & C\left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|\|[b,S_\beta](a_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q\right\}^{\frac{1}{q}}\\
&+C\left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q}\left(\sum_{j=k-1}^{\infty}|\lambda_j|\|[b,S_\beta](a_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q\right\}^{\frac{1}{q}}\\
&+C\left\{\sum_{k=0}^{\infty} 2^{k\alpha_\infty q}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|\|[b,S_\beta](a_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q\right\}^{\frac{1}{q}}\\
&+C\left\{\sum_{k=0}^{\infty} 2^{k\alpha_\infty q}\left(\sum_{j=k-1}^{\infty}|\lambda_j|\|[b,S_\beta](a_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^q\right\}^{\frac{1}{q}}\\
:= & P_1+P_2+P_3+P_4.
\end{aligned}$$

For P_1, noticing that if $x \in R_k$, $(y,t) \in \Gamma(x)$ and $z \in B_j \bigcap \{z : |y-z| \le t\}$ with $j \le k-2$, then we have $t \ge \frac{1}{4}|x|$. Using the vanishing moments of a_j, we deduce that

(16) $\quad |[b,S_\beta](a_j)(x)|$

$$
\begin{aligned}
&= \left(\int \int_{\Gamma(x)} \sup_{\phi \in \mathcal{C}_\beta} \left| \int_{\mathbb{R}^n} (b(x) - b(z)) \phi_t(y-z) a_j(z) dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}} \\
&= \left(\int \int_{\Gamma(x)} \sup_{\phi \in \mathcal{C}_\beta} \left| \int_{\mathbb{R}^n} (b(x) - b(z)) (\phi_t(y-z) - \phi_t(y)) a_j(z) dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}} \\
&\leq C 2^{j\beta} \left(\int_{B_j} |b(x) - b(z)||a_j(z)| dz \right) \left(\int_{\frac{|x|}{4}}^{\infty} \int_{|x-y|<t} \frac{dydt}{t^{3n+1+2\beta}} \right)^{\frac{1}{2}} \\
&\leq C 2^{j\beta} |x|^{-(n+\beta)} \int_{B_j} |b(x) - b(z)||a_j(z)| dz \\
&\leq C 2^{(j-k)\beta - kn} \|a_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\quad \left(|b(x) - b_{B_j}| \|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} + \|(b - b_{B_j}) \chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \right).
\end{aligned}
$$

From (16), Lemmas 4, 5 and 7, it follows that

$$
\begin{aligned}
(17) \qquad & \|[b, S_\beta](a_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\leq C 2^{(j-k)\beta - kn} \|a_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \Big(\|(b - b_{B_j})\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
&\quad + \|(b - b_{B_j})\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \Big) \\
&\leq C(k-j) 2^{(j-k)\beta - kn} \|a_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \\
&\leq C(k-j) 2^{(j-k)\beta} \|a_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \\
&\leq C(k-j) 2^{(j-k)(\beta + n\delta_2)} \|a_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

For convenience below we put $\xi = \beta + n\delta_2 - \alpha(0) > 0$ and $\eta = \beta + n\delta_2 - \alpha_\infty > 0$. If $0 < q \leq 1$, noting that $\|a_j\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq 2^{-j\alpha(0)}$ with $j \leq k-2$ and $k < 0$, by (17) and $n\delta_2 \leq \alpha(0) < n\delta_2 + \beta$, then we get

$$
\begin{aligned}
P_1^q &= C \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| \|[b, S_\beta](a_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^q \\
&\leq C \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| (k-j) 2^{(j-k)(\beta + n\delta_2) - j\alpha(0)} \right)^q \\
&\leq C \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \left(\sum_{j=-\infty}^{k-2} |\lambda_j|^q (k-j)^q 2^{(j-k)(\beta + n\delta_2)q - j\alpha(0)q} \right) \\
&= C \sum_{j=-\infty}^{-3} |\lambda_j|^q \sum_{k=j+2}^{-1} (k-j)^q 2^{(j-k)\xi q} \leq C \sum_{j=-\infty}^{-1} |\lambda_j|^q.
\end{aligned}
$$

If $1 < q < \infty$, by the Hölder inequality, we have

$$\begin{aligned}
P_1^q &= C\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q}\bigg(\sum_{j=-\infty}^{k-2} |\lambda_j| \|[b,S_\beta](a_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\bigg)^q \\
&\le C\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q}\bigg(\sum_{j=-\infty}^{k-2} |\lambda_j|(k-j)2^{(j-k)(\beta+n\delta_2)-j\alpha(0)}\bigg)^q \\
&\le C\sum_{k=-\infty}^{-1} \bigg(\sum_{j=-\infty}^{k-2} |\lambda_j|^q 2^{(j-k)\xi q/2}\bigg)\bigg(\sum_{j=-\infty}^{k-2} (k-j)^{q'} 2^{(j-k)\xi q'/2}\bigg)^{q/q'} \\
&\le C\sum_{j=-\infty}^{-3} |\lambda_j|^q \sum_{k=j+2}^{-1} 2^{(j-k)\xi q/2} \le C\sum_{j=-\infty}^{-1} |\lambda_j|^q.
\end{aligned}$$

For P_3, if $0 < q \le 1$, from (17) and the size conditions of a_j, it follows that

$$\begin{aligned}
P_3^q &= C\sum_{k=0}^{\infty} 2^{k\alpha_\infty q}\bigg(\sum_{j=-\infty}^{k-2} |\lambda_j| \|[b,S_\beta](a_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\bigg)^q \\
&\le C\sum_{k=0}^{\infty} 2^{k\alpha_\infty q}\bigg(\sum_{j=-\infty}^{-1} |\lambda_j|^q (k-j)^q 2^{(j-k)(\beta+n\delta_2)q-j\alpha(0)q}\bigg) \\
&\quad + C\sum_{k=0}^{\infty} 2^{k\alpha_\infty q}\bigg(\sum_{j=0}^{k-2} |\lambda_j|^q (k-j)^q 2^{(j-k)(\beta+n\delta_2)q-j\alpha_\infty q}\bigg) \\
&\le C\sum_{k=0}^{\infty} 2^{k(\alpha_\infty-\beta-n\delta_2)q} \sum_{j=-\infty}^{-1} |\lambda_j|^q (k-j)^q 2^{j\xi q} \\
&\quad + C\sum_{j=0}^{\infty} |\lambda_j|^q \sum_{k=j+2}^{\infty} (k-j)^q 2^{(j-k)\eta q} \\
&\le C\bigg(\sum_{j=-\infty}^{-1} |\lambda_j|^q + \sum_{j=0}^{\infty} |\lambda_j|^q\bigg) = C\sum_{j=-\infty}^{\infty} |\lambda_j|^q.
\end{aligned}$$

If $1 < q < \infty$, by the Hölder inequality, we have

$$\begin{aligned}
P_3^q &= C\sum_{k=0}^{\infty} 2^{k\alpha_\infty q}\bigg(\sum_{j=-\infty}^{k-2} |\lambda_j| \|[b,S_\beta](a_j)\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\bigg)^q \\
&\le C\sum_{k=0}^{\infty} 2^{k\alpha_\infty q}\bigg(\sum_{j=-\infty}^{-1} |\lambda_j|(k-j)2^{(j-k)(\beta+n\delta_2)-j\alpha(0)}\bigg)^q \\
&\quad + C\sum_{k=0}^{\infty} 2^{k\alpha_\infty q}\bigg(\sum_{j=0}^{k-2} |\lambda_j|(k-j)2^{(j-k)(\beta+n\delta_2)-j\alpha_\infty}\bigg)^q
\end{aligned}$$

$$\le C\sum_{k=0}^{\infty}2^{k(\alpha_\infty-\beta-n\delta_2)q}\bigg(\sum_{j=-\infty}^{-1}|\lambda_j|(k-j)2^{j\xi}\bigg)^q+C\sum_{k=0}^{\infty}\bigg(\sum_{j=0}^{k-2}|\lambda_j|(k-j)2^{(j-k)\eta}\bigg)^q$$
$$\le C\bigg(\sum_{j=-\infty}^{-1}|\lambda_j|^q2^{j\xi q/2}\bigg)\bigg(\sum_{j=-\infty}^{-1}(k-j)^{q'}2^{j\xi q'/2}\bigg)^{q/q'}$$
$$+C\sum_{k=0}^{\infty}\bigg(\sum_{j=0}^{k-2}|\lambda_j|^q2^{(j-k)\eta q/2}\bigg)^q\bigg(\sum_{j=0}^{k-2}(k-j)^{q'}2^{(j-k)\eta q'/2}\bigg)^{q/q'}$$
$$\le C\sum_{j=-\infty}^{-1}|\lambda_j|^q2^{j\xi q/2}+C\sum_{j=0}^{\infty}|\lambda_j|^q\sum_{k=j+2}^{\infty}2^{(j-k)\eta q/2}\le C\sum_{j=-\infty}^{\infty}|\lambda_j|^q.$$

For P_2 and P_4, using a combination of the $L^{p(\cdot)}(\mathbb{R}^n)$-boundedness of $[b, S_\beta]$ and the arguments used in the estimation of I_1 and I_3 in [9, Page 338], we can easily get

$$P_2^q\le C\sum_{j=-\infty}^{\infty}|\lambda_j|^q \quad\text{and}\quad P_4^q\le C\sum_{j=0}^{\infty}|\lambda_j|^q.$$

Hence, combining the estimates above, the proof of Theorem 25 is complete. □

Acknowledgement. The author would like to express his deep gratitude to the referee for giving many valuable comments and suggestions.

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