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PETTIS CONDITIONAL EXPECTATION OF CLOSED CONVEX RANDOM SETS IN A BANACH SPACE WITHOUT *RNP*

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Abstract. In this paper we study the existence of conditional expectation for closed and convex valued Pettis-integrable random sets without assuming the Radon Nikodym property of the Banach space. New version of multivalued dominated convergence theorem of conditional expectation and multivalued Lévy's martingale convergence theorem for integrable and Pettis integrable random sets are proved.

1. Introduction

The conditional expectation of random variables (resp. random sets) is the main tool of the martingales theory. The existence of conditional expectation for Bochner integrable random variables and integrably bounded random sets were proved without any additional conditions (see Dudley [8], Neveu [21], Hiai-Umegaki [17], Amri-Hess [10]). However, examples of Rybakov [23] and Talagrand [24] (Example 6-4-1) showed that the conditional expectation of Pettis integrable function does not generally exist. The existence of conditional expectation of vector (resp. multi-)valued Pettis integrable random variable (resp. random sets) is proved by several authors under the assumption that the Banach space possesses the Radon Nikodym property (RNP) (see Musial [20], Faik [13], Ziat [26], Amrani [2]). Recently two results of existence of the Pettis conditional expectation of convex and weakly compact Pettis integrable random sets have been proved in Harami-Ezzaki [11] and Castaing-Ezzaki-Akhiat [1] without assuming the RNP property of Banach space. The main purpose of this present paper is to prove the existence of conditional expectation of Pettis integrable closed and convex random sets in the Banach space without Radon Nikodym property. The results stated here are more general than the case where the random sets are convex and weakly compact. This paper contains also several new properties of Pettis integrable random sets as Jensen's

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inequality dealing with distance, norm, and support functions. As applications we give new results of dominated convergence theorem for conditional expectation of integrably bounded (resp. Pettis integrable) closed and convex random sets. Our convergence results are established with respect to linear topology. Notice that the linear convergence is more general than the Mosco convergence (see Beer [5] and Mosco [19]). Remark that dominated convergence theorem of conditional expectation asserts that $E^{\mathcal{F}_n}(X_n)$ converges to $E^{\mathcal{F}_\infty}(X_\infty)$ where $(\mathcal{F}_n)_{n\in\mathbf{N}^*}$ is an increasing sequence of sub-σ-algebra, and $(X_n)_{n\in\mathbf{N}^*}$ is a sequence of closed and convex random sets. We end this paper by presenting new versions of multivalued Lévy's martingale convergence theorem. These convergence theorems extend several results demonstrated by other authors (see Castaing-Ezzaki-Hess [6], Castaing-Ezzaki-Akhiat [1], Papageorgiou [22], Zhen-Xing [25]).

2. Elementary material

Throughout this paper $(\Omega, \mathcal{F}, \mu)$ will denote a probability measure space, E a real Banach space, E' the topological dual of E, and $\overline{B}$ (resp. B') the closed unit ball of E (resp. of E'). Denote by $cl(E)$ (resp. $cc(E)$) (resp. $ccb(E)$) (resp. $cwk(E)$) the family of all nonempty closed (resp. closed convex) (resp. closed convex bounded) (resp. weakly compact convex) subsets of E. For a subset $C \in 2^E \setminus \emptyset$, $\overline{C}$ (resp. $\overline{co}(C)$) is the norm-closure (resp. the close convex hull) of C. The support function and radius of a subset C are defined as follows:

$$\delta^*(x^{'}, C) = \sup_{x\in C}\langle x', x\rangle, \quad |C| = \sup_{x\in C}\|x\|.$$

The topology determined by convergence of support functional is denoted by T_{scalar}, a sequence (C_n) is T_{scalar} convergent to some subset C if

$$\lim_n \delta^*(x^{'}, C_n) = \delta^*(x^{'}, C) \quad \text{for all } x' \in E'.$$

The distance functional is the mapping: $d : E \times 2^E \setminus \emptyset \longrightarrow \mathbb{R}^+$ such that

$$d(x, C) = \inf_{a\in C}\|x - a\|.$$

The topology determined by the convergence of distance functionals is called the Wijsman topology and is denoted by T_w.

The linear topology T_B introduced by Beer [5] is the upper bound of the following topologies:

(1) the one of simple convergence of distance functions on E.

(2) the one of simple convergence of support functions on E'.

From Theorem 3.4 in Beer [5] a sequence (C_n) in $2^E \setminus \emptyset$ converges to some subset C in linear topology if and on if

$$\lim_n d(x, C_n) = d(x, C) \quad \text{and} \quad \lim_n \delta^*(x^{'}, C_n) = \delta^*(x^{'}, C) \text{ for all } x' \in E'.$$

This topology is stronger than the Mosco topology.

Next denote by $L^1_E(\Omega, \mathcal{F}, \mu)$ the space of all (equivalence classes) of $\mathcal{F}$-measurable and Bochner integrable functions $X : \Omega \to E$. $L^\infty_{R^+}(\Omega, \mathcal{F}, \mu)$ is the space of all equivalence classes of $\mathcal{F}$-measurable essentially bounded functions $X : \Omega \to \mathbb{R}^+$.

The map $X : \Omega \longrightarrow 2^E \setminus \emptyset$ is called a multifunction (or set-valued function, correspondence, etc). We say that it is scalarly measurable if for all $x' \in E'$, the map $\delta^*(x', X(\cdot))$ is measurable. X is said to be Effros measurable (or measurable) if for every open subset U of E, the subset $X^-(U) = \{\omega, X(\omega) \cap U \neq \emptyset\}$ is a member of $\mathcal{F}$. The Effros measurability is stronger than the scalar measurability. Both notions of measurability coincide for more general classes of multifunctions (see Amri-Hess [10]). A measurable multifunction is called a random set. A function f from Ω to E is called a selection of X if $f(\omega) \in X(\omega)$ for any $\omega \in \Omega$, we denote by S_X the set of all measurable selections of X. It is well known that every measurable $cl(E)$-valued random set X admits at least one measurable selection. Furthermore, a random set $X : \Omega \longrightarrow cl(E)$ is measurable if and only if there is a countable family of measurable selections (f_n) such that for each $\omega \in \Omega$, $X(\omega) = \overline{\{f_n(\omega), n \in N\}}$ where the closure is taken with respect to the norm in E (see Castaing-Valadier [7, §3. p. 67]).

Let $\mathcal{L}^1_{cl(E)}(\mathcal{F}) := \mathcal{L}^1_{cl(E)}(\Omega, \mathcal{F}, \mu)$ the space of all random sets X with values in $cl(E)$ such that $|X(\cdot)| \in L^1_R(\Omega, \mathcal{F}, \mu)$, and is called the space of integrably bounded random sets with values in $cl(E)$. Denote by S^1_X the set of all measurable and integrable selections of a random set X. It is not empty if and only if $d(0, X(\cdot)) \in L^1_R(\Omega, \mathcal{F}, \mu)$. In such a case, we shall say that the random set X is integrable (see Lemma 5.1 in Hess [16]).

A random set with values in $cl(E)$ is Aumann integrable if $S^1_X \neq \emptyset$, the Aumann integral of X is defined by

$$\int_A X d\mu = \{\int_A f d\mu, f \in S^1_X\}.$$

Let $\mathcal{B}$ be a sub-σ-algebra of $\mathcal{F}$ and X be an integrable random variable defined in $(\Omega, \mathcal{F}, \mu)$ with values in a Banach space E. A conditional expectation of X with respect to $\mathcal{B}$ is the unique $\mathcal{B}$-measurable and integrable random set Y such that

$$\int_A X d\mu = \int_A Y d\mu \quad \text{for all} \quad A \in \mathcal{B}.$$

The conditional expectation Y of X is denoted by $E^{\mathcal{B}}(X)$.

Hiai-Umegaki [17] in Theorem 5.1 showed the existence of conditional expectation of the random set $X \in \mathcal{L}^1_{cl(E)}(\mathcal{F})$ which is a $\mathcal{B}$-measurable random set denoted by $E^{\mathcal{B}}(X)$, and such that

$$S^1_{E^{\mathcal{B}}(X)}(\mathcal{B}) = cl\{E^{\mathcal{B}}(f) : f \in S^1_X\}.$$

From Klei-Assani [4] if X is $cwk(E)$-valued, then S^1_X is convex and weakly compact, furthermore $E^{\mathcal{B}}(X)$ exists and satisfies the following property

$$S^1_{E^{\mathcal{B}}(X)}(\mathcal{B}) = \{E^{\mathcal{B}}(f) : f \in S^1_X\}.$$

A random set $X : \Omega \to cl(E)$ is Pettis integrable whenever

(1) $(\delta^*(x^{'}, X))^- \in L^1_R(\Omega, \mathcal{F}, \mu)$, where $(\delta^*(x^{'}, X))^- = \max(0, -\delta^*(x^{'}, X))$.

(2) For all $A \in \Sigma$, there exists $C_A(X) \in cl(E)$ such that

$$\delta^*(x^{'}, C_A(X)) = \int_A \delta^*(x^{'}, X) d\mu \quad \forall x^{'} \in E^{'}.$$

$C_A(X)$ is called the Pettis or the weak integral of X over A. A measurable E-valued function f is Pettis integrable if and only if the set $\{\langle x', f\rangle, x' \in B'\}$ is uniformly integrable. This equivalence is not true in the multivalued case (see Amri-Hess [10]). Let us denote by S^{Pe}_X the set of all Pettis integrable selections of X. A $cl(E)$-valued random set X is said to be Aumann-Pettis integrable if $S^{Pe}_X \neq \emptyset$. The Aumann-Pettis integral of X over A is denoted by $I_A(X)$ and is defined by

$$I_A(X) = \{\int_A f d\mu, f \in S^{Pe}_X\}.$$

It is well known from Theorem 3.7 in Amri-Hess [10] that if a $cc(E)$-valued random set X is Aumann Pettis integrable and $(\delta^*(x^{'}, X))^-$ is integrable, then X is Pettis integrable.

A subset H in P^1_E is called uniformly Pettis continuous if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $A \in \mathcal{F}$ with $\mu(A) < \delta$, then we have $\|f \cdot \chi_A\|_{Pe} < \epsilon$ for each $f \in H$. And H is called uniformly Pettis integrable if it is uniformly Pettis continuous and bounded in P^1_E. It is well known from Amrani-Castaing-Valadier [3] that the uniformly Pettis integrable property implies uniformly Pettis continuous property, and from Edgar-Sucheston [9] if μ is atomless, then both properties are equivalent. Remark that some authors call a family of functions uniformly Pettis integrable if it is only uniformly Pettis continuous.

Definition 2.1. A Banach space E possesses the weak Radon-Nikodym property ($WRNP$) if every E-valued measure m defined on a probability measure space $(\Omega, \mathcal{F}, \mu)$, which is μ-continuous and of σ-bounded variation has a Pettis integrable density $f : \Omega \to E$.

For the RNP property we replace σ-bounded by bounded and Pettis by Bochner in the definition. It is known from Musial [20, p. 157] that if E is separable both properties RNP and $WRNP$ are equivalent.

3. Conditional expectation of $cc(E)$-valued Pettis integrable random sets

Since the existence result of conditional expectation is the foundation of many convergence theorems of random processes, in this section we shall present a sufficient condition of the existence of this operator in the case where the Banach space E is not $WRNP$. This result extends the Hiai and Umegaki [17] integrably bounded results to the Pettis integrable random sets. Also notice that here, our result is also an extension of the results obtained by Ziat [26], Castaing-Ezzaki-Akhiat [1], Harami-Ezzaki [11] in the Pettis case.

We begin by the following lemma which is needed in the proof of the next theorem.

Lemma 3.1. *Let X be a $\mathcal{B}$-measurable and $cc(E)$-valued Aumann Pettis integrable random set. Assume that there exists a partition $(B_m)_{m\geq 1}$ of Ω in $\mathcal{B}$ such that $\int_{B_m} |X|d\mu < +\infty$ for each $m \geq 1$. If each single element of S_X is Pettis integrable, then for each $m \geq 1$,*

$$S^1_{X\cdot\chi_{B_m}}(\mathcal{B}) = S^{Pe}_{X\cdot\chi_{B_m}}(\mathcal{B}) = S^{Pe}_X(\mathcal{B}) \cdot \chi_{B_m}.$$

Proof. We begin to prove that

$$S^1_{X.\chi_{B_m}}(\mathcal{B}) = \{f \cdot \chi_{B_m} : f \in S^{Pe}_X(\mathcal{B})\}.$$

It is clear that

$$\{f \cdot \chi_{B_m} : f \in S^{Pe}_X(\mathcal{B})\} \subset S^1_{X\cdot\chi_{B_m}}(\mathcal{B}).$$

Now for each m, let $g \in S^1_{X\cdot\chi_{B_m}}(\mathcal{B})$, then $g(\omega) \in X \cdot \chi_{B_m}(\omega)$ and there exists $f \in S_X$ such that $g = f \cdot \chi_{B_m}$. Since every single of S_X is Pettis integrable, then f is Pettis integrable. This shows that

$$g \in \{f \cdot \chi_{B_m} : f \in S^{Pe}_X(\mathcal{B})\}.$$

Since for each $m \geq 1$,

$$\int_\Omega |X \cdot \chi_{B_m}|d\mu = \int_{B_m} |X|d\mu < +\infty$$

thus we have

$$S^{Pe}_{X\cdot\chi_{B_m}}(\mathcal{B}) = S^1_{X\cdot\chi_{B_m}}(\mathcal{B}) = \{f \cdot \chi_{B_m} : f \in S^{Pe}_X(\mathcal{B})\}.$$

Consequently

$$S^1_{X\cdot\chi_{B_m}}(\mathcal{B}) = S^{Pe}_{X\cdot\chi_{B_m}}(\mathcal{B}) = S^{Pe}_X(\mathcal{B}) \cdot \chi_{B_m}. \qquad \square$$

In the next theorem we prove that the existence of conditional expectation of $cc(E)$-valued Pettis integrable random sets in a Banach space without the RNP property. Before starting this theorem remark that if X is $cc(E)$-valued Pettis integrable random set, $\mathcal{B}$ a sub-σ-algebra of $\mathcal{F}$, and M the measure defined on $\mathcal{F}$ by $M(A) = \int_A |X|d\mu$. Then the following properties are equivalent:

1) $E^{\mathcal{B}}(|X|) < +\infty$ μ a.s.;

2) The restriction of M to $\mathcal{B}$ is a σ-finite measure;

3) There exists a partition $(B_m)_{m\geq 1}$ of Ω in $\mathcal{B}$ such that for each m, $M(B_m) = \int_{B_m} |X|d\mu < +\infty$.

Theorem 3.2. *Assume that $(\Omega, \mathcal{F}, \mu)$ is a probability measure space, and E' a strongly separable dual of E. Let $\mathcal{B}$ be a sub-σ-algebra of $\mathcal{F}$ and X a $cc(E)$-valued scalarly integrable random set with $S_X^{Pe} \neq \emptyset$, and such that*:

(i) $E^{\mathcal{B}}(|X|) < +\infty \quad \mu$ *a.s.*

(ii) *Every countable subset of S_X is uniformly Pettis continuous.*

Then there exists a unique $\mathcal{B}$-measurable and Aumann Pettis integrable random set Y such that

$$S_Y^{Pe}(\mathcal{B}) = cl\{E^{\mathcal{B}}(f) : f \in S_X^{Pe}\}.$$

Consequently Y is Pettis integrable and satisfies

$$\overline{\int_A X d\mu} = \overline{\int_A Y d\mu} \qquad \forall A \in \mathcal{B}.$$

Proof. The Pettis integrability of X is an immediate consequence of Theorem 3.7 in El Amri-Hess [10]. Now for each $m \geq 1$, set $B_m = \{\omega, m \leq E^{\mathcal{B}}(|X|) < m+1\}$, then it is clear from (ii) that $(B_m)_{m\geq 0}$ is a partition of Ω in $\mathcal{B}$. Define $X_m := X \cdot \chi_{B_m}$, since

$$\begin{aligned}\int_\Omega |X_m| d\mu = \int_{B_m} |X| d\mu &= \int_{B_m} E^{\mathcal{B}}(|X|) d\mu \\ &\leq (m+1) \cdot \mu(B_m) < +\infty.\end{aligned}$$

Then each $X_m \in \mathcal{L}^1_{cc(E)}(\Omega, \mathcal{F}, \mu)$. Now by applying Theorem 5.1 and Corollary 1.6 in Hiai-Umegaki [17] for each $m \geq 1$, there exists a unique $Y_m \in \mathcal{L}^1_{cc(E)}(\Omega, \mathcal{B}, \mu)$ such that

$$S^1_{Y_m}(\mathcal{B}) = cl\{E^{\mathcal{B}}(g_m) : g_m \in S^1_{X_m}\}.$$

Applying Lemma 3.1 we can write

$$\begin{aligned}S^1_{Y_m}(\mathcal{B}) &= cl\{E^{\mathcal{B}}(g_m) : g_m \in S_X^{Pe} \cdot \chi_{B_m}\} \\ &= cl\{E^{\mathcal{B}}(f \cdot \chi_{B_m}) : f \in S_X^{Pe}\}.\end{aligned}$$

Then from (i) and Proposition 3.1 in Harami-Ezzaki [11], $E^{\mathcal{B}}(f)$ exists for every $f \in S_X^{Pe}$ and

$$E^{\mathcal{B}}(f \cdot \chi_{B_m}) = E^{\mathcal{B}}(f) \cdot \chi_{B_m} \quad \mu \text{ a.s.}$$

Then

$$S^1_{Y_m}(\mathcal{B}) = cl\{E^{\mathcal{B}}(f) \cdot \chi_{B_m} : f \in S_X^{Pe}\}. \tag{3.2.1}$$

Now let us define a multifunction Y by: $Y(\omega) = Y_m(\omega)$ if $\omega \in B_m$ (i.e., $Y \cdot \chi_{B_m}(\omega) = Y_m \cdot \chi_{B_m}(\omega)$), then Y is $\mathcal{B}$-measurable, $cc(E)$-valued random set. Since $S_X^{Pe} \neq \emptyset$, there is $h \in S_X^{Pe}$. From condition (i) and Harami-Ezzaki [11], $E^{\mathcal{B}}(h)$ exists. Now let us define $g(\omega) = E^{\mathcal{B}}(h)(\omega)$ for all $\omega \in \Omega$, so g is a $\mathcal{B}$-measurable and Pettis integrable function. Now we prove that g is a selection of Y. Since for each $m \geq 1$,

$$E^{\mathcal{B}}(h \cdot \chi_{B_m}) = E^{\mathcal{B}}(h) \cdot \chi_{B_m}$$

and

$$S^1_{Y_m}(\mathcal{B}) = cl\{E^{\mathcal{B}}(h\cdot\chi_{B_m}) : h\in S^{Pe}_X\}.$$

Then for each $m\geq 1$,

$$g\cdot\chi_{B_m} = E^{\mathcal{B}}(h)\cdot\chi_{B_m} = E^{\mathcal{B}}(h\cdot\chi_{B_m})\in S^1_{Y_m}(\mathcal{B}).$$

This shows that $g\in S^{Pe}_Y$, then Y is Aumann Pettis integrable. Now by using $Y\cdot\chi_{B_m} = Y_m\cdot\chi_{B_m}$ and applying Lemma 3.1, it is clear that the formula (3.2.1) becomes

$$S^{Pe}_Y(\mathcal{B})\cdot\chi_{B_m} = cl\{E^{\mathcal{B}}(f)\cdot\chi_{B_m} : f\in S^{Pe}_X\}. \tag{3.2.2}$$

Now we prove that

$$S^{Pe}_Y(\mathcal{B}) = cl\{E^{\mathcal{B}}(f) : f\in S^{Pe}_X\}.$$

Let $g\in S^{Pe}_Y(\mathcal{B})$, then from (3.2.2) for each $m\geq 1$,

$$g\cdot\chi_{B_m}\in cl\{E^{\mathcal{B}}(f)\cdot\chi_{B_m} : f\in S^{Pe}_X\}.$$

So for each $m\geq 1$, there is a sequence $(f^m_n)_{n\geq 0}$ in $S^{P_e}_X$ such that

$$\lim_n \|g\cdot\chi_{B_m} - E^{\mathcal{B}}(f^m_n)\cdot\chi_{B_m}\|_1 = 0.$$

Then by extracting a subsequence $(f^m_{\varphi^m(n)})_{n\geq 0}$ of $(f^m_n)_{n\geq 0}$ we have

$$\lim_n E^{\mathcal{B}}(f^m_{\varphi^m(n)})\cdot\chi_{B_m} = g\cdot\chi_{B_m} \quad \mu\,\text{a.s.}$$

Let us define a sequence $(f_{\varphi(n)})_{n\geq 0}$ such that

$$f_{\varphi(n)}(\omega) = f^m_{\varphi^m(n)}(\omega)\ \text{ if } \omega\in B_m.$$

Clearly that $(f_{\varphi(n)})_{n\geq 0}$ is a sequence of measurable and Pettis integrable selections of X and we have

$$\lim_n E^{\mathcal{B}}(f_{\varphi(n)}) = g \quad \mu\,\text{a.s.} \tag{3.2.3}$$

From condition (ii) the sequence $(f_{\varphi(n)})_{n\geq 0}$ is uniformly Pettis continuous, then for every $\epsilon > 0$, there is $\eta > 0$ such that for every $A\in\mathcal{B}$ with $\mu(A) < \eta$ we have

$$\begin{aligned}\|E^{\mathcal{B}}(f_{\varphi(n)})\cdot\chi_A\|_{P_e} &= \sup_{x'\in B'}\int_A |\langle x', E^{\mathcal{B}}(f_{\varphi(n)})\rangle| d\mu\\ &\leq \sup_{x'\in B'}\int_A E^{\mathcal{B}}(|\langle x', f_{\varphi(n)}\rangle|) d\mu\\ &= \sup_{x'\in B'}\int_A |\langle x', f_{\varphi(n)}\rangle| d\mu < \epsilon.\end{aligned}$$

Which prove that the sequence $(E^{\mathcal{B}}(f_{\varphi(n)}))_{n\geq 0}$ is uniformly Pettis continuous. So from formula (3.2.3), and Proposition 2.1 in Amrani-Castaing-Valadier [3] we have

$$\lim_n \|E^{\mathcal{B}}(f_{\varphi(n)}) - g\|_{P_e} = 0.$$

This shows that $g\in cl\{E^{\mathcal{B}}(f) : f\in S^{Pe}_X\}$.

Conversely, by construction

$$\{E^{\mathcal{B}}(f) : f \in S_X^{Pe}\} \subset S_Y^{Pe}(\mathcal{B}). \tag{3.2.4}$$

Now we prove that Y is Pettis integrable. Indeed, since for every $f \in S_Y$ we have $(\delta^*(x', Y))^- \leq (\langle x', f\rangle)^-$, and the fact $S_Y^{Pe} \neq \emptyset$, by Theorem 3.7 in Amri-Hess [10], Y is Pettis integrable. Consequently by using the Pettis integrability of X and Y, the existence of $E^{\mathcal{B}}(\delta^*(x', X))$ and $Y = E^{\mathcal{B}}(X)$, and the result of Hiai [17] in the integrable case, we get $\forall x' \in E'$,

$$\begin{aligned}
\delta^*(x', \int_A Y d\mu) &= \int_A \delta^*(x', Y) d\mu \\
&= \sum_{m\geq 1} \int_{A\cap B_m} \delta^*(x', Y_m) d\mu = \sum_{m\geq 1} \int_{A\cap B_m} (\delta^*(x', E^{\mathcal{B}}(X_m)) d\mu \\
&= \sum_{m\geq 1} \int_{A\cap B_m} E^{\mathcal{B}}(\delta^*(x', X_m)) d\mu = \sum_{m\geq 1} \int_{A\cap B_m} E^{\mathcal{B}}(\delta^*(x', X)) d\mu \\
&= \int_A E^{\mathcal{B}}(\delta^*(x', X)) d\mu = \int_A \delta^*(x', X) d\mu \\
&= \delta^*(x', \int_A X d\mu) \quad \forall\, A \in \mathcal{B}. \qquad (3.2.5)
\end{aligned}$$

By uniqueness of $E^{\mathcal{B}}(\delta^*(x', X))$ we have

$$\delta^*(x', Y) = E^{\mathcal{B}}(\delta^*(x', X)) \quad \mu \, \text{a.s.}$$

Now apply the monotonicity of $E^{\mathcal{B}}$ we get $\forall x' \in E'$,

$$\begin{aligned}
|\delta^*(x', Y)| = |E^{\mathcal{B}}(\delta^*(x', X))| &\leq E^{\mathcal{B}}(|\delta^*(x', X)|) \\
&\leq \|x'\| \cdot E^{\mathcal{B}}(|X|) < +\infty.
\end{aligned}$$

Then from Banach-Steinhaus theorem Y takes values in $ccb(E)$. Using the argument in (Amri-Hess [10, §4. p. 349]) and the separability of E' we conclude that Y is countably supported. Then $S_Y^{Pe}(\mathcal{B})$ is norm closed in $P_E^1(\mathcal{B})$ and (3.2.4) becomes

$$cl\{E^{\mathcal{B}}(f) : f \in S_X^{Pe}\} \subset S_Y^{Pe}(\mathcal{B}).$$

Then

$$S_Y^{Pe}(\mathcal{B}) = cl\{E^{\mathcal{B}}(f) : f \in S_X^{Pe}\}.$$

The equality

$$\overline{\int_A X d\mu} = \overline{\int_A Y d\mu} \qquad \forall A \in \mathcal{B}$$

is an immediate consequence of (3.2.5). □

The following lemmas are extensions of the classical Jensen inequality to the Pettis integrable random variables (resp. sets).

Lemma 3.3. *Assume that E is a separable Banach space. Let $f : \Omega \longrightarrow E$ be a scalarly integrable and Pettis integrable function, and $\mathcal{B}$ a sub-σ-algebra of $\mathcal{F}$ such that $E^{\mathcal{B}}(f)$ exists. Then we have*

$$\|E^{\mathcal{B}}(f)\| \leq E^{\mathcal{B}}(\|f\|) \quad \mu \text{ a.s.}$$

Proof. Let $D' = \{e'_n, n \geq 1\}$ be a countable dense subset in B' with respect to the weak* topology in E'. Then for all $e'_n \in D'$,

$$\langle e'_n, f\rangle \in L^1_R \text{ and } E^{\mathcal{B}}(\langle e'_n, f\rangle) = \langle e'_n, E^{\mathcal{B}}(f)\rangle.$$

Thus by applying the classical Jensen's inequality in L^1_R for every $n \geq 1$, there exists a negligible N_n such that

$$|\langle e'_n, E^{\mathcal{B}}(f)(\omega)\rangle| = |E^{\mathcal{B}}(\langle e'_n, f\rangle)(\omega)| \leq E^{\mathcal{B}}(|\langle e'_n, f\rangle|)(\omega) \quad \forall \omega \in N^c_n.$$

Let $N = \cup_{n\geq 1} N_n$, then we have

$$|\langle e'_n, E^{\mathcal{B}}(f)(\omega)\rangle| \leq E^{\mathcal{B}}(|\langle e'_n, f\rangle|)(\omega) \quad \text{for all } \omega \in N^c.$$

By taking the supremum in n we get

$$\begin{aligned}\sup_{n\geq 1} |\langle e'_n, E^{\mathcal{B}}(f)\rangle| &\leq \sup_{n\geq 1} E^{\mathcal{B}}(|\langle e'_n, f\rangle|)\\ &\leq \sup_{n\geq 1} E^{\mathcal{B}}(\sup_{n\geq 1} |\langle e'_n, f\rangle|) = E^{\mathcal{B}}(\|f\|)\ ; \quad \mu \text{ a.s.}\end{aligned}$$

Thus

$$\|E^{\mathcal{B}}(f)\| \leq E^{\mathcal{B}}(\|f\|) \quad \mu \text{ a.s.} \qquad \square$$

In the multivalued case we state the following:

Lemma 3.4. *Let X be a random set satisfying all conditions of Theorem 3.2. Then we have the following*:
(1) $\forall x' \in E'$, $E^{\mathcal{B}}(\delta^*(x', X)) = \delta^*(x', E^{\mathcal{B}}(X))$ μ *a.s.*
(2) $\forall x \in E$, $d(x, E^{\mathcal{B}}(X)) \leq E^{\mathcal{B}}(d(x, X))$ μ *a.s.*
(3) $|E^{\mathcal{B}}(X)| \leq E^{\mathcal{B}}(|X|)$ μ *a.s.*

Proof. Note first that, for all $x' \in E'$, for all $A \in \mathcal{B}$,

$$\begin{aligned}\int_A \delta^*(x', E^{\mathcal{B}}(X))d\mu = \delta^*(x', \int_A E^{\mathcal{B}}(X)d\mu) = \delta^*(x', \int_A X d\mu)\\ = \int_A \delta^*(x', X)d\mu = \int_A E^{\mathcal{B}}(\delta^*(x', X))d\mu.\end{aligned}$$

Then from the uniqueness of conditional expectation we have, for all $x' \in E'$

$$E^{\mathcal{B}}(\delta^*(x', X)) = \delta^*(x', E^{\mathcal{B}}(X)) \quad \mu \text{ a.s.}$$

Next we prove (2): since X is measurable, then the mapping: $\omega \longmapsto d(x, X(\omega))$ is measurable for every $x \in E$ (see [7, §3. p. 67]). Now the fact that X has a

Castaing representation in S_X^{Pe} such that $X(\omega) = \overline{\{f_n(\omega), n \geq 0\}}$ with respect to the norm in E (see Lemma 2.3 in Godet-Thobie [14]). Thus

$$d(x, X(\omega)) = \inf\{\|x-y\|, y \in X\} = \inf_{n \in \mathbf{N}} d(x, f_n(\omega)).$$

Given $\epsilon > 0$, for every x let us define a reel positive random variable r as

$$r(\omega) = d(x, X(\omega)) + \epsilon \quad \forall \omega \in \Omega$$

and a multifunction G as

$$G(\omega) = X(\omega) \cap \overline{B}(x, r(\omega)) \quad \forall \omega \in \Omega.$$

Then $G(\omega) \neq \emptyset$, because for each $\omega \in \Omega$, there exists $n_0 \in \mathbf{N}$ such that

$$\|x - f_{n_0}(\omega)\| < d(x, X(\omega)) + \epsilon.$$

So G is with non-empty and closed values in a complete space E, and from Hess [15, Prop. 3.3.3] we get the measurability of the multifunction G. Moreover, applying theorem of Kuratowski and Ryll-Nardzewski [18] (see also Theorem III.6 in [7]) there is a measurable selection g of G. So it is clear that

$$\|x - g(\omega)\| \leq d(x, X(\omega)) + \epsilon. \tag{3.4.1}$$

Condition (ii) in Theorem 3.2 and the scalar integrability of X show that g is Pettis integrable. Now by taking the conditional expectation in (3.4.1) we have

$$E^{\mathcal{B}}(\|x-g\|) \leq E^{\mathcal{B}}(d(x, X)) + \epsilon.$$

Furthermore, from Lemma 3.4 we get

$$\begin{aligned} \|x - E^{\mathcal{B}}(g)\| = \|E^{\mathcal{B}}(x-g)\| &\leq E^{\mathcal{B}}(\|x-g\|) \\ &\leq E^{\mathcal{B}}(d(x, X)) + \epsilon \quad \mu \text{ a.s.} \end{aligned}$$

Thus the property $E^{\mathcal{B}}(g) \in E^{\mathcal{B}}(X)$ implies

$$d(x, E^{\mathcal{B}}(X)) \leq E^{\mathcal{B}}(d(x, X)) + \epsilon \quad \mu \text{ a.s.},$$

where ϵ is arbitrary. Then for all $x \in E$,

$$d(x, E^{\mathcal{B}}(X)) \leq E^{\mathcal{B}}(d(x, X)) \quad \mu \text{ a.s.}$$

Finally to prove (3), let D'_1 be a dense subset in B' with respect to the norm topology in E', Then we have

$$\begin{aligned} E^{\mathcal{B}}|X| &= E^{\mathcal{B}}(\sup_{x' \in B'}(\delta^*(x', X)) = E^{\mathcal{B}}(\sup_{x' \in D'_1}(\delta^*(x', X)) \\ &\geq E^{\mathcal{B}}(\delta^*(x', X)) = \delta^*(x', E^{\mathcal{B}}(X)) \quad \mu \text{ a.s.} \quad \forall\, x' \in D'_1. \end{aligned}$$

Thus

$$E^{\mathcal{B}}|X| \geq \sup_{x' \in D'_1} \delta^*(x', E^{\mathcal{B}}(X)) = |E^{\mathcal{B}}(X)| \quad \mu \text{ a.s.}$$

□

4. Dominated convergence theorem and Lévy's convergence theorem

In this section we present some applications of existence for the conditional expectation of Pettis integrable random sets. So we extend the dominated convergence theorem and Lévy's martingale convergence theorem of conditional expectation to the $cc(E)$-valued Pettis integrable random sets defined on a probability measure space $(\Omega, \mathcal{F}, \mu)$. The two convergence theorems are treated by considering the linear topology which is more general than the Mosco topology.

4.1. Dominated convergence theorem of conditional expectation

The study of almost sure Mosco convergence for sequences of random sets of the form $E^{\mathcal{F}_n}(X_n)$ where (X_n) is a sequence of $cc(E)$-valued and integrably bounded random sets has given in Ezzaki [12] and Castaing-Ezzaki-Hess [6]. The purpose in this subsection is to go in this study of convergence of conditional expectation dealing with Pettis integration. In first we begin by an extension of this result to the integrably bounded random sequences where the linear topology is considered.

Proposition 4.1. *Assume that E' is strongly separable and $(\Omega, \mathcal{F}, \mu)$ is a probability measure space. Let $(\mathcal{F}_n)_{n\geq 1}$ be an increasing sequence of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma(\bigcup_{n\geq 1} \mathcal{F}_n)$, $(X_n)_{n\in N^*\cup\{\infty\}}$ be a sequence in $\mathcal{L}^1_{cc(E)}(\Omega, \mathcal{F}, \mu)$, and Y a positive random variable such that $\sup_n E^{\mathcal{F}_n}(Y) < +\infty$ μ a.s. Assume that*

(i) $\forall n \in N^*$, $\|X_n\| \leq Y$ μ *a.s.*

(ii) $\forall x' \in E'$, $\delta^*(x', X_n)$ *converges* μ *a.s. to* $\delta^*(x', X_\infty)$.

(iii) $\forall x \in E$, $\lim_n d(x, X_n) = d(x, X_\infty)$ μ *a.s.*

Then

$$T_B - \lim_n E^{\mathcal{F}_n}(X_n) = X_\infty \quad \mu \text{ a.s.}$$

Proof. Step (1) $\lim_n \delta^*(x', E^{\mathcal{F}_n}(X_n)) = \delta^*(x', X_\infty)$ μ a.s.

Since $X_n \in \mathcal{L}^1_{cc(E)}(\Omega, \mathcal{F}, \mu)$ then $E^{\mathcal{F}_m}(X_n)$ exists for any $m \geq 1$. Condition (i) implies that $\forall x' \in E'$, $\forall n \in N^*$,

$$|\delta^*(x', X_n)| \leq \|x'\| \cdot |X_n| \leq \|x'\| \cdot Y \quad \mu \text{ a.s.}$$

Then by applying condition (ii), Lemma 3.4(1), and Lemma 2.2 in Ezzaki [12] we have $\forall x' \in E'$

$$\lim_n \delta^*(x', E^{\mathcal{F}_n}(X_n)) = \lim_n E^{\mathcal{F}_n}(\delta^*(x', X_n)) = \delta^*(x', X_\infty) \quad \mu \text{ a.s.}$$

Let $(x_k^{'})_{k\geq 1}$ be a dense sequence in E' with respect to the norm topology in E', then there is a negligible N such that $\forall \omega \in \Omega \setminus N$, $\forall k \geq 1$,

$$\lim_n \delta^*(x_k^{'}, E^{\mathcal{F}_n}(X_n)(\omega)) = \delta^*(x_k^{'}, X_\infty(\omega))).$$

Since

$$\sup_n |E^{\mathcal{F}_n}(X_n)| \leq \sup_n E^{\mathcal{F}_n}(|X_n|) \leq \sup_n E^{\mathcal{F}_n}(Y) < +\infty \quad \mu \text{ a.s.}$$

there is a negligible N' such that $\sup_n |E^{\mathcal{F}_n}(X_n)(\omega)| < +\infty \ \forall \omega \in \Omega \setminus N'$. Then $(\delta^*(\cdot, E^{\mathcal{F}_n}(X_n))(\omega))_{n\geq 1}$ is for each $\omega \in \Omega \setminus (N \cup N')$ uniformly continuous with respect to norm topology in E'. Since the sequence $(x_k^{'})_{k\geq 1}$ is strongly dense, it follows that $\forall \omega \in \Omega \setminus (N \cup N')$, $\forall x' \in E'$,

$$\lim_n \delta^*(x', E^{\mathcal{F}_n}(X_n)(\omega)) = \delta^*(x', X_\infty(\omega)). \tag{4.1.1}$$

Step (2) $\lim_n d(x, E^{\mathcal{F}_n}(X_n)) = d(x, X_\infty)$ μ a.s.

By applying conditions (i), (iii), and Lemma 2.2 in Ezzaki [12], it follows that $\forall x \in E$,

$$\lim_n E^{\mathcal{F}_n}(d(x, X_n)) = d(x, X_\infty) \quad \mu \text{ a.s.}$$

Since from Lemma 3.4(2) we have

$$d(x, E^{\mathcal{F}_n}(X_n)) \leq E^{\mathcal{F}_n}(d(x, X_n) \quad \mu \text{ a.s.}$$

Then

$$\limsup_n d(x, E^{\mathcal{F}_n}(X_n)) \leq d(x, X_\infty) \quad \mu \text{ a.s.} \tag{4.1.2}$$

On the other hand since for each $n \in N^*$, and $x \in E$, we have

$$d(x, E^{\mathcal{F}_n}(X_n)) = \sup_{x \in B'} (\langle x', x\rangle - \delta^*(x', E^{\mathcal{F}_n}(X_n))).$$

Then $\forall x' \in B'$,

$$d(x, E^{\mathcal{F}_n}(X_n)) \geq (\langle x', x\rangle - \delta^*(x', E^{\mathcal{F}_n}(X_n)))$$

and from (4.1.1) we have

$$(\langle x', x\rangle - \delta^*(x', X_\infty)) \leq \liminf_n d(x, E^{\mathcal{F}_n}(X_n)) \ \ \mu \text{ a.s.}$$

By taking the supremum in x' we get

$$d(x, X_\infty) \leq \liminf_n d(x, E^{\mathcal{F}_n}(X_n)) \ \ \mu \text{ a.s.} \tag{4.1.3}$$

Thus from (4.1.2) and (4.1.3) we have $\forall x \in E$,

$$\lim_n d(x, E^{\mathcal{F}_n}(X_n)) = d(x, X_\infty) \quad \mu \text{ a.s.}$$

The μ a.s. convergence follows from the fact that the family of distance functionals $d(\cdot, C), C \in cc(E)$ is equicontinuous and E is separable. Finally from this and formula (4.1.1) we have

$$T_B - \lim_n E^{\mathcal{F}_n}(X_n) = X_\infty \quad \mu \text{ a.s.}$$

□

The following theorem is an extension of Proposition 4.1 to the $cc(E)$-valued Aumann-Pettis integrable sequences of random sets.

Theorem 4.2. *Assume that E' is strongly separable and $(\Omega, \mathcal{F}, \mu)$ is a probability measure space. Let $(\mathcal{F}_n)_{n\geq 1}$ be an increasing sequence of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma(\bigcup_{n\geq 1} \mathcal{F}_n)$. Let $(X_n)_{n\in N^*\cup\{\infty\}}$ be a sequence of $cc(E)$-valued Aumann-Pettis integrable random set, and Y a positive random variable such that $\sup_n E^{\mathcal{F}_n}(Y) < +\infty$. Assume that*

(i) *the set $\{\delta^*(x', X_n), x' \in B', n \geq 1\}$ is uniformly integrable.*

(ii) *$\forall n \in N^*$, $|X_n| \leq Y$ μ a.s.*

(iii) *$\forall x' \in E'$, $\delta^*(x', X_n)$ converge μ a.s. to $\delta^*(x', X_\infty)$.*

(iv) *$\forall x \in E$, $\lim_n d(x, X_n) = d(x, X_\infty)$ μ a.s.*

Then

$$T_B - \lim_n E^{\mathcal{F}_n}(X_n) = X_\infty \quad \mu \text{ a.s.}$$

Proof. By applying conditions (i), (ii) and Theorem 3.2 is not hard to see that for each $n \geq 1$, $E^{\mathcal{F}_n}(X_n)$ exists and is a Pettis integrable random set. Set $A_m = \{\omega, m-1 \leq E^{\mathcal{F}_1}(Y) < m\}$, then $(A_m)_{m\geq 1}$ is a partition of Ω in $\mathcal{F}_1$ and for each $m \geq 1$,

$$\int_{A_m} \|X_n\| d\mu \leq \int_{A_m} Y d\mu = \int_{A_m} E^{\mathcal{F}_1}(Y) d\mu \leq (m+1)\cdot \mu(A_m) < +\infty.$$

Then for each $m \geq 1$, the random set $X_n.\chi_{A_m}$ is in $\mathcal{L}^1_{cc(E)}(\Omega, \mathcal{F}, \mu)$, and from conditions (iii) and (iv) we have

$$\forall x' \in E' \quad \lim_n \delta^*(x', X_n \cdot \chi_{A_m}) = \delta^*(x', X_\infty \cdot \chi_{A_m}) \quad \mu \text{ a.s.}$$

$$\forall x \in E \quad \lim_n d(x, X_n \cdot \chi_{A_m}) = d(x, X_\infty . \chi_{A_m}) \quad \mu \text{ a.s.}$$

Now by applying Proposition 4.1 to each $X_n \cdot \chi_{A_m}$ we have

$$T_B - \lim_n E^{\mathcal{F}_n}(X_n \cdot \chi_{A_m}) = X_\infty \cdot \chi_{A_m} \quad \mu \text{ a.s.}$$

The fact that $(A_m)_{m\geq 1}$ is partition of Ω, it follows that

$$X_\infty = \sum_{m=1}^{+\infty} X_\infty \cdot \chi_{A_m} = T_B - \lim_n E^{\mathcal{F}_n}(X_n) \quad \mu \text{ a.s.}$$

□

Now given an increasing sequence $(\mathcal{F}_n)_{n\geq 1}$ of sub-σ-algebras and an integrable vector valued random variable X, then the Lévy's martingale convergence theorem asserts that $E^{\mathcal{F}_n}(X)$ converges to $E^{\mathcal{F}}(X)$ almost surely and in L^1, where $\mathcal{F}$ is the σ-algebra generated by $(\mathcal{F}_n)_{n\geq 1}$. Here, we get similar results for $cc(E)$-valued Pettis integrable random sets (dealing with linear topology) that extend some results proved by the authors (see Castaing-Ezzaki-Akhiat [1], Papageorgiou [22], Zhen-Xing [25]). Then if the sequence of random sets $(X_n)_{n\in\mathbf{N}^*}$ in the last theorem does not depend on n we obtain the following results:

Corollary 4.3. *Let X be a $cc(E)$-valued scalarly integrable random set such that $S_X^{Pe} \neq \emptyset$, and E' a strongly separable Banach space. Let $(\mathcal{F}_n)_{n\geq 1}$ be an increasing sequence of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma(\bigcup_{n\geq 1} \mathcal{F}_n)$. Assume that*

(i) $\sup_n E^{\mathcal{F}_n}(|X|) < +\infty \quad \mu$ *a.s.*

(ii) *Every countable subset of S_X is uniformly Pettis continuous.*

Then

$$T_B - \lim_n E^{\mathcal{F}_n}(X) = X \quad \mu \text{ a.s.}$$

Proof. Since X satisfies all conditions of Theorem 3.2, then for every $n \geq 1$, $E^{\mathcal{F}_n}(X)$ exists and by applying Lemma 3.4(1) for every $x' \in E'$ we have

$$\delta^*(x', E^{\mathcal{F}_n}(X)) = E^{\mathcal{F}_n}(\delta^*(x', X)) \qquad \mu \text{ a.s.}$$

Now set $Y = |X|$, then Y is a positive random variable and by applying (i) we have $\sup_n E^{\mathcal{F}_n}(Y) < +\infty$ μ a.s. Then by applying Theorem 4.2 to $X = X_\infty = X_n$ we have

$$T_B - \lim_n E^{\mathcal{F}_n}(X) = X \qquad \mu \text{ a.s.}$$ □

Remark 4.4. In the case of $cc(E)$-valued integrably bounded multifunctions the condition $\sup_n E^{\mathcal{F}_n}(|X|) < +\infty$ may be omitted and we obtain the following.

Corollary 4.5. *Assume that E' is strongly separable. Let $(\mathcal{F}_n)_n$ be an increasing sequence of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma(\bigcup_{n\geq 1} \mathcal{F}_n)$, and $X \in \mathcal{L}^1_{cc(E)}(\mathcal{F})$. Then we have*

$$T_B - \lim_n E^{\mathcal{F}_n}(X) = X \quad \mu \text{ a.s.}$$

Proof. From Theorem 5.1 in Hiai-Umegaki [17] for every $n \geq 1$, $E^{\mathcal{F}_n}(X)$ exists, and by applying the classical Lévy's theorem in L^1_R we have

$$\lim_n E^{\mathcal{F}_n}(|X|) = |X| < +\infty \qquad \mu \text{ a.s.}$$

Since convergent sequences are bounded, we have

$$\sup_n E^{\mathcal{F}_n}(|X|) < +\infty \quad \mu \text{ a.s.}$$

By the fact that E' is strongly separable the rest of the proof is similar to the proof of the previous corollary. □

Remark 4.6. Remark that if X is a $cwk(E)$-valued Pettis integrable multifunction, the separability of E' may be omitted and we have this corollary.

Corollary 4.7. *Let E be a separable Banach space, $(\mathcal{F}_n)_{n\geq 1}$ be an increasing sequence of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma(\bigcup_{n\geq 1} \mathcal{F}_n)$. Let X be a $cwk(E)$-valued scalarly integrable random set which is Pettis integrable in $cwk(E)$, and such that*

$$\sup_n E^{\mathcal{F}_n}(|X|) < +\infty \quad \mu \text{ a.s.}$$

Then

$$T_B - \lim_n E^{\mathcal{F}_n}(X) = X \quad \mu \; a.s.$$

Proof. From Proposition 5.3 and Theorem 5.4 in Amri Hess [10], we conclude that X satisfies all conditions of Theorem 3.2, then for every $n \geq 1$, $E^{\mathcal{F}_n}(X)$ exists and is $cwk(E)$-valued. So by Lemma 3.4(1) for every $x' \in E'$, we have

$$\delta^*(x', E^{\mathcal{F}_n}(X)) = E^{\mathcal{F}_n}(\delta^*(x', X)) \quad \mu \text{ a.s.}$$

If we set $Y = |X|$, then Y is a positive random variable and we have

$$\sup_n E^{\mathcal{F}_n}(Y) < +\infty \quad \mu \text{ a.s.}$$

Then by applying Theorem 4.2 to $X = X_\infty = X_n$ we have

$$T_B - \lim_n E^{\mathcal{F}_n}(X) = X \qquad \mu \text{ a.s.}$$

□

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