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ON UNICITY OF MEROMORPHIC SOLUTIONS OF DIFFERENTIAL-DIFFERENCE EQUATIONS

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Abstract. In this paper, we give a uniqueness theorem on meromorphic solutions f of finite order of a class of differential-difference equations such that solutions f are uniquely determined by their poles and two distinct values.

1. Introduction and main results

Let $\mathcal{M}(\mathbb{C})$ be the fields of meromorphic functions on the complex plane $\mathbb{C}$ and let $\mathbb{Z}_+$ (resp., $\mathbb{Z}^+$) denote the set of non-negative (resp., positive) integers. Take two integers $m \in \mathbb{Z}_+,\ n \in \mathbb{Z}^+$ and take n multi-indexes

$$\mathbf{j}_k = (j_{k0}, \ldots, j_{km}) \in \mathbb{Z}_+^{m+1},\ k = 1, \ldots, n$$

associated to n elements

$$\mathbf{c}_k = (c_{k0}, \ldots, c_{km}) \in \mathbb{C}^{m+1},\ k = 1, \ldots, n.$$

We define a differential-difference operator $D : \mathcal{M}(\mathbb{C}) \longrightarrow \mathcal{M}(\mathbb{C})$ as follows:

$$Df = \sum_{k=1}^{n} a_k f_{c_{k0}}^{j_{k0}} (f'_{c_{k1}})^{j_{k1}} \cdots (f_{c_{km}}^{(m)})^{j_{km}}, \tag{1.1}$$

where $a_k \in \mathcal{M}(\mathbb{C}) - \{0\}$ for each $k \in \{1, \ldots, n\}$, and where the function f_c associated to $f \in \mathcal{M}(\mathbb{C})$ and a constant c is defined by

$$f_c(z) = f(c + z),\ z \in \mathbb{C}.$$

Further, take two coprime polynomials over $\mathcal{M}(\mathbb{C})$

$$P(w) = \sum_{i=0}^{p} b_i w^i,\ Q(w) = \sum_{l=0}^{q} d_l w^l \tag{1.2}$$

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with $b_p d_q \neq 0$. We will study admissible meromorphic solutions of the differential-difference equation

$$Df = \frac{P(f)}{Q(f)}. \tag{1.3}$$

A meromorphic solution f of (1.3) is said to be *admissible* if f is non-constant such that the Nevanlinna's characteristic functions of f, a_k, b_i, d_l satisfy

$$\sum_{k=1}^{n} T(r, a_k) + \sum_{i=0}^{p} T(r, b_i) + \sum_{l=0}^{q} T(r, d_l) = S(r, f), \tag{1.4}$$

where $S(r, f)$ denotes any function of r with the following property

$$S(r, f) = o(T(r, f)) \tag{1.5}$$

for all r outside of a possible exceptional set with finite logarithmic measure.

If (1.3) is only a differential equation, that is, $\mathbf{c}_1 = \cdots = \mathbf{c}_n = 0$, the general Malmquist's theorem shows that if (1.3) has an admissible meromorphic solution f, then we must have

$$q = 0, \ p \leq \max_{1 \leq k \leq n} \lambda_k,$$

where

$$\lambda_k = \text{Weight}(\mathbf{j}_k) := j_{k0} + 2j_{k1} + \cdots + (m+1)j_{km}. \tag{1.6}$$

More results related to this topic are referred to Tu [6], Brosch [1], Yang [7].

However, if (1.3) contains really differences, that is, $\mathbf{c}_k \neq 0$ for some k, there are different results. For example, Li [3] notes that (1.3) has admissible meromorphic solutions (or see Remarks below). Some works related to the topics are referred to [2], [7].

Write

$$H[f] = Q(f)Df - P(f), \ \Lambda = \sum_{k=1}^{n} \lambda_k.$$

In this paper, we prove the following main theorem:

Theorem 1.1. *Let f be an admissible meromorphic solution of* (1.3) *and further assume that the order of f is finite. Suppose that $p \leq q = \Lambda$ and take two distinct complex numbers e_1, e_2 with*

$$H[e_1] \neq 0, \ H[e_2] \neq 0.$$

If $g \in \mathcal{M}(\mathbb{C})$ and f share the values e_1, e_2 and ∞ CM, then $f = g$.

By definition, f and g are said to share a value e CM if $f^{-1}(e) = g^{-1}(e)$ counting multiplicity. For the special case $m = 0$, $\mathbf{j}_1 = \cdots = \mathbf{j}_n = 1$, Lü, Han and Lü [5] proved Theorem 1.1 by applying main ideas due to [1].

Remark 1.2. The number of shared values in Theorem 1.1 cannot be reduced. For example, define a differential-difference operator $D : \mathcal{M}(\mathbb{C}) \longrightarrow \mathcal{M}(\mathbb{C})$ as follows:

$$Df = f'_c + f'_{c'}$$

with $c = \frac{\pi}{4}, c' = -\frac{\pi}{4}$, and take

$$P(f) = 4(f^2 + 1)^2,\ Q(f) = (f^2 - 1)^2.$$

Obviously, we have

$$H[\pm 1] = -16,\ p = q = \Lambda = 4.$$

Equation (1.3) has an admissible meromorphic solution $f(z) = \frac{1}{\tan z}$ of order 1. However, the solution f and a different meromorphic function $g(z) = \tan z$ share two values ± 1 CM.

Remark 1.3. The condition $H[e_1] \neq 0, H[e_2] \neq 0$ cannot be dropped. Take in (1.3)

$$Df = f'_c,\ P(f) = 2 + 2f^2,\ Q(f) = (f - 1)^2$$

with $c = \frac{\pi}{4}$, $p = q = \Lambda = 2$, $H[\pm i] = 0$. Equation (1.3) has an admissible meromorphic solution $f(z) = \tan z$ of order 1 such that $f(z)$ and $g(z) = -\tan z$ share the values $\pm i$ and ∞ CM, but $f \neq g$.

Remark 1.4. The condition $p \leq q$ is sharp in the following meanings. Take in (1.3)

$$Df = f_c f'_{c'},\ P(f) = f^2,\ Q(f) = 1$$

with $c = -1, c' = 1$, $p = 2$, $q = 0$, $\Lambda = 3$, $H[\pm 1] = -1$. Equation (1.3) has an admissible entire solution $f(z) = e^z$ of order 1 such that $f(z)$ and $g(z) = e^{-z}$ share the values ± 1 and ∞ CM, but $f \neq g$.

Remark 1.5. The condition $q = \Lambda$ is necessary. Take in (1.3)

$$Df = -e^2 f_c + f'_{c'},\ P(f) = -e^2,\ Q(f) = 1$$

with $c = -1, c' = 1$, $p = 0$, $q = 0$, $\Lambda = 3$, $H[0] = e^2$, $H[2] = -e^2$. Equation (1.3) has an admissible meromorphic solution $f(z) = e^z + 1$ of order 1 such that $f(z)$ and $g(z) = e^{-z} + 1$ share the values 0, 2 and ∞ CM, but $f \neq g$.

Remark 1.6. The assumption that f is of finite order is necessary. Take in (1.3)

$$Df = f'_c - f'_{c'},\ P(f)(z) = 3e^z f(z) - 4e^z,\ Q(f) = f^4$$

with $e^c = -4, e^{c'} = -3, p = 1, q = \Lambda = 4$, $H[0](z) = 4e^z$, $H[e](z) = 4e^z - 3e^{z+1}$. Equation (1.3) has an admissible entire solution $f(z) = e^{e^z}$ of order ∞, $f(z)$ and $g(z) = e^{2-e^z}$ share the values $0, e$ and ∞ CM, but $f \neq g$.

2. Preliminary

We assume that the reader is familiar with the standard notations and fundamental results in Nevanlinna theory, Refer to the book [4].

The following lemma is referred to Lemma 2.4 and Lemma 2.5 in [3].

Lemma 2.1. *If f is a non-constant meromorphic function of finite order, then*

$$m\left(r, \frac{f_c^{(k)}}{f}\right) = S(r, f)$$

holds for $c \in \mathbb{C}, k \in \mathbb{Z}_+$.

Lemma 2.2. *Let f be an admissible meromorphic solution of finite order to the equation* (1.3)*. If $b \in \mathcal{M}(\mathbb{C})$ is a small function of f, that is,*

$$T(r, b) = S(r, f),$$

with $H[b] \neq 0$, then

$$m\left(r, \frac{1}{f-b}\right) = S(r, f). \tag{2.1}$$

Proof. Substituting $f = h + b$ into (1.3), we obtain

$$A[h] + H[b] = 0, \tag{2.2}$$

where

$$A[h] = H[h+b] - H[b] = \sum_{1 \le k_0, k_1, \ldots, k_m \le n} \sum_{\mathbf{i}} c_{\mathbf{i}} h_{c_{k_0}0}^{i_0} (h'_{c_{k_1}1})^{i_1} \cdots (h_{c_{k_m}m}^{(m)})^{i_m}$$

in which $\mathbf{i} = (i_0, \ldots, i_m)$ runs on a finite set of $\mathbb{Z}_+^{m+1} - \{0\}$, and $c_{\mathbf{i}}$ is a combination of a_k, b_i, d_l, $b_{c_{k_0}0}, \ldots, b_{c_{k_m}m}^{(m)}$ satisfying

$$T(r, c_{\mathbf{i}}) = S(r, f).$$

Then, when $|h(z)| \le 1$ with $|z| = r$, we obtain an estimate

$$\left|\frac{A[h](z)}{h(z)}\right| \le \sum_{1 \le k_0, k_1, \ldots, k_m \le n} \sum_{\mathbf{i}} |c_{\mathbf{i}}(z)| \left|\frac{h_{c_{k_0}0}(z)}{h(z)}\right|^{i_0} \cdots \left|\frac{h_{c_{k_m}m}^{(m)}(z)}{h(z)}\right|^{i_m}.$$

By using (2.2) and Lemma 2.1, it follows that

$$\begin{aligned} m\left(r, \frac{1}{f-b}\right) &= m\left(r, \frac{1}{h}\right) \le m\left(r, \frac{H[b]}{h}\right) + m\left(r, \frac{1}{H[b]}\right) \\ &= m\left(r, \frac{A[h]}{h}\right) + m\left(r, \frac{1}{H[b]}\right) = S(r, f) \end{aligned}$$

since $T(r, h) = T(r, f) + S(r, f)$. When $|h(z)| > 1$ with $|z| = r$, we know $m\left(r, \frac{1}{f-b}\right) = m\left(r, \frac{1}{h}\right) = S(r, f)$ is obvious. Hence Lemma 2.2 is proved. □

Lemma 2.3. *If f is an admissible meromorphic solution of finite order of the equation* (1.3) *with $p \leq q = \Lambda$, then we have*

$$m(r,f) = S(r,f). \tag{2.3}$$

Proof. Put

$$d = \max_{1\leq l\leq q}\left(1, 2\left|\frac{d_{q-l}}{d_q}\right|^{\frac{1}{l}}\right). \tag{2.4}$$

Take $z \in \mathbb{C}$ and write $z = re^{i\theta}$. Set

$$E_1 := \left\{\theta \in [0,2\pi) : \left|f(re^{i\theta})\right| \leq d(re^{i\theta})\right\},\ E_2 := [0,2\pi)\backslash E_1. \tag{2.5}$$

In the set E_1, we have the following estimate

$$\begin{aligned} |Df| &\leq \sum_{k=1}^{n}\left|a_k f^{j_{k0}+j_{k1}+\cdots+j_{km}}\right|\left|\frac{f_{c_{k0}}}{f}\right|^{j_{k0}}\cdots\left|\frac{f^{(m)}_{c_{km}}}{f}\right|^{j_{km}} \\ &\leq d^{\gamma}\sum_{k=1}^{n}|a_k|\left|\frac{f_{c_{k0}}}{f}\right|^{j_{k0}}\cdots\left|\frac{f^{(m)}_{c_{km}}}{f}\right|^{j_{km}}, \end{aligned} \tag{2.6}$$

where

$$\gamma = \max_{1\leq k\leq n}\{j_{k0}+j_{k1}+\cdots+j_{km}\}.$$

In the set E_2, noting that

$$|f| > d \geq 2\left|\frac{d_{q-l}}{d_q}\right|^{\frac{1}{l}},$$

and hence

$$\left|\frac{d_{q-l}}{d_q f^l}\right| \leq \frac{1}{2^l}$$

for $l = 1, \ldots, q$, which means

$$\begin{aligned} |Q(f)| &= \left|d_q f^q + d_{q-1}f^{q-1} + \cdots + d_1 f + d_0\right| \\ &\geq |d_q f^q|\left(1 - \sum_{l=1}^{q}\frac{|d_{q-l}|}{|d_q f^l|}\right) \geq \frac{|d_q|\,|f|^q}{2^q}, \end{aligned}$$

we also obtain an estimate

$$\begin{aligned} |Df| &= \left|\frac{P(f)}{Q(f)}\right| \leq \frac{2^q}{|d_q|\,|f|^q}\sum_{i=0}^{p}|b_i|\left|f^i\right| \\ &= \frac{2^q}{|d_q|}\sum_{i=0}^{p}|b_i|\left|f^{i-q}\right| \leq \frac{2^q}{|d_q|}\sum_{i=0}^{p}|b_i|. \end{aligned} \tag{2.7}$$

Combing (2.6) and (2.7), we obtain a complete estimate

$$|Df| \leq \frac{2^q}{|d_q|}\sum_{i=0}^{p}|b_i| + d^{\gamma}\sum_{k=1}^{n}|a_k|\left|\frac{f_{c_{k0}}}{f}\right|^{j_{k0}}\cdots\left|\frac{f^{(m)}_{c_{km}}}{f}\right|^{j_{km}},$$

which yields immediately

$$m(r, Df) \le (\gamma+1)m\left(r, \frac{1}{d_q}\right) + \sum_{i=0}^{p} m(r, b_i) + \gamma \sum_{l=0}^{q} m(r, d_l) + \sum_{k=1}^{n} m(r, a_k) + \sum_{k=1}^{n}\sum_{\nu=0}^{m} j_{k\nu} m\left(r, \frac{f_{c_{k\nu}}^{(\nu)}}{f}\right) + O(1).$$

Further, by using Lemma 2.1, it follows that

$$m(r, Df) = S(r, f)$$

since a_k, b_i, d_l are small functions of f.

Theorem 2.2 of Chiang and Feng [2] implies

$$N(r, f_{c_{k\nu}}) = N(r, f) + S(r, f)$$

which further yields

$$N(r, f_{c_{k\nu}}^{(\nu)}) \le (\nu+1)N(r, f_{c_{k\nu}}) = (\nu+1)N(r, f) + S(r, f).$$

It follows that

$$T(r, Df) = m(r, Df) + N(r, Df) \le \sum_{k=1}^{n}\sum_{\nu=0}^{m} j_{k\nu} N(r, f_{c_{k\nu}}^{(\nu)}) + S(r, f) \le \sum_{k=1}^{n}\sum_{\nu=0}^{m} (\nu+1) j_{k\nu} N(r, f) + S(r, f) = qN(r, f) + S(r, f).$$

On the other hand, it is well knew that

$$T(r, Df) = T\left(r, \frac{P(f)}{Q(f)}\right) = qT(r, f) + S(r, f).$$

Therefore, we have

$$m(r, f) = \frac{1}{q}\{T(r, Df) - qN(r, f)\} + S(r, f),$$

and hence Lemma 2.3 follows. □

3. Proof of Theorem 1.1: special cases $m \le 1$

Now (1.3) has the following form

$$Df = \sum_{k=1}^{n} a_k f_{c_{k0}}^{j_{k0}} (f'_{c_{k1}})^{j_{k1}} = \frac{P(f)}{Q(f)}. \tag{3.1}$$

Since f is an admissible meromorphic solution of (3.1), it follows that f must be non-constant. By using Theorem 5.25 in [8], we have

$$T(r, g) = \{1 + o(1)\}T(r, f)$$

as $r \to \infty$, which particularly implies that g and f have the same order. Hence, there exist two polynomials α, β satisfying

$$\frac{f - e_1}{g - e_1} = e^{\alpha}, \ \frac{f - e_2}{g - e_2} = e^{\beta}. \tag{3.2}$$

Assume, to the contrary, that $g \neq f$. Then we obtain easily

$$e^{\alpha} \neq 1, \ e^{\beta} \neq 1, \ e^{\alpha} \neq e^{\beta}$$

and

$$f = e_1 + (e_2 - e_1)\frac{e^{\beta} - 1}{e^{\gamma} - 1} = e_2 + (e_1 - e_2)\frac{e^{\alpha} - 1}{e^{-\gamma} - 1}, \tag{3.3}$$

where $\gamma = \beta - \alpha$ is not a constant (see Lemma 2.3). Thus one of α and β at least is not constant. Moreover, by Lemma 2.2 and the first main theorem of Nevanlinna, we have

$$T(r, f) = N\left(r, \frac{1}{f - e_j}\right) + S(r, f) \tag{3.4}$$

for $j = 1, 2$. If one of α and β is constant, it follows that

$$T(r, f) = S(r, f).$$

This is a contradiction. Hence α, β are not constants.

Further, we claim

$$d := \operatorname{ord}(f) = \deg \alpha = \deg \beta = \deg \gamma > 0. \tag{3.5}$$

The first main theorem due to Nevanlinna yields immediately

$$N\left(r, \frac{1}{e^{\alpha} - 1}\right) \leq T(r, e^{\alpha}) + O(1),$$

and the second main theorem applied to three values $0, 1, \infty$ implies

$$T(r, e^{\alpha}) = N\left(r, \frac{1}{e^{\alpha} - 1}\right) + S(r, e^{\alpha}).$$

Note that

$$T(r, e^{\alpha}) \leq T(r, f) + T(r, g) + O(1) \leq 2T(r, f) + S(r, f). \tag{3.6}$$

We obtain

$$T(r, e^{\alpha}) = N\left(r, \frac{1}{e^{\alpha} - 1}\right) + S(r, f). \tag{3.7}$$

Similarly, we can obtain

$$T(r, e^{\beta}) = N\left(r, \frac{1}{e^{\beta} - 1}\right) + S(r, f), \tag{3.8}$$

and

$$T(r, e^{\gamma}) = N\left(r, \frac{1}{e^{\gamma} - 1}\right) + S(r, f). \tag{3.9}$$

It follows from (3.9) that

$$T(r, e^{\gamma}) = N(r, f) + N\left(r, \frac{1}{\xi}\right) + S(r, f),$$

where ξ is an entire function determined by common zeros of $e^{\beta} - 1$ and $e^{\gamma} - 1$. By using Lemma 2.3, we see

$$T(r, e^{\gamma}) = T(r, f) + N\left(r, \frac{1}{\xi}\right) + S(r, f).$$

Note that (3.8) and (3.4) yield

$$\begin{aligned} T(r, e^{\beta}) &= N\left(r, \frac{1}{f - e_1}\right) + N\left(r, \frac{1}{\xi}\right) + S(r, f) \\ &= T(r, f) + N\left(r, \frac{1}{\xi}\right) + S(r, f). \end{aligned}$$

Therefore, we have

$$T(r, e^{\beta}) = T(r, e^{\gamma}) + S(r, f)$$

which means

$$\deg \beta = \operatorname{ord}(e^{\beta}) := \limsup_{r\to\infty} \frac{\log T(r, e^{\beta})}{\log r} = \operatorname{ord}(e^{\gamma}) = \deg \gamma > 0.$$

According to the arguments above, we can prove

$$T(r, e^{\alpha}) = T(r, e^{\gamma}) + S(r, f)$$

and hence

$$\deg \alpha = \deg \gamma.$$

Now (3.6) implies

$$\deg \alpha = \operatorname{ord}(e^{\alpha}) \leq \operatorname{ord}(f).$$

But, we also have

$$T(r, f) \leq T(r, e^{\alpha}) + 2T(r, e^{\beta}) + S(r, f) \leq 3T(r, e^{\alpha}) + S(r, f)$$

which means

$$\operatorname{ord}(f) \leq \operatorname{ord}(e^{\alpha}) = \deg \alpha.$$

The claim (3.5) is proved.

Substituting the representation (3.3) of f into (3.1), we have

$$\tag{3.10} \begin{aligned} &\sum_{i=0}^{p} b_i \left[e_1 + (e_2 - e_1)\frac{e^{\beta} - 1}{e^{\gamma} - 1}\right]^i = \sum_{l=0}^{q} d_l \left[e_1 + (e_2 - e_1)\frac{e^{\beta} - 1}{e^{\gamma} - 1}\right]^l \\ &\sum_{k=1}^{n} a_k \left[e_1 + (e_2 - e_1)\frac{e^{\beta_{c_{k0}}} - 1}{e^{\gamma_{c_{k0}}} - 1}\right]^{j_{k0}} \left[(e_2 - e_1)\left(\frac{e^{\beta_{c_{k1}}} - 1}{e^{\gamma_{c_{k1}}} - 1}\right)'\right]^{j_{k1}}. \end{aligned}$$

Write

$$\beta_{c_{k0}} = \beta + s_{k0},\ \gamma_{c_{k0}} = \gamma + t_{k0},\ \beta_{c_{k1}} = \beta + s_{k1},\ \gamma_{c_{k1}} = \gamma + t_{k1}$$

for $1 \leq k \leq n$, where $s_{k0}, t_{k0}, s_{k1}, t_{k1}$ are polynomials of degrees $\leq d-1$. Then (3.10) becomes the following form

$$\sum_{\mu=0}^{M}\sum_{v=0}^{N} a_{\mu,v} e^{\mu\beta+v\gamma} - \sum_{\mu=0}^{p}\sum_{v=0}^{2q} b_{\mu,v} e^{\mu\beta+v\gamma} = 0, \tag{3.11}$$

where $a_{\mu,v}$ (resp., $b_{\mu,v}$) are combinations of a_k, d_l, $e^{s_{k0}}$, $e^{t_{k0}}$, $e^{s_{k1}}$, $e^{t_{k1}}$ (resp., b_i, $e^{t_{k0}}$, $e^{t_{k1}}$) with polynomial coefficients (resp., constant coefficients) depending on β', γ', s'_{k1}, t'_{k1}, and where

$$M = q + \max_{1\leq k\leq n}\{j_{k0} + j_{k1}\}, \ N = 2q - \min_{1\leq k\leq n}\{j_{k1}\},$$

or further

$$\sum_{\mu=0}^{M}\sum_{v=0}^{2q} A_{\mu,v} e^{\mu\beta+v\gamma} = 0, \tag{3.12}$$

where $A_{\mu,v}$ are completely determined by $a_{\mu,v}, b_{\mu,v}$ or 0. Moreover, it is not difficult to show that

$$\begin{aligned} A_{0,0} &= H[e_2] \neq 0, \\ A_{0,2q} &= H[e_1] \prod_{k=1}^{n} e^{j_{k0}t_{k0}+2j_{k1}t_{k1}} \neq 0. \end{aligned} \tag{3.13}$$

We claim that

$$\deg(\mu\beta + v\gamma) = \deg(\mu\beta - v\gamma) = d \tag{3.14}$$

for $(\mu, v) \in \mathbb{Z}_+^2 - \{(0,0)\}$, which follows from (3.5) if one of μ and ν is zero.

Now we consider the cases $\mu v \neq 0$. First of all, assume, to the contrary, that $\deg(\mu\beta + v\gamma) < d$, so that the entire function $U_1 = e^{\mu\beta+v\gamma}$ is a small function of e^{α}. We have

$$T(r, U_1 e^{-\mu\alpha}) = T(r, e^{-\mu\alpha}) + S(r, e^{\alpha}) = \mu T(r, e^{\alpha}) + S(r, f).$$

On the other hand, we also have

$$\begin{aligned} T(r, U_1 e^{-\mu\alpha}) &= T\left(r, e^{(\mu+v)\gamma}\right) = (\mu + v)T(r, e^{\gamma}) \\ &= (\mu + v)T(r, e^{\alpha}) + S(r, f). \end{aligned}$$

So that $v = 0$. This is a contradiction. The first part of (3.14) is confirmed.

Next, assume, to the contrary, that $\deg(\mu\beta - v\gamma) < d$, so that the entire function $U_2 = e^{\mu\beta-v\gamma}$ is a small function of e^{α}. Thus if $\mu \geq v$, we have

$$T(r, U_2 e^{-\mu\alpha}) = T(r, e^{-\mu\alpha}) + S(r, e^{\alpha}) = \mu T(r, e^{\alpha}) + S(r, f).$$

On the other hand, we also have

$$\begin{aligned} T(r, U_2 e^{-\mu\alpha}) &= T\left(r, e^{(\mu-v)\gamma}\right) = (\mu - v)T(r, e^{\gamma}) \\ &= (\mu - v)T(r, e^{\alpha}) + S(r, f), \end{aligned}$$

and hence either $\mu = 0$ (if $\mu = v$) or $v = 0$ (if $\mu > v$). This is a contradiction.

If $v > \mu$, we can get $\mu = 0$ in the same way. This is also a contradiction. Therefore, the claim (3.14) is confirmed completely.

By using (3.14), we find that each $A_{\mu,v}$ satisfies the following estimate

$$T(r, A_{\mu,v}) = S\left(r, e^{(\mu_1\beta+v_1\gamma)-(\mu_2\beta+v_2\gamma)}\right) \tag{3.15}$$

for two distinct elements (μ_1, v_1) and (μ_2, v_2) in $\mathbb{Z}_+^2$. Thus, by Theorem 1.51 in [8], we have $A_{\mu,v} \equiv 0$. It contradicts to (3.13). Therefore, we complete the proof of Theorem 1.1.

4. Proof of Theorem 1.1: general cases

We can copy completely the procedure of proof in last section up to the claim (3.5). Now a change of proof is to substitute the representation (3.3) of f into the general equation (1.3), so that we obtain

$$\begin{aligned} \sum_{i=0}^{p} b_i\left[e_1 + (e_2 - e_1)\frac{e^{\beta}-1}{e^{\gamma}-1}\right]^i &= \sum_{l=0}^{q} d_l\left[e_1 + (e_2 - e_1)\frac{e^{\beta}-1}{e^{\gamma}-1}\right]^l \\ \sum_{k=1}^{n} a_k\left[e_1 + (e_2 - e_1)\frac{e^{\beta_{c_{k0}}}-1}{e^{\gamma_{c_{k0}}}-1}\right]^{j_{k0}} &\prod_{\nu=1}^{m}\left[(e_2 - e_1)\left(\frac{e^{\beta_{c_{k\nu}}}-1}{e^{\gamma_{c_{k\nu}}}-1}\right)^{(\nu)}\right]^{j_{k\nu}}. \end{aligned} \tag{4.1}$$

Write

$$\beta_{c_{k\nu}} = \beta + s_{k\nu},\ \gamma_{c_{k\nu}} = \gamma + t_{k\nu}$$

for $1 \le k \le n, 0 \le \nu \le m$, where $s_{k\nu}, t_{k\nu}$ are polynomials of degrees $\le d-1$. Then (4.1) becomes the following form

$$\sum_{\mu=0}^{M}\sum_{v=0}^{N} a_{\mu,v}e^{\mu\beta+v\gamma} - \sum_{\mu=0}^{p}\sum_{v=0}^{2q} b_{\mu,v}e^{\mu\beta+v\gamma} = 0, \tag{4.2}$$

where $a_{\mu,v}$ (resp., $b_{\mu,v}$) are combinations of a_k, d_l, $e^{s_{k\nu}}$, $e^{t_{k\nu}}$ (resp., b_i, $e^{t_{k\nu}}$) with polynomial coefficients (resp., constant coefficients) depending on derivatives $\beta_{c_{k\nu}}$ and $\gamma_{c_{k\nu}}$, and where

$$M = q + \max_{1\le k\le n}\{j_{k0} + j_{k1} + \cdots + j_{km}\},$$

$$N = 2q - \min_{1\le k\le n}\{j_{k1} + 2j_{k2} + \cdots + mj_{km}\},$$

or further

$$\sum_{\mu=0}^{M}\sum_{v=0}^{2q} A_{\mu,v}e^{\mu\beta+v\gamma} = 0, \tag{4.3}$$

where $A_{\mu,v}$ are completely determined by $a_{\mu,v}, b_{\mu,v}$ or 0. Moreover, it is not difficult to show that

$$
\begin{aligned}
&A_{0,0} = H[e_2] \neq 0, \\
&A_{0,2q} = H[e_1] \prod_{k=1}^{n} e^{j_{k0}t_{k0}+2j_{k1}t_{k1}+\cdots+(m+1)j_{km}t_{km}} \neq 0.
\end{aligned} \tag{4.4}
$$

Thus, according to the arguments in last section, we obtain a contradiction, so that the proof of Theorem 1.1 is completed.

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