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AN OPTIMAL CONSUMPTION AND INVESTMENT PROBLEM WITH CES UTILITY AND NEGATIVE WEALTH CONSTRAINTS

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Abstract. We investigate the optimal consumption and portfolio strategies of an agent who has a constant elasticity of substitution (CES) utility function under the negative wealth constraint. We use the martingale method to derive the closed-form solution, and we give some numerical implications.

1. Introduction

The continuous-time portfolio selection problem is one of the most interesting fields in mathematical finance/financial economics after the pioneer research works of Merton [10, 11]. There are two famous approaches to solve the optimization problem obtained from the optimal consumption and investment problem. One is the dynamic programming method ([6]) and the other is the martingale method ([7, 2]). In this paper we use the martingale approach to solve our optimization problem.

A borrowing constraint is one of the most important issues when we consider the portfolio selection problem with labor income. This constraint means that the agent is required to maintain a non-negative wealth level (see [5, 4, 1, 3, 9] etc). More general version of the borrowing constraint is a negative wealth constraint. This constraint means that the agent can borrow the partial amount of the future labor income, that is, the agent is required to maintain a negative wealth level which is determined by the negative wealth constraint ratio ν, for $\nu \in [0, 1]$ (see [13, 12] etc). In this paper we focus on the negative wealth constraint.

When we investigate the portfolio selection problem, we need to consider a utility function as an objective function. Generally the constant relative risk aversion (CRRA) utility function or the constant absolute risk aversion (CARA)

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utility function is used. Sometimes the Cobb-Douglas utility function (see [4, 13] etc) is considered when there are two goods (for example consumption and leisure). In this paper we apply the constant elasticity of substitution (CES) utility function which is the general version of the Cobb-Douglas utility function. Although the CES utility function is very important in economics, there are fewer research works (see [1, 8] etc) about portfolio selection with CES utility because it is difficult to handle the CES utility function.

In this paper we use the martingale method to obtain the closed-form solution to the optimal consumption and investment problem with CES utility and a negative wealth constraint. This paper is organized as follows. In Section 2 the financial market is described. Section 3 gives our main optimization problem with the closed-form solution. Also we supply the numerical results for our solutions. In Section 4 we concludes.

2. The Financial Market

We consider a continuous-time financial market and assume that two assets are traded. There are one riskless asset with constant interest rate $r > 0$ and one risky asset S_t, which follows the geometric Brownian motion $dS_t = \mu S_t dt + \sigma S_t dB_t$, where $\mu > r$ and $\sigma > 0$ are assumed to be constant, and B_t is a standard Brownian motion on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\{\mathcal{F}_t\}_{t\geq 0}$ be the augmentation under $\mathbb{P}$ of the filtration generated by the standard Brownian motion $\{B_t\}_{t\geq 0}$.

Let $\boldsymbol{\pi} := \{\pi_t\}_{t\geq 0}$ be the dollar-amount of money can be invested in the risky asset S_t and $\boldsymbol{c} := \{c_t\}_{t\geq 0}$ be the nonnegative consumption rate process of the agent. They are measurable processes adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ and satisfy the integrability conditions:

$$\int_0^t \pi_s^2 ds < \infty, \text{ and } \int_0^t c_s ds < \infty, \text{ for all } t \geq 0, \text{ a.s.}$$

We also consider the leisure rate process $l_t \geq 0$. In order to obtain the closed-form solutions, we assume that the leisure rate process l_t is equal to a constant L. Let $I > 0$ be the agent's constant labor income stream. Thus the agent's wealth process x_t evolves according to

$$dx_t = [rx_t + \pi_t(\mu - r) - c_t + I]\, dt + \sigma \pi_t dB_t, \tag{1}$$

with an initial endowment $x_0 = x$.

We assume that the agent has a utility function of constant elasticity of substitution (CES) type of consumption and leisure

$$u(c,l) := \frac{\{\alpha c^\rho + (1-\alpha) l^\rho\}^{\frac{1-\gamma}{\rho}}}{\alpha(1-\gamma)}, \ \rho < 1, \ \rho \neq 0 \ , 0 < \alpha < 1, \gamma > 0 \text{ and } \gamma \neq 1, \tag{2}$$

where $1/(1-\rho)$ is the elasticity of substitution between consumption and leisure, α is a share parameter of consumption's contribution to the agent's utility, and

γ is the agent's coefficient of relative risk aversion. Also we assume that the leisure rate process is constant i.e. $l_t = L$ for all $t \geq 0$. Under the constant leisure L, the CES utility function (2) can be rewritten as

$$u_L(c) := u(c, L) = \frac{\{\alpha c^\rho + (1-\alpha)L^\rho\}^{\frac{1-\gamma}{\rho}}}{\alpha(1-\gamma)}. \tag{3}$$

Also we assume that there is a negative wealth constraint ([13, 12]) given by

$$x_t \geq -\nu \frac{I}{r}, \quad \text{for all} \quad t \geq 0 \quad \text{and} \quad \nu \in [0,1]. \tag{4}$$

This constraint means that the agent can borrow partially against her future labor income. Especially, if $\nu = 1$, then she can borrow fully against her future labor income, and $\nu = 0$ means that she cannot borrow against her future labor income, which is called the borrowing constraint. The market price of risk and the state price density are defined, respectively, as

$$\theta := \frac{\mu - r}{\sigma} \text{ and } H_t := \exp\left\{-\left(r + \frac{1}{2}\theta^2\right)t - \theta B_t\right\}.$$

From the wealth process (1), we derive the budget constraint as follows:

$$\mathbb{E}\left[\int_0^\infty H_t(c_t - I)dt\right] \leq x. \tag{5}$$

3. The Optimization Problem with Negative Wealth Constraints

Now we consider the optimization problem as follows:

$$V(x) = \sup_{(c,\pi)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^\infty e^{-\beta t} u_L(c_t)dt\right] \tag{6}$$

with the negative wealth constraint (4) and the budget constraint (5). Here $\beta > 0$ is a subjective discount factor, $\mathcal{A}(x)$ is the class of all admissible controls (c, π), and the utility $u_L(c_t)$ is a CES utility function defined in (3).

Using a Lagrange mutiplier $\lambda > 0$, we define a dual value function as follow:

$$\begin{aligned}
\tilde{V}(\lambda) &= \sup_c \mathbb{E}\left[\int_0^\infty e^{-\beta t} u_L(c_t)dt - \lambda \int_0^\infty H_t(c_t - I)dt\right] \\
&= \sup_c \mathbb{E}\left[\int_0^\infty e^{-\beta t}\left\{u_L(c_t) - \lambda e^{\beta t} H_t(c_t - I)\right\}dt\right] \\
&= \mathbb{E}\left[\int_0^\infty e^{-\beta t}\left(\tilde{u}_L(y_t) + I y_t\right)dt\right],
\end{aligned}$$

where $\tilde{u}_L(\cdot)$ is the dual utility function of the CES utility function and $y_t := \lambda e^{\beta t} H_t$. The dual utility $\tilde{u}_L(y)$ is defined by as follows:

$$\tilde{u}_L(y) = \sup_c \{u_L(c) - yc\} = u_L(c^*) - yc^*,$$

where c^* satisfies the equation

$$(c^*)^{\rho-1}\left\{\alpha(c^*)^\rho+(1-\alpha)L^\rho\right\}^{\frac{1-\gamma-\rho}{\rho}}=y.$$

Remark 1. For later use, we define a quadratic equation,

$$f(m):=\frac{1}{2}\theta^2m^2+\left(\beta-r-\frac{1}{2}\theta^2\right)m-\beta=0, \tag{7}$$

with two roots $m_+>1$ and $m_-<0$.

Now we define a function

$$\phi(t,y):=\mathbb{E}\left[\int_t^\infty e^{-\beta s}\left(\tilde{u}_L(y_s)+Iy_s\right)ds\,\middle|\,y_t=y\right].$$

By Feymann-Kac formula, we derive the partial differential equation (PDE) as follows:

$$\mathcal{L}\phi(t,y)+e^{-\beta t}\left(\tilde{u}_L(y)+Iy\right)=0,$$

where the partial differential operator is given by

$$\mathcal{L}:=\frac{\partial}{\partial t}+(\beta-r)y\frac{\partial}{\partial y}+\frac{1}{2}\theta^2y^2\frac{\partial^2}{\partial y^2}.$$

Let $\phi(t,y)=e^{-\beta t}v(y)$, then we obtain the following ODE with respect to y,

$$\frac{1}{2}\theta^2y^2v''(y)+(\beta-r)yv'(y)-\beta v(y)+\tilde{u}_L(y)+Iy=0. \tag{8}$$

Theorem 3.1. *The value function of our optimization problem* (6) *is given by*

$$\begin{aligned}V(x)=&\,C_1(y^*)^{m_+}-\frac{2}{\theta^2(m_+-1)(m_--1)}Iy^*+y^*x\\&+\frac{2}{\theta^2(m_+-m_-)}\left\{(y^*)^{m_-}\int_0^{y^*}z^{-1-m_-}\tilde{u}_L(z)dz\right.\\&\left.+(y^*)^{m_+}\int_{y^*}^{\hat{y}}z^{-1-m_+}\tilde{u}_L(z)dz+\frac{\hat{y}^{1-m_+}I}{1-m_+}(y^*)^{m_+}\right\},\end{aligned}$$

where $\hat{y}$ satisfies the following algebraic equation

$$\tilde{u}_L(\hat{y})=-m_-\hat{y}^{m_-}\int_0^{\hat{y}}z^{-1-m_-}\tilde{u}_L(z)dz+\left(\frac{\theta^2(m_+-1)\nu}{2r}+\frac{1}{m_--1}\right)I\hat{y},$$

$$\begin{aligned}C_1=&-\frac{2m_-\hat{y}^{m_--m_+}}{\theta^2m_+(m_+-m_-)}\int_0^{\hat{y}}z^{-1-m_-}\tilde{u}_L(z)dz\\&+\left(\frac{\nu}{r}+\frac{2m_-}{\theta^2(m_+-m_-)(m_--1)}\right)\frac{I}{m_+}\hat{y}^{1-m_+},\end{aligned}$$

and y^* is determined from the following algebraic equation

$$x = -m_+ C_1 (y^*)^{m_+ - 1} + \frac{2}{\theta^2(m_+ - 1)(m_- - 1)} I$$
$$- \frac{2}{\theta^2(m_+ - m_-)} \left\{ m_-(y^*)^{m_- - 1} \int_0^{y^*} z^{-1-m_-} \tilde{u}_L(z) dz \right.$$
$$\left. + m_+(y^*)^{m_+ - 1} \int_{y^*}^{\hat{y}} z^{-1-m_+} \tilde{u}_L(z) dz + \frac{m_+ \hat{y}^{1-m_+} I}{1 - m_+} (y^*)^{m_+ - 1} \right\}.$$

Proof. We use the method of variation of parameters to derive the solution to the Cauchy equation (8) as follows:

$$v(y) = C_1 y^{m_+} + \frac{2}{\theta^2(m_+ - m_-)} \left\{ y^{m_-} \int_0^y z^{-1-m_-} \tilde{u}_L(z) dz \right.$$
$$\left. + y^{m_+} \int_y^{\hat{y}} z^{-1-m_+} \tilde{u}_L(z) dz + \frac{\hat{y}^{1-m_+}}{1 - m_+} I y^{m_+} \right\}$$
$$- \frac{2}{\theta^2(m_+ - 1)(m_- - 1)} I y,$$

where $m_+ > 1$ and $m_- < 0$ are two roots of the quadratic equation (7), and $\hat{y} > 0$ is the dual variable level corresponding to the negative wealth level $\nu I/r$. The negative wealth constraint (4) implies two free boundary conditions (see Dybvig and Liu [3] and Lim and Shin [9])

$$v'(\hat{y}) = \nu \frac{I}{r}, \quad v''(\hat{y}) = 0. \tag{9}$$

The boundary conditions (9) implies that $\hat{y}$ is the solution to the following algebraic equation

$$\tilde{u}_L(\hat{y}) = -m_- \hat{y}^{m_-} \int_0^{\hat{y}} z^{-1-m_-} \tilde{u}_L(z) dz + \left(\frac{\theta^2(m_+ - 1)\nu}{2r} + \frac{1}{m_- - 1} \right) I \hat{y}$$

and

$$C_1 = -\frac{2 m_- \hat{y}^{m_- - m_+}}{\theta^2 m_+ (m_+ - m_-)} \int_0^{\hat{y}} z^{-1-m_-} \tilde{u}_L(z) dz$$
$$+ \left(\frac{\nu}{r} + \frac{2 m_-}{\theta^2(m_+ - m_-)(m_- - 1)} \right) \frac{I}{m_+} \hat{y}^{1-m_+}.$$

Now we use the Legendre inverse transform formula,

$$V(x) = \inf_{y>0} \{ v(y) + yx \}$$

to derive the value function

$$\begin{aligned} V(x) &= C_1(y^*)^{m_+} - \frac{2}{\theta^2(m_+-1)(m_--1)} I y^* + y^* x \\ &\quad + \frac{2}{\theta^2(m_+-m_-)} \left\{ (y^*)^{m_-} \int_0^{y^*} z^{-1-m_-} \tilde{u}_L(z) dz \right. \\ &\qquad \left. + (y^*)^{m_+} \int_{y^*}^{\hat{y}} z^{-1-m_+} \tilde{u}_L(z) dz + \frac{\hat{y}^{1-m_+} I}{1-m_+} (y^*)^{m_+} \right\}, \end{aligned}$$

where y^* is determined from the following algebraic equation

$$\begin{aligned} x &= -m_+ C_1 (y^*)^{m_+-1} + \frac{2}{\theta^2(m_+-1)(m_--1)} I \\ &\quad - \frac{2}{\theta^2(m_+-m_-)} \left\{ m_-(y^*)^{m_--1} \int_0^{y^*} z^{-1-m_-} \tilde{u}_L(z) dz \right. \\ &\qquad \left. + m_+(y^*)^{m_+-1} \int_{y^*}^{\hat{y}} z^{-1-m_+} \tilde{u}_L(z) dz + \frac{m_+ \hat{y}^{1-m_+} I}{1-m_+} (y^*)^{m_+-1} \right\}. \end{aligned}$$

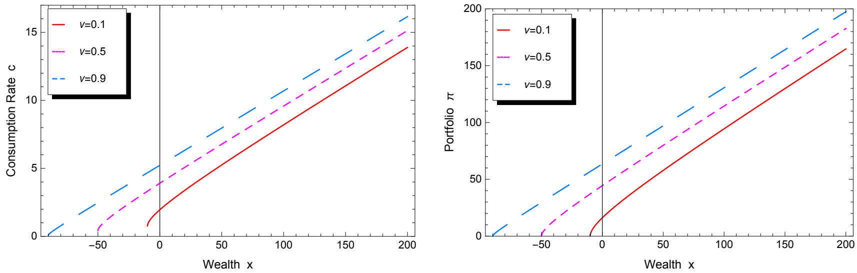
□

Theorem 3.2. *The optimal consumption c_t^* and portfolio π_t^* are given by*

$$c_t^* = \xi_t$$

and

$$\begin{aligned} \pi_t^* &= \frac{\theta}{\sigma} y_t^* v''(y_t^*) \\ &= \frac{\theta}{\sigma} \Bigg[m_+(m_+-1) C_1 (y_t^*)^{m_+-1} \\ &+ \frac{2}{\theta^2(m_+-m_-)} \Bigg\{ m_-(m_--1)(y_t^*)^{m_--1} \int_0^{y_t^*} z^{-1-m_-} \tilde{u}_L(z) dz \\ &+ m_+(m_+-1)(y_t^*)^{m_+-1} \int_{y_t^*}^{\hat{y}} z^{-1-m_+} \tilde{u}_L(z) dz - (m_+-m_-)(y_t^*)^{-1} \tilde{u}_L(y_t^*) \\ &- m_+ I \hat{y}^{1-m_+} (y_t^*)^{m_+-1} \Bigg\} \Bigg], \end{aligned}$$



(a) Optimal consumption ($\beta = 0.07,\ r = 0.01,\ \mu = 0.05,\ \sigma = 0.2,\ \gamma = 2,\ \alpha = 0.5,\ L = 0.5,\ I = 1,\ \rho \to 0$)

(b) Optimal portfolio ($\beta = 0.07,\ r = 0.01,\ \mu = 0.05,\ \sigma = 0.2,\ \gamma = 2,\ \alpha = 0.5,\ L = 0.5,\ I = 1,\ \rho \to 0$)

FIGURE 1. Optimal consumption/portfolio with various values of ν

where y_t^* and ξ_t satisfy the following algebraic equations

$$x_t = -m_+ C_1 (y_t^*)^{m_+-1} + \frac{2}{\theta^2(m_+-1)(m_--1)} I - \frac{2}{\theta^2(m_+-m_-)} \Big\{ m_-(y_t^*)^{m_--1} \int_0^{y_t^*} z^{-1-m_-} \tilde{u}_L(z) dz + m_+(y_t^*)^{m_+-1} \int_{y_t^*}^{\hat{y}} z^{-1-m_+} \tilde{u}_L(z) dz + \frac{m_+ \hat{y}^{1-m_+} I}{1-m_+} (y_t^*)^{m_+-1} \Big\}$$

and

$$(\xi_t)^{\rho-1} \left\{\alpha(\xi_t)^\rho + (1-\alpha)L^\rho\right\}^{\frac{1-\gamma-\rho}{\rho}} = y_t^*,$$

respectively.

Remark 2 (Numerical Implications).
From Figures 1(a) and 1(b), we obtain the optimal consumption and investment under various values of ν. We see that the optimal consumption and investment become lower, as ν decreases, i.e., the negative wealth constraint becomes tighter. This is because the strong constraint limits the financial behavior of the agent much more.

4. Concluding Remarks

We consider the optimal consumption and portfolio selection problem with CES utility and a negative wealth constraint. We obtain the closed-form solution using the martingale approach and give some numerical results. In this paper, basically we extend the works of Merton [10, 11] with CES utility and a negative wealth constraint. Also it is meaningful to consider this problem with a voluntary retirement option as a future research work.

References

[1] K.J. Choi, G. Shim and Y.H. Shin, *Optimal Portfolio, Consumption-Leisure and Retirement Choice Problem with CES Utility*, Mathematical Finance **18** (2008), no. 3, 445–472.

[2] J.C. Cox and C.F. Huang, *Optimum Consumption and Portfolio Policies When Asset Prices Follow a Diffusion Process*, Journal of Economic Theory **49** (1989), no. 1, 33–83.

[3] P.H. Dybvig and H. Liu, *Lifetime Consumption and Investment: Retirement and Constrained Borrowing*, Journal of Economic Theory **145** (2010), no. 3, 885–907.

[4] E. Farhi and S. Panageas, *Saving and Investing for Early Retirement: A Theoretical Analysis*, Journal of Financial Economics **83** (2007), no. 1, 87–121.

[5] H. He and H. Pagés, *Labor Income, Borrowing Constraints, and Equilibrium Asset Prices*, Economic Theory **3** (1993), no. 4, 663–696.

[6] I. Karatzas, J.P. Lehoczky, S.P. Sethi and S.E. Shreve, *Explicit Solution of a General Consumption/Investment Problem*, Mathematics of Operations Research **11** (1986), no. 2, 261–294.

[7] I. Karatzas, J.P. Lehoczky and S.E. Shreve, *Optimal Portfolio and Consumption Decisions for a "Small Investor" on a Finite Horizon*, SIAM Journal on Control and Optimization **25** (1987), no. 6, 1557–1586.

[8] H.-S. Lee and Y.H. Shin, *An optimal portfolio, consumption-leisure and retirement choice problem with CES utility: a dynamic programming approach*, Journal of Inequalities and Applications **319** (2015), 13 pages.

[9] B.H. Lim and Y.H. Shin, *Optimal Investment, Consumption and Retirement Decision with Disutility and Borrowing Constraints*, Quantitative Finance **11** (2011), no. 10, 1581–1592.

[10] R.C. Merton, *Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case*, Review of Economics and Statistics **51** (1969), no. 3, 247–257.

[11] R.C. Merton, *Optimum Consumption and Portfolio Rules in a Continuous-Time Model*, Journal of Economic Theory **3** (1971), no. 4, 373–413.

[12] K. Park, M. Kang and Y.H. Shin, *An Optimal Consumption, Leisure, and Investment Problem with an Option to Retire and Negative Wealth Constraints*, Chaos, Solitons and Fractals **103** (2017), 374–381.

[13] S. Park and B.-G. Jang, *Optimal retirement strategy with a negative wealth constraint*, Operations Research Letters **42** (2014), 208–212.

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