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THE AVERAGING VALUE OF A SAMPLING OF THE RIEMANN ZETA FUNCTION ON THE CRITICAL LINE USING POISSON DISTRIBUTION

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Abstract. We investigate the averaging value of a random sampling $\zeta(1/2+iX_t)$ of the Riemann zeta function on the critical line. Our result is that if X_t is an increasing random sampling with Poisson distribution, then

$$\mathbb{E}\zeta(1/2+iX_t) = O(\sqrt{\log t}),$$

for all sufficiently large t in $\mathbb{R}$.

1. Introduction

The behaviour of the Riemann zeta function on the critical strip is one of the most important subjects in Number theory. Especially, the behaviour of the Riemann zeta function along the $\mathrm{Re}(z) = \frac{1}{2}$ has received many Number theorists' attention. The famous conjecture about the behaviour along $\mathrm{Re}(z) = \frac{1}{2}$ is known as Lindelöf Hypothesis that the absolute value of $\zeta(\frac{1}{2}+it)$ is less than t^{ϵ} as $t \to \infty$. (cf. [3], [4])

To overcome difficulties about estimations of $\zeta(\frac{1}{2}+it)$, there are various attempts using probabilitic theory. Lifshits and Weber [2] researched the behaviour of the Riemann zeta function $\zeta(\frac{1}{2}+it)$, when t is sampled by the Cauchy random walk. After that, Jo and Yang [1] studied the behaviour of the Riemann zeta function $\zeta(\frac{1}{2}+it)$, when t is sampled by the Gamma distribution. In this paper, we study the behaviour of the Riemann zeta function $\zeta(s)$ along the critical strip $s = 1/2+it$, when t is sampled by the Poisson distribution.

The following is the main result.

Theorem 1.1. *Let X_t denote the Poisson process with $\mathbb{E}(X_t) = t$ and $\mathrm{Var}(X_t) = t$. Then for all sufficiently large t,*

$$\mathbb{E}\zeta(1/2+iX_t) = O(\sqrt{\log t}).$$

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Because the Poisson process is increasing with mean value t and its variance t, we can use this process to explain the behaviour of $\zeta(\frac{1}{2}+it)$ as $t\to\infty$. The definition and basic properties of the Poisson process are explained in the next section. In this paper, we use the Landau notation $f=O(g)$, which means that $|f(x)|\le Cg(x)$ for some constant C and the Vinogradov notation $f\ll g$ which is equivalent to $f=O(g)$.

2. Preliminaries

2.1. Poisson process

The Poisson distribution is the discrete probability distribution of the number of events that occur in an interval time period.

It is said that a discrete random variable X_t has a Poisson distribution with parameter $t>0$ if the probability mass function of X is given by

$$P(X_t=k)=\frac{t^k e^{-t}}{k!}$$

for $k=0,1,2,\cdots$.

We can get that its average value is $\mathbb{E}(X_t)=\sum_{k=0}^{\infty}k\frac{t^k e^{-t}}{k!}=t$, and its variance is $\mathbb{E}|X_t|^2-|\mathbb{E}X_t|^2=t$. And it is well known that the characteristic function has the following form:

$$\varphi_{X_t}(u):=\mathbb{E}(e^{iuX_t})=\exp(t(e^{iu}-1)). \tag{1}$$

3. Proof of Theorem

First, we prove an analytic continuation of the Riemann zeta function as follows:

Lemma 3.1. *Let $\{u\}$ be the fractional part of u. Then for $0<\sigma<1$, we have*

$$\zeta(s)=1-\int_0^1 u^{-s}du+\int_1^{\infty}\{u\}\frac{d}{du}u^{-s}du.$$

Proof. Note that for $\sigma>1$,

$$\zeta(s)=1+\int_1^{\infty}u^{-s}d[u]=1+\int_1^{\infty}u^{-s}du-\int_1^{\infty}u^{-s}d\{u\}.$$

Using an integration by parts, we get an analytic continuation of $\zeta(s)$ into the half-plane $\sigma>0$ as follows:

$$\zeta(s)=1+\left[\frac{u^{1-s}}{1-s}\right]_1^{\infty}+\int_1^{\infty}\{u\}\frac{d}{du}u^{-s}du=1-\frac{1}{1-s}+\int_1^{\infty}\{u\}\frac{d}{du}u^{-s}du.$$

$\square$

From this, we estimate the mean value of the sampling of the Riemann zeta function using Poisson distribution.

Proof of Theorem 1.1. By Lemma 3.1 and the fact $\mathbb{E}(e^{iuX_t}) = \exp(t(e^{iu} - 1))$, we have that

$$\begin{aligned}
&\mathbb{E}\zeta(1/2 + iX_t) \\
&= 1 - \int_0^1 u^{-1/2} \exp(t(e^{-i\log u} - 1))du + \int_1^\infty \{u\} \frac{d}{du}\left(u^{-1/2}\exp(t(e^{-i\log u} - 1))\right) du \\
&= 1 - \int_0^1 u^{-1/2} \exp(t(u^{-i} - 1))du + \int_1^\infty \{u\} \frac{d}{du}\left(u^{-1/2}\exp(t(u^{-i} - 1))\right) du \\
&=: 1 + A + B.
\end{aligned}$$

(i) We estimate the integral A.

Note that

$$\frac{d}{du}\exp(t(u^{-i} - 1)) = -i\exp(t(u^{-i} - 1))tu^{-i-1}. \tag{2}$$

From an integration by part, we have

$$\begin{aligned}
&\int_0^1 u^{-1/2} \exp(t(u^{-i} - 1))du \\
&= \frac{i}{t}\int_0^1 u^{1/2+i} \frac{d}{du}\exp(t(u^{-i} - 1))du \\
&= \frac{i}{t}\Big[u^{1/2+i}\exp(t(u^{-i} - 1))\Big]_0^1 - \frac{i/2 - 1}{t}\int_0^1 u^{-1/2+i}\exp(t(u^{-i} - 1))du \\
&= \frac{i}{t} - \frac{i/2 - 1}{t}\int_0^1 u^{-1/2+i}\exp(t(u^{-i} - 1))du = O\left(t^{-1}\right).
\end{aligned}$$

(ii) We consider the integral B.

Note that

$$\exp(t(u^{-i} - 1)) = \exp\Big(t\big(\cos(\log u) - 1 - i\sin(\log u)\big)\Big).$$

If, for all $m \in \mathbb{Z}$,

$$|\log u - 2\pi m| \geq \frac{\sqrt{2\log t}}{\sqrt{t}},$$

then

$$|\exp(t(u^{-i} - 1))| = \exp\Big(t\big(\cos(\log u) + 1\big)\Big) \ll \exp\left(-t\frac{\log t}{t}\right) = t^{-1}, \tag{3}$$

because

$$\cos\left(\frac{\sqrt{2\log t}}{\sqrt{t}}\right) = 1 - \frac{1}{2}\left(\frac{\sqrt{2\log t}}{\sqrt{t}}\right)^2 + O\Big((\log t)^2/t^2\Big).$$

Let

$$S=\bigcup_{m=0}^{\infty}\left\{u\in\mathbb{R}\;\middle|\; u\geq 1,\ |\log u-2\pi m|<\frac{\sqrt{2\log t}}{\sqrt{t}}\right\}.$$

We divide B into two parts.

$$\begin{aligned}B&=\int_S\{u\}\frac{d}{du}\left(u^{-1/2}\exp(t(u^{-i}-1))\right)du+\int_{S^C}\{u\}\frac{d}{du}\left(u^{-1/2}\exp(t(u^{-i}-1))\right)du\\&=: B_1+B_2\end{aligned}$$

case 1) First, we consider the integral for S^C.

From (3), we can get

$$\begin{aligned}B_2&=\int_{S^C}\{u\}\left(-\frac{1}{2}-itu^{-i}\right)u^{-3/2}\exp(t(u^{-i}-1))du\\&\ll t\int_1^{\infty}u^{-\frac{3}{2}}\frac{1}{t}du=O(1).\end{aligned}$$

case 2) The integral for S is the following:

$$B_1=\sum_{m=1}^{\infty}\int_{e^{2\pi m-\sqrt{2\log t/t}}}^{e^{2\pi m+\sqrt{2\log t/t}}}\{u\}\frac{d}{du}\left(u^{-1/2}\exp(t(u^{-i}-1))\right)du.$$

We divide B_1 into two parts as following:

$$\begin{aligned}B_1=&-\sum_{m<\frac{1}{2\pi}\log t}\int_{e^{2\pi m-\sqrt{2\log t/t}}}^{e^{2\pi m+\sqrt{2\log t/t}}}\left(\frac{1}{2}+itu^{-i}\right)\{u\}u^{-3/2}\exp(t(u^{-i}-1))du\\&-\sum_{m\geq\frac{1}{2\pi}\log t}\int_{e^{2\pi m-\sqrt{2\log t/t}}}^{e^{2\pi m+\sqrt{2\log t/t}}}\left(\frac{1}{2}+itu^{-i}\right)\{u\}u^{-3/2}\exp(t(u^{-i}-1))du\\&=: M+E.\end{aligned}$$

First, we calculate the integral E.

$$\begin{aligned}E&\ll t\sum_{m\geq\frac{1}{2\pi}\log t}\int_{e^{2\pi m-\sqrt{2\log t/t}}}^{e^{2\pi m+\sqrt{2\log t/t}}}u^{-3/2}du\ll 2t\sum_{m\geq\frac{1}{2\pi}\log t}e^{-\pi m}\left(\frac{\sqrt{\log t}}{\sqrt{t}}\right)\\&\ll\sqrt{\log t}.\end{aligned}$$

Next, we calculate the integral M.

Note that

$$\begin{aligned}
&\int_{e^{2\pi m-\sqrt{2\log t/t}}}^{e^{2\pi m+\sqrt{2\log t/t}}} \left(\frac{1}{2}+itu^{-i}\right)\{u\}u^{-3/2}\exp(t(u^{-i}-1))du \\
&= \int_{[e^{2\pi m-\sqrt{2\log t/t}}]}^{[e^{2\pi m+\sqrt{2\log t/t}}]} \left(\frac{1}{2}+itu^{-i}\right)\{u\}u^{-3/2}\exp(t(u^{-i}-1))du \\
&\quad + \int_{[e^{2\pi m+\sqrt{2\log t/t}}]}^{e^{2\pi m+\sqrt{2\log t/t}}} \left(\frac{1}{2}+itu^{-i}\right)\{u\}u^{-3/2}\exp(t(u^{-i}-1))du \\
&\quad - \int_{[e^{2\pi m-\sqrt{2\log t/t}}]}^{e^{2\pi m-\sqrt{2\log t/t}}} \left(\frac{1}{2}+itu^{-i}\right)\{u\}u^{-3/2}\exp(t(u^{-i}-1))du \\
&=: M_1+M_2+M_3.
\end{aligned}$$

From an integration by parts, we get

$$\begin{aligned}
M_2 &= \frac{1}{2}\int_{[e^{2\pi m+\sqrt{2\log t/t}}]}^{e^{2\pi m+\sqrt{2\log t/t}}} \left(u-[e^{2\pi m+\sqrt{2\log t/t}}]\right)u^{-3/2}\exp(t(u^{-i}-1))du \\
&\quad + it\int_{[e^{2\pi m+\sqrt{2\log t/t}}]}^{e^{2\pi m+\sqrt{2\log t/t}}} \left(u-[e^{2\pi m+\sqrt{2\log t/t}}]\right)u^{-3/2-i}\exp(t(u^{-i}-1))du \\
&\ll \frac{1}{t}e^{-\pi m}\left(\frac{1}{2}+t\right) \ll e^{-\pi m}.
\end{aligned}$$

using (2).
Similarly, we have $M_3 \ll e^{-\pi m}$. M_1 is the following:

$$\begin{aligned}
M_1 &= -\frac{1}{2}\int_{[e^{2\pi m-\sqrt{2\log t/t}}]}^{[e^{2\pi m+\sqrt{2\log t/t}}]} \{u\}u^{-3/2}\exp(t(u^{-i}-1))du \\
&\quad - it\int_{[e^{2\pi m-\sqrt{2\log t/t}}]}^{[e^{2\pi m+\sqrt{2\log t/t}}]} \{u\}u^{-3/2-i}\exp(t(u^{-i}-1))du \\
&=: M_1^1+M_1^2.
\end{aligned}$$

Using (2), we have

$$\begin{aligned}
M_1^2 &= -it \sum_{k=[e^{2\pi m-\sqrt{2\log t/t}}]}^{[e^{2\pi m+\sqrt{2\log t/t}}]-1} \int_k^{k+1} (u-k)u^{-3/2-i}\exp(t(u^{-i}-1))du \\
&= -it \sum_{k=[e^{2\pi m-\sqrt{2\log t/t}}]}^{[e^{2\pi m+\sqrt{2\log t/t}}]-1} \Bigg(\left[(u-k)\frac{u^{-1/2}i}{t}\exp(t(u^{-i}-1))\right]_k^{k+1} \\
&\qquad - i\int_k^{k+1} \frac{u^{-1/2}}{t}\left(1-\frac{(u-k)u^{-1}}{2}\right)\exp(t(u^{-i}-1))du\Bigg) \\
&= -i \sum_{k=[e^{2\pi m-\sqrt{2\log t/t}}]}^{[e^{2\pi m+\sqrt{2\log t/t}}]-1} k^{-\frac{1}{2}} \ll e^{\pi m}\frac{\sqrt{\log t}}{\sqrt{t}}.
\end{aligned}$$

Similarly, we have

$$M_1^1 \ll \frac{1}{t}e^{\pi m}\frac{\sqrt{\log t}}{\sqrt{t}}.$$

From these facts, we have

$$M \ll \sum_{m<\frac{1}{2\pi}\log t}\left(e^{\pi m}\frac{\sqrt{\log t}}{\sqrt{t}} + e^{-\pi m}\right) \ll \sqrt{\log t}.$$

Because $E \ll \sqrt{\log t}$, we have

$$B_1 = M + E \ll \sqrt{\log t}.$$

Hence, from case 1 and case 2, we can get that $B \ll \sqrt{\log t}$.
Therefore we can know that

$$\mathbb{E}\zeta(1/2+iX_t) \ll \sqrt{\log t}$$

and the proof is complete. □

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