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ITERATIVE ALGORITHMS FOR A SYSTEM OF RANDOM NONLINEAR EQUATIONS WITH FUZZY MAPPINGS

Jong Kyu Kim and Salahuddin

Abstract. The main purpose of this paper, by using the resolvent operator technique associated with randomly (A, η, m)-accretive operator is to establish an existence and convergence theorem for a class of system of random nonlinear equations with fuzzy mappings in Banach spaces. Our works are improvements and generalizations of the corresponding well-known results.

1. Introduction

The fuzzy sets theory introduced by Zadeh [34] is an extension of a crisp set by enlarging the truth valued set $\{0, 1\}$ to the real unit interval $[0, 1]$. A fuzzy set characterized by and identified with a mapping called a membership grade function from the whole set into $[0, 1]$. Heilpern [16] introduced the concepts of fuzzy mappings and proved a fixed point theorem for fuzzy contraction mappings which is a fuzzy analogue of Nadler's fixed point theorem for multi-valued mappings.

In 1989, Chang and Zhu [6] first introduced and studied a class of variational inequalities for fuzzy mappings. Since then several classes of variational inequalities, quasi variational inequalities and complementarity problems with fuzzy mappings were considered by Agarwal *et al.* [1], Chang and Huang [8], Dai [9], Ding [10], Ding *et al.* [11], Huang [13], Lee *et al.* [24], Salahuddin [28], in the setting of Hilbert spaces and Banach spaces.

Lan *et al.* [23] introduced a new concepts of (A, η)-accretive mappings which generalizes the (H, η)-accretive and A-accretive in Banach spaces and studied some properties of (A, η)-accretive mappings and applied resolvent operators associated with (A, η)-accretive mappings to approximate solution of a new class of nonlinear (A, η)-accretive operator inclusion problems with relaxed cocoercive operators in Banach spaces. Recently Kim *et al.* [19] introduced the (A, η, m)-proximal operator to study the system of equations in Hilbert spaces.

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Recently some systems of variational inequalities, variational inclusions, complementarity problems and equilibrium problems have been studied by some authors in recent years because of their close relation to Nash equilibrium problems. Huang and Fang [14] introduced a system of order complementarity problems and established some existence results for these using fixed point theory. Kim and Kim [21] introduced and studied some system of variational inequalities and developed some iterative algorithms for approximate solutions of system of variational inequalities.

On the other hand, random variational inequality problems, random quasi variational inequality problems, random variational inclusions and complementarity problems have been studied by Chang [5], Chang and Huang [7], Huang [13], Khan and Salahuddin [18], Tan [30], Yuan [33] and Bharucha-Reid [3], *etc.*.

The concepts of random fuzzy mapping was first introduced by Huang [13]. Subsequently the random variational inclusion problems for random fuzzy mappings is studied by Anastassiou *et al.* [2] and Salahuddin [28].

Inspired and motivated by the works [2, 15, 17, 20, 26, 31]. In this paper, we established the existence and convergence theorem for system of random nonlinear equations with fuzzy mapping in Banach spaces by using randomly (A_t, η_t, m_t)-proximal operator equations.

2. Preliminaries

Throughout this paper (Ω, Σ) is a complete σ-finite measurable space, X is a real separable Banach space with a norm $\|\cdot\|$ and dual pairing $\langle\cdot,\cdot\rangle$ between X and X^*. $B(X)$, 2^X and $CB(X)$ denote the class of Borel σ fields in X, the family of all nonempty subsets of X, the family of all nonempty closed bounded subset of X, respectively.

The generalized dual mapping $j_q : X \to 2^X$ is defined by

$$J_q(x) = \{f^* \in X^* : \langle x, f^*\rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\},\ \forall x \in X$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is known that $J_q(x) = \|x\|^{q-2}J_2(x)$ for all $x \neq 0$, J_q is single valued if X^* is strictly convex and if $X = H$, the Hilbert space, then J_2 becomes the identity mapping on H. The modulus of smoothness of X is the function $\pi_X : [0,\infty) \to [0,\infty)$ defined by

$$\pi_X(t) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\| - 1 : \|x\| \leq 1, \|y\| \leq t\}.$$

A Banach space X is called uniformly smooth if

$$\lim_{t\to 0} \frac{\pi_x(t)}{t} = 0.$$

X is called q-uniformly smooth if there exists a constant $c > 0$ such that

$$\pi_X(t) \leq ct^q,\ (q > 1).$$

Definition 1. A mapping $u : \Omega \to X$ is said to be measurable if for any $B \in B(X)$, $u^{-1} = \{t \in \Omega, u(t) \in B \in \Sigma\}$.

Definition 2. A mapping $f : \Omega \times X \to X$ is called a random mapping if for each fixed $x \in X$, a mapping $f(\cdot, x) : \Omega \to X$ is measurable. A random mapping f is said to be continuous if for each fixed $t \in \Omega$, a mapping $f(t, \cdot) : X \to X$ is continuous.

Definition 3. A multi-valued mapping $T : \Omega \to 2^X$ is said to be measurable if for any $B \in B(X)$, $T^{-1}(B) = \{t \in \Omega : T(t) \cap B \neq \emptyset\} \in \Sigma$.

Definition 4. A mapping $u : \Omega \to X$ is called a measurable selection of a measurable multi-valued mapping $T : \Omega \to 2^X$, if u is measurable and for any $t \in \Omega$, $u(t) \in T_t(u(t))$.

Definition 5. A mapping $T : \Omega \times X \to 2^X$ is called a random multi-valued mapping if for each fixed $x \in X$, $T(\cdot, x) : \Omega \to 2^X$ is a measurable multi-valued mapping. A random multi-valued mapping $T : \Omega \times X \to CB(X)$ is said to be D-continuous if for each fixed $t \in \Omega$, $T(t, \cdot) : \Omega \times X \to 2^X$ is randomly continuous with respect to the Hausdorff metric on D.

Definition 6. A multi-valued mapping $T : \Omega \times X \to 2^X$ is called a random multi-valued mapping if for any $x \in X, T(\cdot, x)$ is measurable (denoted by $T_{t,x}$ or T_t).

Let Ω be a set and $\mathfrak{F}(X)$ be a collection of fuzzy sets over X. A mapping $F : \Omega \times X \to \mathfrak{F}(X)$ is called a fuzzy mapping on X. If F is a fuzzy mapping on X then for any $t \in \Omega, F(t)$ (denote it by F_t in the sequel) is a fuzzy mapping on X and $F_t(x)$ is the membership-grade of x in F_t. Let $A \in \mathfrak{F}(X)$, $a \in (0, 1]$. Then the set

$$A_a = \{x \in X : A(x) \geq a\}$$

is called an a-cut of A.

Definition 7. A fuzzy mapping $F : \Omega \times X \to \mathfrak{F}(X)$ is said to be measurable, if for any $a \in (0, 1]$, $(F(\cdot))_a : \Omega \to 2^X$ is a measurable multi-valued mapping.

Definition 8. A fuzzy mapping $F : \Omega \times X \to \mathfrak{F}(X)$ is a random fuzzy mapping if for any $x \in X, F(\cdot, x) : \Omega \times X \to \mathfrak{F}(X)$ is a measurable fuzzy mapping (denoted by $F_{t,x}$ short down $F_t(x)$).

We give the condition (**C**) for the random fuzzy mapping $T : \Omega \times X \to \mathfrak{F}(X)$.

(C): There exists a function $a : X \to (0, 1]$ such that for all $(t, x) \in \Omega \times X$, we have

$$\left(T_{t,x(t)}\right)_{a(x(t))} \in CB(X),$$

where $T_{t,x}$ denotes the value of T at (t, x).

Induced multi-valued random mapping $\tilde{T}_t$ from T is as follows:

$$\tilde{T} : \Omega \times X \to CB(X), T_t = \left(T_{t,x(t)}\right)_{a(x(t))},\ (t, x) \to \left(T_{t,x(t)}\right)_{a(x(t))}, \forall (t, x) \in \Omega \times X.$$

In this paper, we consider the following randomly (A_t, η_t, m_t)-proximal operator equation system with fuzzy mappings, for each fixed $t \in \Omega$ finding $(x(t), y(t)), (z(t), w(t)) \in X_1 \times X_2, T_{t,x(t)}(u(t)) \geq a(x(t))$ and

$$\begin{cases} E_t(x(t), y(t)) + \rho^{-1} R^{M_t(\cdot, x(t))}_{\rho, A_{1,t}}(z(t)) = 0, \\ G_t(u(t), y(t)) + \varrho^{-1} R^{N_t(\cdot, y(t))}_{\varrho, A_{2,t}}(w(t)) = 0, \end{cases} \tag{1}$$

where $T : X_1 \times \Omega \to \mathfrak{F}(X_1)$ is a fuzzy mapping, $E : X_1 \times X_2 \times \Omega \to X_1, G : X_1 \times X_2 \times \Omega \to X_2, g : X_1 \times \Omega \to X_1, h : X_2 \times \Omega \to X_2, \eta_1 : X_1 \times X_1 \times \Omega \to X_1$ and $\eta_2 : X_2 \times X_2 \times \Omega \to X_2$ are nonlinear random single-valued mappings, $A_1 : X_1 \times \Omega \to X_1,\ A_2 : X_2 \times \Omega \to X_2, M : X_1 \times X_1 \times \Omega \to 2^{X_1}$ and $N : X_2 \times X_2 \times \Omega \to 2^{X_2}$ are any nonlinear mappings such that for all $(z(t), t) \in X_1 \times \Omega, M_t(\cdot, z(t)) : X_1 \to 2^{X_1}$ is a randomly $(A_{1,t}, \eta_{1,t}, m_{1,t})$-accretive mapping with $f_t(X_1) \cap dom(M_t(\cdot, z(t))) \neq \emptyset$ and for all $(w(t), t) \in X_2 \times \Omega : N_t(\cdot, w(t)) : X_2 \to 2^{X_2}$ is a randomly $(A_{2,t}, \eta_{2,t}, m_{2,t})$-accretive mapping with $g_t(X_2) \cap dom(N_t(\cdot, w(t)) \neq \emptyset$, $R^{M_t(\cdot, x(t))}_{\rho_t, A_{1,t}} = I - A_{1,t}\left(J^{M_t(\cdot, x(t))}_{\rho_t, A_{1,t}}\right)$, $R^{N_t(\cdot, y(t))}_{\varrho_t, A_{2,t}} = I - A_{2,t}\left(J^{N_t(\cdot, y(t))}_{\varrho_t, A_{2,t}}\right)$, I is the identity mapping, $A_{1,t}\left(J^{M_t(\cdot, x(t))}_{\rho_t, A_{1,t}}(z(t))\right) = A_{1,t}\left(J^{M_t(\cdot, x(t))}_{\rho_t, A_{1,t}}\right)(z(t))$,
$A_{2,t}\left(J^{N_t(\cdot, y(t))}_{\varrho_t, A_{2,t}}(w(t))\right) = A_{2,t}\left(J^{N_t(\cdot, y(t))}_{\varrho_t, A_{2,t}}\right)(w(t))$,
$J^{M_t(\cdot, x(t))}_{\rho_t, A_{1,t}} = (A_{1,t} + \rho_t M_t(\cdot, x(t)))^{-1}$ and $J^{N_t(\cdot, y(t))}_{\varrho_t, A_{2,t}} = (A_{2,t} + \varrho_t N_t(\cdot, y(t)))^{-1}$ for all $(x(t), z(t)) \in X_1, (y(t), w(t)) \in X_2$ and $\rho, \varrho : \Omega \to (0, 1)$ are measurable mappings.

For appropriate and suitable choice of $T, E, G, M, N, f, g, A_i, \eta_i$ and X_i for $i = 1, 2$, we see that (1) is a generalized version of some problems which includes the system (random) variational inclusions, (random) generalized quasi-variational inequalities and (random) implicit quasi-variational inequalities for fuzzy mappings (see [20, 21]).

Lemma 2.1. [4] *Let $M : \Omega \times X \to CB(X)$ be a D-continuous random multi-valued mapping. Then for a measurable mapping $x : \Omega \to X$, a multi-valued mapping $M(\cdot, x(\cdot)) : \Omega \to CB(X)$ is measurable.*

Lemma 2.2. [4] *Let $M, V : \Omega \to CB(X)$ be two measurable multi-valued mappings and $\epsilon > 0$ be a constant and $x : \Omega \to X$ be a measurable selection of M. Then there exists a measurable selection $y : \Omega \to X$ of V such that for all $t \in \Omega$*

$$\|x(t) - y(t)\| \ \leq \ (1 + \epsilon)\, D(M(t), V(t)).$$

Lemma 2.3. [25] *Let (X, d) be a complete metric space. Suppose that $G : X \to CB(X)$ satisfies*

$$\widetilde{D}(G(x), G(y)) \leq \omega d(x, y), \forall x, y \in X,$$

where $\omega \in (0, 1)$ is a constant. Then the mapping G has a fixed point in X.

Definition 9. Let $x, y, w : \Omega \to X$ be random mappings and $t \in \Omega$. A random mapping $T : X \times X \times \Omega \to X$ is said to be

(i) randomly accretive in the first argument of T if $j_q(x(t)-y(t)) \in J_q(x(t)-y(t))$ and

$$\langle T_t(x(t),w(t)) - T_t(y(t),w(t)), j_q(x(t)-y(t))\rangle \geq 0,$$

for all $x(t), y(t) \in X$;

(ii) randomly r_t-strongly accretive in the first argument of T if there exists a measurable function $r_t : \Omega \to (0,\infty)$ such that

$$\langle T_t(x(t),w(t)) - T_t(y(t),w(t)), j_q(x(t)-y(t))\rangle \geq r_t\|x(t)-y(t)\|^q,$$

for all $x(t), y(t) \in X$;

(iii) randomly m_t-relaxed accretive in the first argument if there exists a measurable function $m_t : \Omega \to (0,\infty)$ such that

$$\langle T_t(x(t),w(t)) - T_t(y(t),w(t)), j_q(x(t)-y(t))\rangle \geq -m_t\|x(t)-y(t)\|^q,$$

for all $x(t), y(t) \in X$;

(iv) randomly s_t-cocoercive in the first argument of T if there exists a measurable function $s_t : \Omega \to (0,\infty)$ such that

$$\begin{aligned}&\langle T_t(x(t),w(t)) - T_t(y(t),w(t)), j_q(x(t)-y(t))\rangle \\ &\qquad \geq s_t\|T_t(x(t),w(t)) - T_t(y(t),w(t))\|^q,\end{aligned}$$

for all $x(t), y(t), w(t) \in X$;

(v) randomly γ_t-relaxed cocoercive with respect to A_t in the first argument of T if there exists a measurable function $\gamma_t \to (0,\infty)$ such that

$$\begin{aligned}&\langle T_t(x(t),w(t)) - T_t(y(t),w(t)), j_q(A_t(x(t)) - A_t(y(t)))\rangle \\ &\qquad \geq -\gamma_t\|T_t(x(t),w(t)) - T_t(y(t),w(t))\|^q,\end{aligned}$$

for all $x(t), y(t), w(t) \in X$;

(vi) randomly (γ_t, α_t)-relaxed cocoercive with respect to A_t in the first argument of T if there exists a measurable function $\gamma_t, \alpha_t : \Omega \to (0,\infty)$ such that

$$\begin{aligned}&\langle T_t(x(t),w(t)) - T_t(y(t),w(t)), j_q(A_t(x(t)) - A_t(y(t)))\rangle \\ &\qquad \geq -\gamma_t\|T_t(x(t),w(t)) - T_t(y(t),w(t))\|^q + \alpha_t\|x(t)-y(t)\|^q,\end{aligned}$$

for all $x(t), y(t), w(t) \in X$;

(vii) randomly μ_t-Lipschitz continuous in the first argument if there exists a measurable function $\mu_t : \Omega \to (0,\infty)$ such that

$$\|T_t(x(t),w(t)) - T_t(y(t),w(t))\| \leq \mu_t\|x(t)-y(t)\|,$$

for all $x(t), y(t), w(t) \in X$.

Definition 10. Let $T : X \times \Omega \to 2^X$ be a random multi-valued mapping. Then T is said to be randomly τ_t-$\widetilde{D}$-Lipschitz continuous in the first argument if there exists a measurable mapping $\tau : \Omega \to (0,1)$ such that

$$\widetilde{D}(T_t(x(t)), T_t(y(t))) \leq \tau_t\|x(t)-y(t)\|, \ \forall x(t), y(t) \in X, t \in \Omega,$$

where $\widetilde{D} : 2^X \times 2^X \to (-\infty, +\infty) \cup \{+\infty\}$ is the Hausdorff metric, that is,

$$\widetilde{D}(A, B) = \max\left\{ \sup_{x(t)\in A} \inf_{y(t)\in B} \|x(t) - y(t)\|, \sup_{x(t)\in B} \inf_{y(t)\in A} \|x(t) - y(t)\| \right\}, \forall A, B \in 2^X.$$

Lemma 2.4. *Let (X, d) be a complete metric space and $T_1, T_2 : X \to CB(X)$ be two set-valued contractive mappings with same contractive constant $t \in (0, 1)$, that is,*

$$\widetilde{D}(T_i(x), T_i(y)) \le td(x, y), \forall x, y \in X, i = 1, 2.$$

Then

$$\widetilde{D}(F(T_i), F(T_i)) \le \frac{1}{1-t} \sup_{x\in H} \widetilde{D}(T_1(x), T_2(x)),$$

where $F(T_1)$ and $F(T_2)$ are fixed point sets of T_1 and T_2, respectively.

Definition 11. Let $A : X \times \Omega \to X, \eta : X \times X \times \Omega \to X$ be two random single-valued mappings. The set-valued mapping $M : X \times X \times \Omega \to 2^X$ is said to be randomly (A_t, η_t, m_t)-accretive if

(1) M is a randomly m_t-relaxed η_t-accretive mapping;
(2) $(A_t + \rho_t M_t)(X) = X$, where $\rho : \Omega \to (0, 1)$ is a measurable mapping.

Definition 12. Let $A : \Omega \times X \to X$ be a randomly r_t-strongly η_t-accretive mapping and $M : \Omega \times X \to 2^X$ be a randomly (A_t, η_t)-accretive mapping. Then random operator $(A_t + \rho_t M_t)^{-1}$ is a single-valued random mapping for any measurable mapping $\rho : X \to (0, \infty)$.

Definition 13. Let $A : \Omega\times X \to X$ be a randomly strictly η_t-accretive mapping and $M : \Omega \times X \to 2^X$ be a randomly (A_t, η_t, m_t)-accretive mapping. Then for any given measurable mapping $\rho : \Omega \to (0, 1)$, the random resolvent operator $J^{\eta_t,M_t}_{\rho_t,A_t} : X \to X$ is defined by

$$J^{\eta_t,M_t}_{\rho_t,A_t}(x(t)) = (A_t + \rho_t M_t)^{-1}(x(t)), \forall t \in \Omega, x(t) \in X.$$

Proposition 2.5. [22] *Let X be a q-uniformly smooth Banach spaces and $\eta : \Omega\times X\times X \to X$ be a randomly τ_t-Lipschitz continuous mapping, $A : \Omega\times X \to X$ be a randomly r_t-strongly η_t-accretive mapping and $M : \Omega \times X \to 2^X$ be a randomly (A_t, η_t, m_t)-accretive mapping. Then the random resolvent operator $J^{\eta_t,M_t}_{\rho_t,A_t} : X \to X$ is a randomly $\frac{\tau_t^{q-1}}{r_t-\rho_t m_t}$-Lipschitz continuous, i.e.,*

$$\|J^{\eta_t,M_t}_{\rho_t,A_t}(x(t)) - J^{\eta_t,M_t}_{\rho_t,A_t}(y(t))\| \le \frac{\tau_t^{q-1}}{r_t - \rho_t m_t}\|x(t) - y(t)\|,$$

where $\rho_t \in (0, \frac{r_t}{m_t})$ is a real-valued random variable for all $t \in \Omega$.

In connection with randomly (A_t, η_t, m_t)-proximal operator equation systems (1), we consider the system of random nonlinear equations with fuzzy mappings for finding measurable mappings $x, u : \Omega :\to X_1, y : \Omega \to X_2$ such that for all $t \in \Omega$ and each fixed $T_{t,x(t)}(u(t)) \geq a(x(t))$ and

$$\begin{cases} 0 \in E_t(x(t), y(t)) + M_t(x(t), x(t)), \\ 0 \in G_t(u(t), y(t)) + N_t(y(t), y(t)). \end{cases} \tag{2}$$

Lemma 2.6. *For $t \in \Omega$, $x, u : \Omega \to X_1$ and $y : \Omega \to X_2, (x(t), y(t), u(t))$ is a solution of problem (2) if and only if $x(t), u(t) \in X_1, y(t) \in X_2$ such that*

$$\begin{cases} x(t) = J^{M_t(\cdot,x(t))}_{\rho_t,A_{1,t}}[A_{1,t}(x(t)) - \rho_t E_t(x(t), y(t))], \\ y(t) = J^{N_t(\cdot,y(t))}_{\varrho_t,A_{2,t}}[A_{2,t}(y(t)) - \varrho_t G_t(u(t), y(t))], \end{cases} \tag{3}$$

where $J^{M_t(\cdot,x(t))}_{\rho_t,A_{1,t}} = (A_{1,t}+\rho_t M_t(\cdot, x(t)))^{-1}$ and $J^{N_t(\cdot,y(t))}_{\varrho_t,A_{2,t}} = (A_{2,t}+\varrho_t N_t(\cdot, y(t)))^{-1}$ are corresponding random resolvent operators in the first argument of a random $(A_{1,t}, \eta_{1,t})$-accretive mapping $M_t(\cdot,\cdot)$, random $(A_{2,t}, \eta_{2,t})$-accretive mapping $N_t(\cdot,\cdot)$, respectively, $A_{i,t}$ is randomly $r_{i,t}$-strongly accretive mapping for each $i = 1, 2$ and $\rho, \varrho : \Omega \to (0, 1)$ are measurable mappings.

Proof. From the definition of the random resolvent operator $J^{M_t(\cdot,x(t))}_{\rho_t,A_{1,t}} = (A_{1,t}+\rho_t M_t(\cdot, x(t)))^{-1}$ of $M_t(\cdot, x(t))$ and $J^{N_t(\cdot,y(t))}_{\varrho_t,A_{2,t}} = (A_{2,t}+\varrho_t N_t(\cdot, y(t)))^{-1}$ of $N_t(\cdot, y(t))$, for each $t \in \Omega$, respectively, we know that there exists $t \in \Omega, x(t) \in X_1, y(t) \in X_2, u(t) \in \tilde{T}_t(x(t))$ such that (3) holds if and only if

$$\begin{cases} A_{1,t}(x(t)) - \rho_t E_t(x(t), y(t)) \in A_{1,t}(x(t)) + \rho_t M_t(x(t), x(t)), \\ A_{2,t}(y(t)) - \varrho_t G_t(u(t), y(t)) \in A_{2,t}(y(t)) + \varrho_t N_t(y(t), y(t)), \end{cases}$$

that is,

$$\begin{cases} 0 \in E_t(x(t), y(t)) + M_t(x(t), x(t)), \\ 0 \in G_t(u(t), y(t)) + N_t(y(t), y(t)), \end{cases}$$

where $\rho, \varrho : \Omega \to (0, 1)$ are measurable mappings. □

Now we prove that problem (1) is equivalent to problem (3).

Lemma 2.7. *For $t \in \Omega$, problem (1) has a solution $(x(t), y(t), u(t))$ with $u(t) \in \tilde{T}_t(x(t))$ if and only if problem (3) has a solution $(x(t), y(t), u(t))$ with $u(t) \in \tilde{T}_t(x(t))$, where*

$$x(t) = J^{M_t(\cdot,x(t))}_{\rho_t,A_{1,t}}(z(t)),\ y(t) = J^{N_t(\cdot,y(t))}_{\varrho_t,A_{2,t}}(w(t)) \tag{4}$$

and

$$\begin{cases} z(t) = A_{1,t}(x(t)) - \rho_t E_t(x(t), y(t)), \\ w(t) = A_{2,t}(y(t)) - \varrho_t G(u(t), y(t)), \end{cases}$$

where $\rho, \varrho : \Omega \to (0, 1)$ are measurable mappings.

Proof. Let $x, u : \Omega \to X_1$ and $y : \Omega \to X_2$ be measurable mappings, and $(x(t), y(t), u(t))$ with $u(t) \in \tilde{T}_t(x(t))$ be the solution of problem (3) for $t \in \Omega$,. Then by Lemma 2.6, it is a solution of the following system of random nonlinear equations:

$$\begin{cases} x(t) = J_{\rho_t, A_{1,t}}^{M_t(\cdot, x(t))}[A_{1,t}(x(t)) - \rho_t E_t(x(t), y(t))], \\ y(t) = J_{\varrho_t, A_{2,t}}^{N_t(\cdot, y(t))}[A_{2,t}(y(t)) - \varrho_t G_t(u(t), y(t))]. \end{cases}$$

By using the fact $R_{\rho_t, A_{1,t}}^{M_t(\cdot, x(t))} = I - A_{1,t}(J_{\rho_t, A_{1,t}}^{M_t(\cdot, x(t))})$, $R_{\varrho_t, A_{2,t}}^{N_t(\cdot, y(t))} = I - A_{2,t}(J_{\varrho_t, A_{2,t}}^{N_t(\cdot, y(t))})$ and (3), we have

$$\begin{aligned} & R_{\rho_t, A_{1,t}}^{M_t(\cdot, x(t))}[A_{1,t}(x(t)) - \rho_t E_t(x_t, y(t))] \\ & = [A_{1,t}(x(t)) - \rho_t E_t(x(t), y(t))] \\ & \quad - A_{1,t}(J_{\rho_t, A_{1,t}}^{M_t(\cdot, x(t))}[A_{1,t}(x(t)) - \rho_t E_t(x(t), y(t))]) \\ & = A_{1,t}(x(t)) - \rho_t E_t(x(t), y(t)) - A_{1,t}(x(t)) \\ & = -\rho_t E_t(x(t), y(t)) \end{aligned}$$

and

$$\begin{aligned} & R_{\varrho_t, A_{2,t}}^{N_t(\cdot, y(t))}[A_{2,t}(y(t)) - \varrho_t G_t(u(t), y(t))] \\ & = A_{2,t}(y(t)) - \varrho_t G_t(u(t), y(t)) \\ & \quad - A_{2,t}(J_{\varrho_t, A_{2,t}}^{N_t(\cdot, y(t))}[A_{2,t}(y(t)) - \varrho_t G_t(u(t), y(t))]) \\ & = A_{2,t}(y(t)) - \varrho_t G_t(u(t), y(t)) - A_{2,t}(y(t)) \\ & = -\varrho_t G_t(u(t), y(t)), \end{aligned}$$

which imply that

$$\begin{cases} E_t(x(t), y(t)) + \rho_t^{-1} R_{\rho_t, A_{1,t}}^{M_t(\cdot, x(t))}(z(t)) = 0, \\ G_t(u(t), y(t)) + \varrho_t^{-1} R_{\varrho_t, A_{2,t}}^{N_t(\cdot, y(t))}(w(t)) = 0 \end{cases}$$

with

$$\begin{cases} z(t) = A_{1,t}(x(t)) - \rho_t E_t(x(t), y(t)), \\ w(t) = A_{2,t}(y(t)) - \varrho_t G_t(u(t), y(t)), \end{cases}$$

that is, $(z(t), w(t), x(t), y(t), u(t))$ with $u_t \in \tilde{T}_t(x(t))$ is a solution of problem (1) and $\rho, \varrho : \Omega \to (0, \infty)$ are measurable mappings.

Conversely, letting $(z(t), w(t), x(t), y(t), u(t))$ with $u(t) \in \tilde{T}_t(x(t))$ is a solution of problem (1), then we have

$$\rho_t E_t(x(t), y(t)) = -R_{\rho_t, A_{1,t}}^{M_t(\cdot, x(t))}(z(t)), \ \varrho_t G_t(u(t), y(t)) = -R_{\varrho_t, A_{2,t}}^{N_t(\cdot, y(t))}(w(t)),$$

$$\rho_t E_t(x(t), y(t)) = -R_{\rho_t, A_{1,t}}^{M_t(\cdot, x(t))}(z(t)) = A_{1,t}(J_{\rho_t, A_{1,t}}^{M_t(\cdot, x(t))}(z(t))) - z(t),$$

$$\varrho_t G_t(u(t), y(t)) = -R_{\varrho_t, A_{2,t}}^{N_t(\cdot, y(t))}(w(t)) = A_{2,t}(J_{\varrho_t, A_{2,t}}^{N_t(\cdot, y(t))}(w(t))) - w(t). \tag{5}$$

It follows from (3) and (5) that

$$\begin{aligned}\rho_t E_t(x(t),y(t)) = A_{1,t}(J^{M_t(\cdot,x(t))}_{\rho_t,A_{1,t}}&(A_{1,t}(x(t)) - \rho_t E_t(x(t),y(t))))\\ &- A_{1,t}(x(t)) + \rho_t E_t(x(t),y(t))\end{aligned}$$

and

$$\begin{aligned}\varrho_t G_t(u(t),y(t)) = A_{2,t}(J^{N_t(\cdot,y(t))}_{\varrho_t,A_{2,t}}&(A_{2,t}(y(t)) - \varrho_t G_t(u(t),y(t))))\\ &- A_{2,t}(y(t)) + \varrho_t G_t(u(t),y(t)),\end{aligned}$$

which imply that

$$\begin{cases} A_{1,t}(x(t)) = A_{1,t}(J^{M_t(\cdot,x(t))}_{\rho_t,A_{1,t}}(A_{1,t}(x(t)) - \rho_t E_t(x(t),y(t)))),\\ A_{2,t}(y(t)) = A_{2,t}(J^{N_t(\cdot,y(t))}_{\varrho_t,A_{2,t}}(A_{2,t}(y(t)) - \varrho_t G_t(u(t),y(t)))),\end{cases}$$

and so

$$\begin{cases} x(t) = J^{M_t(\cdot,x(t))}_{\rho_t,A_{1,t}}(A_{1,t}(x(t)) - \rho_t E_t(x(t),y(t))),\\ y(t) = J^{N_t(\cdot,y(t))}_{\varrho_t,A_{2,t}}(A_{2,t}(y(t)) - \varrho_t G_t(u(t),y(t))),\end{cases}$$

that is, $(x(t),y(t),u(t))$ with $u(t)\in \tilde{T}_t(x(t))$ is a solution of problem (2).

Alternative Proof.

Let

$$z(t) = A_{1,t}(x(t)) - \rho_t E_t(x(t),y(t)),\ w(t) = A_{2,t}(y(t)) - \varrho_t G_t(u(t),y(t)).$$

Then by (4) we have

$$x(t) = J^{M_t(\cdot,x(t))}_{\rho_t,A_{1,t}}(z(t)),\ y(t) = J^{N_t(\cdot,y(t))}_{\varrho_t,A_{2,t}}(w(t)),$$

$$z(t) = A_{1,t}(J^{M_t(\cdot,x(t))}_{\rho_t,A_{1,t}}(z(t))) - \rho_t E_t(x(t),y(t)),$$

and

$$w(t) = A_{2,t}(J^{N_t(\cdot,y(t))}_{\varrho_t,A_{2,t}}(w(t))) - \varrho_t G_t(u(t),y(t)).$$

Since

$$\begin{cases} A_{1,t}(J^{M_t(\cdot,x(t))}_{\rho_t,A_{1,t}}(z(t))) = A_{1,t}(J^{M_t(\cdot,x(t))}_{\rho_t,A_{1,t}})(z(t)),\\ A_{2,t}(J^{N_t(\cdot,y(t))}_{\varrho_t,A_{2,t}}(w(t))) = A_{2,t}(J^{N_t(\cdot,y(t))}_{\varrho_t,A_{2,t}})(w(t)),\end{cases}$$

we have

$$\begin{cases} E_t(x(t),y(t)) + \rho_t^{-1} R^{M_t(\cdot,x(t))}_{\rho_t,A_{1,t}}(z(t)) = 0,\\ G_t(u(t),y(t)) + \varrho_t^{-1} R^{N_t(\cdot,y(t))}_{\varrho_t,A_{2,t}}(w(t)) = 0,\end{cases}$$

where $\rho,\varrho:\Omega\to(0,1)$ are measurable mappings. □

Lemma 2.8. [32] *Let X be a real uniformly smooth Banach space. Then X is q-uniformly smooth if and only if there exists a constant $c_q>0$ such that for all $x,y\in X$,*

$$\|x+y\|^q \le \|x\|^q + q\langle y, j_q(x)\rangle + c_q\|y\|^q.$$

3. Main Results

In this section, we first discuss the existence theorem, after then we develop an algorithm for the problem and prove the convergence of the random sequences generated by given algorithm.

Theorem 3.1. *Let (Ω, Σ) be a measurable space. Let $A_i : X_i \times \Omega \to X_i$ be a randomly $r_{i,t}$-strongly accretive and randomly $s_{i,t}$-Lipschitz continuous mapping for each $i = 1, 2, T : X_1 \times \Omega \to \mathfrak{F}(X_1)$ be a fuzzy mapping induced by a set-valued mapping $\tilde{T} : X_1 \times \Omega \to CB(X_1)$, and $a : X_1 \to (0, 1)$ and $T_{t,x(t)}(x(t)) \geq a(x(t))$ satisfying the condition* (**C**). *Let $\tilde{T} : X_1 \times \Omega \to CB(X_1)$ be the randomly $\kappa_t - \widetilde{D}$-Lipschitz continuous mapping, where $\widetilde{D}$ is the Hausdorff pseudo metric on 2^{X_i}. Let $M : X_1 \times X_1 \times \Omega \to 2^{X_1}$ be a randomly $(A_{1,t}, \eta_{1,t})$-accretive mapping with measurable mapping $m_1 : \Omega \to (0, 1)$ in the first variable and $N : X_2 \times X_2 \times \Omega \to 2^{X_2}$ be a randomly $(A_{2,t}, \eta_{2,t})$-accretive mapping with measurable mapping $m_2 : \Omega \to (0, 1)$ in the first variable. Let $\eta_1 : X_1 \times X_1 \times \Omega \to X_1$ be a randomly $\tau_{1,t}$-Lipschitz continuous mapping with measure$\tau_1 : \Omega \to (0, 1)$, $\eta_2 : X_2 \times X_2 \times \Omega \to X_2$ be a randomly $\tau_{2,t}$-Lipschitz continuous mapping with measure $\tau_2 : \Omega \to (0, 1)$, $E : X_1 \times X_2 \times \Omega \to X_1$ be the randomly Lipschitz continuous mapping with respect to first variable with measurable mapping $\beta : \Omega \to (0, 1)$, and second argument with respect to measurable mapping $\xi : \Omega \to (0, 1)$ and randomly $(\gamma_{1,t}, \alpha_{1,t})$-relaxed cocoercive with respect to $A_{1,t}$ and first variable of E_t with measurable mappings $\gamma_1, \alpha_1 : \Omega \to (0, 1)$. Let $G : X_1 \times X_2 \times \Omega \to X_2$ be the randomly Lipschitz continuous with respect to first and second variables with measurable mappings $\mu, \zeta : \Omega \to (0, 1)$, respectively. Let G be a randomly $(\gamma_{2,t}, \alpha_{2,t})$-relaxed cocoercive mapping with respect to $A_{2,t}$ with measurable mappings $\gamma_2, \alpha_2 : \Omega \to (0, 1)$, respectively. If in addition $\rho : \Omega \to (0, \frac{r_{1,t}}{m_{1,t}})$ and $\varrho : \Omega \to (0, \frac{r_{2,t}}{m_{2,t}})$ are measurable mappings such that*

$$\tau_{1,t}^{q-1} \sqrt[q]{s_{1,t}^q - q\rho_t(-\gamma_{1,t}\beta_t^q + \alpha_{1,t}s_{2,t}^q) + c_q\rho_t^q\beta_t^q}$$
$$< (r_{1,t} - \rho_t m_{1,t})\left(1 - \upsilon_{1,t} - \frac{\tau_{2,t}^{q-1}\varrho_t\mu_t\kappa_t}{r_{2,t} - \varrho_t m_{2,t}}\right),$$

$$\tau_{2,t}^{q-1} \sqrt[q]{s_{2,t}^q - q\varrho_t(-\gamma_{2,t}\zeta_t^q + \alpha_{2,t}s_{2,t}^q) + c_q\varrho_t^q\zeta_t^q} \tag{6}$$
$$< (r_{2,t} - \varrho_t m_{2,t})\left(1 - \upsilon_{2,t} - \frac{\tau_{1,t}^{q-1}\rho_t\xi_t}{r_{1,t} - \rho_t m_{1,t}}\right),$$

$$\|J_{\rho_t,A_{1,t}}^{M_t(\cdot,x(t))}(z(t)) - J_{\rho_t,A_{1,t}}^{M_t(\cdot,y(t))}(z(t))\| \leq \upsilon_{1,t}\|x(t) - y(t)\|, \tag{7}$$

for all $(x(t), y(t), z(t), t) \in X_1 \times X_1 \times X_1 \times \Omega$ *and*

$$\|J_{\varrho_t,A_{2,t}}^{N_t(\cdot,x(t))}(z(t)) - J_{\varrho_t,A_{2,t}}^{N_t(\cdot,y(t))}(z(t))\| \leq \upsilon_{2,t}\|x(t) - y(t)\|, \tag{8}$$

for all $(x(t), y(t), z(t), t) \in X_2 \times X_2 \times X_2 \times \Omega$, *where* $x, u : \Omega \to X_1$ *and* $y : \Omega \to X_2$ *are measurable mappings, then problem* (1) *has a random solution* $(x^*(t), y^*(t), u^*(t))$.

Proof. For given any measurable mappings $\rho, \varrho : \Omega \to (0,1)$, we first define random mappings $\Phi_{\rho_t} : X_1 \times X_2 \times \Omega \to X_1$ and $\Psi_{\varrho_t} : X_1 \times X_2 \times \Omega \to X_2$ as follows:

$$\begin{cases} \Phi_{\rho_t}(x(t), y(t)) = J_{\rho_t, A_{1,t}}^{M_t(\cdot, x(t))}[A_{1,t}(x(t)) - \rho_t E_t(x(t), y(t))], \\ \Psi_{\varrho_t}(x(t), y(t)) = J_{\varrho_t, A_{2,t}}^{N_t(\cdot, y(t))}[A_{2,t}(y(t)) - \varrho_t G_t(u(t), y(t))], \end{cases} \tag{9}$$

where $x, u : \Omega \to X_1, y : \Omega \to X_2, a : X_1 \to (0,1)$ are random mappings and $T_{t,x(t)}(u(t)) \geq a(x(t))$.

Now we define a norm $\|\cdot,\cdot\|_1$ on $X_1 \times X_2$ by

$$\|(x(t), y(t))\|_1 = \|x(t)\| + \|y(t)\|, \ \forall (x(t), y(t)) \in X_1 \times X_2, t \in \Omega.$$

It is easy to see that $(X_1 \times X_2, \|\cdot,\cdot\|_1)$ is a Banach space [12]. For any given measurable mappings $\rho, \varrho : \Omega \to (0,1)$, define $Q_{\rho_t, \varrho_t} : X_1 \times X_2 \times \Omega \to 2^{X_1 \times X_2}$ by

$$Q_{\rho_t,\varrho_t}(x(t), y(t)) = \left\{ (\Phi_{\rho_t}(x(t), y(t)), \Psi_{\varrho_t}(x(t), y(t))) : u(t) \in \tilde{T}_t(x(t)) \right\},$$

for all $(x(t), y(t)) \in X_1 \times X_2$. For $t \in \Omega, (x(t), y(t)) \in X_1 \times X_2$,

$$T_{t,x(t)}(u(t)) \geq a(x(t)),$$

$\tilde{T}_t(x(t)) \in CB(X_1), A_{1,t}, A_{2,t}, \eta_{1,t}, \eta_{2,t}, E_t, G_t, J_{\rho_t, A_{1,t}}^{M_t(\cdot, x(t))}, J_{\varrho_t, A_{2,t}}^{N_t(\cdot, x(t))}$ are continuous and measurable mappings, we have

$$Q_{\rho_t,\varrho_t}(x(t), y(t)) \in CB(X_1 \times X_2).$$

Now for each fixed $t \in \Omega$, we prove that $Q_{\rho_t,\varrho_t}(x(t), y(t))$ is multi-valued contractive mapping. In fact, for any $t \in \Omega, (x(t), y(t)) \in X_1 \times X_2$ and $(a_1, a_2) \in Q_{\rho_t,\varrho_t}(x(t), y(t))$, there exists $T_{t,x(t)}(u(t)) \geq a(x(t))$ with $u(t) \in \tilde{T}_t(x(t)) \in CB(X_1)$ such that

$$a_1 = J_{\rho_t, A_{1,t}}^{M_t(\cdot, x(t))}[A_{1,t}(x(t)) - \rho_t E_t(x(t), y(t))]$$

and

$$a_2 = J_{\varrho_t, A_{2,t}}^{N_t(\cdot, y(t))}[A_{2,t}(y(t)) - \varrho_t G_t(u(t), y(t))].$$

Since $T_{t,x(t)}(u(t)) \geq a(x(t))$ *i.e.*, $u(t) \in \tilde{T}_t(x(t))$ for any $(\bar{x}(t), \bar{y}(t)) \in X_1 \times X_2$, it follows from Nadler's Theorem [26] that there exists $T_{t,x(t)}(u(t)) \geq a(x(t))$, *i.e.*, $\bar{u}(t) \in \tilde{T}_t(\bar{x}(t))$ such that

$$\|u(t) - \bar{u}(t)\| \leq (1 + \iota)\widetilde{D}(\tilde{T}_t(x(t)), \tilde{T}_t(\bar{x}(t))). \tag{10}$$

Letting $(b_1, b_2) \in Q_{\rho_t,\varrho_t}(\bar{x}(t), \bar{y}(t))$, where

$$b_1 = J_{\rho_t, A_{1,t}}^{M_t(\cdot, \bar{x}(t))}[A_{1,t}(\bar{x}(t)) - \rho_t E_t(\bar{x}(t), \bar{y}(t))]$$

and

$$b_2 = J_{\varrho_t, A_{2,t}}^{N_t(\cdot,\bar{y}(t))}[A_{2,t}(\bar{y}(t)) - \varrho_t G_t(\bar{u}(t), \bar{y}(t))].$$

Then it follows from (7) and Proposition 2.5 that

$$\begin{aligned}
&\|a_1 - b_1\| \\
&= \|J_{\rho_t, A_{1,t}}^{M_t(\cdot,x(t))}[A_{1,t}(x(t)) - \rho_t E_t(x(t), y(t))] \\
&\quad - J_{\rho_t, A_{1,t}}^{M_t(\cdot,\bar{x}(t))}[A_{1,t}(\bar{x}(t)) - \rho_t E_t(\bar{x}(t), \bar{y}(t))]\| \\
&\leq \|J_{\rho_t, A_{1,t}}^{M_t(\cdot,x(t))}[A_{1,t}(x(t)) - \rho_t E_t(x(t), y(t))] \\
&\quad - J_{\rho_t, A_{1,t}}^{M(\cdot,\bar{x}(t))}[A_{1,t}(x(t)) - \rho_t E_t(x(t), y(t))]\| \\
&\quad + \|J_{\rho_t, A_{1,t}}^{M_t(\cdot,\bar{x}(t))}[A_{1,t}(x(t)) - \rho_t E_t(x(t), y(t))] \\
&\quad - J_{\rho_t, A_{1,t}}^{M_t(\cdot,\bar{x}(t))}[A_{1,t}(\bar{x}(t)) - \rho_t E(\bar{x}(t), \bar{y}(t))]\| \\
&\leq \upsilon_{1,t}\|x(t) - \bar{x}(t)\| \\
&\quad + \frac{\tau_{1,t}^{q-1}}{r_{1,t} - \rho_t m_{1,t}}\|A_{1,t}(x(t)) - A_{1,t}(\bar{x}(t)) - \rho_t(E_t(x(t), y(t)) - E_t(\bar{x}(t), \bar{y}(t)))\| \\
&\leq \upsilon_{1,t}\|x(t) - \bar{x}(t)\| \\
&\quad + \frac{\tau_{1,t}^{q-1}}{r_{1,t} - \rho_t m_{1,t}}\|A_{1,t}(x(t)) - A_{1,t}(\bar{x}(t)) - \rho_t(E_t(x(t), y(t)) - E_t(\bar{x}(t), y(t)))\| \\
&\quad + \frac{\tau_{1,t}^{q-1}}{r_{1,t} - \rho_t m_{1,t}}\rho_t\|E_t(\bar{x}(t), y(t)) - E_t(\bar{x}(t), \bar{y}(t))\|,
\end{aligned} \tag{11}$$

where $\rho_t, \tau_{1,t}, r_{1,t}, m_{1,t} : \Omega \to (0,1)$ are measurable mappings. By the assumption of $E_t, A_{1,t}$,

$$\|E_t(\bar{x}(t), y(t)) - E_t(\bar{x}(t), \bar{y}(t))\| \leq \xi_t\|y(t) - \bar{y}(t)\| \tag{12}$$

and

$$\|E_t(x(t), y(t)) - E_t(\bar{x}(t), y(t))\| \leq \beta_t\|x(t) - \bar{x}(t)\|, \tag{13}$$

where $\beta_t, \xi_t : \Omega \to (0,1)$ are measurable mappings. Since $A_{1,t}$ is randomly $s_{1,t}$-Lipschitz continuous and E_t is randomly $(\gamma_{1,t}, \alpha_{1,t})$-relaxed cocoercive with respect to $A_{1,t}$, from (13) and Lemma 2.8, we have

$$\begin{aligned}
&\|A_{1,t}(x(t)) - A_{1,t}(\bar{x}(t)) - \rho_t(E_t(x(t), y(t)) - E_t(\bar{x}(t), y(t)))\|^q \\
&\le \|A_{1,t}(x(t)) - A_{1,t}(\bar{x}(t))\|^q \\
&\quad - q\rho_t\langle E_t(x(t), y(t)) - E_t(\bar{x}(t), y(t)), j_q(A_{1,t}(x(t)) - A_{1,t}(\bar{x}(t)))\rangle \\
&\quad + c_q\rho_t^q\|E_t(x(t), y(t)) - E_t(\bar{x}(t), y(t))\|^q \\
&\le s_{1,t}^q\|x(t) - \bar{x}(t)\|^q \\
&\quad - q\rho_t(-\gamma_{1,t}\|E_t(x(t), y(t)) - E_t(\bar{x}(t), y(t))\|^q + \alpha_{1,t}\|A_{1,t}(x(t)) - A_{1,t}(\bar{x}(t))\|^q) \\
&\quad + c_q\rho_t^q\beta_t^q\|x(t) - \bar{x}(t)\|^q \\
&\le s_{1,t}^q\|x(t) - \bar{x}(t)\|^q \\
&\quad - q\rho_t(-\gamma_{1,t}\beta_t^q\|x(t) - \bar{x}(t)\|^q + \alpha_{1,t}s_{1,t}^q\|x(t) - \bar{x}(t)\|^q) + c_q\rho_t^q\beta_t^q\|x(t) - \bar{x}(t)\|^q \\
&\le \left[s_{1,t}^q - q\rho_t(-\gamma_{1,t}\beta_t^q + \alpha_{1,t}s_{1,t}^q) + c_q\rho_t^q\beta_t^q\right]\|x(t) - \bar{x}(t)\|^q.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\|A_{1,t}(x(t)) - A_{1,t}(\bar{x}(t)) - \rho_t((E_t(x(t), y(t)) - E_t(\bar{x}(t), y(t))))\| \\
&\le \sqrt[q]{s_{1,t}^q - q\rho_t(-\gamma_{1,t}\beta_t^q + \alpha_{1,t}s_{1,t}^q) + c_q\rho_t^q\beta_t^q}\,\|x(t) - \bar{x}(t)\|. \qquad (14)
\end{aligned}$$

Combining (11)-(14), we have

$$\|a_1 - b_1\| \le \vartheta_{1,t}\|y(t) - \bar{y}(t)\| + \theta_{1,t}\|x(t) - \bar{x}(t)\|, \qquad (15)$$

where

$$\begin{aligned}
\theta_{1,t} &= \upsilon_{1,t} + \frac{\tau_{1,t}^{q-1}}{r_{1,t} - \rho_t m_{1,t}}\sqrt[q]{s_{1,t}^q - q\rho_t(-\gamma_{1,t}\beta_t^q + \alpha_{1,t}s_{1,t}^q) + c_q\rho_t^q\beta_t^q}, \\
\vartheta_{1,t} &= \frac{\tau_{1,t}^{q-1}\rho_t\xi_t}{r_{1,t} - \rho_t m_{1,t}}.
\end{aligned}$$

Since G_t is randomly μ_t-Lipschitz continuous for first variable with measurable mapping $\mu : \Omega \to (0, 1)$ and randomly ζ_t-Lipschitz continuous for second variable with measurable mapping $\zeta : \Omega \to (0, 1)$ and $\tilde{T}$ is randomly $\kappa_t - \widetilde{D}$-Lipschitz continuous with measurable mapping $\kappa : \Omega \to (0, 1)$, we have

$$\|G_t(u(t), y(t)) - G_t(u(t), \bar{y}(t))\| \le \zeta_t\|y(t) - \bar{y}(t)\| \qquad (16)$$

and

$$\begin{aligned}
\|G_t(u(t), y(t)) - G_t(\bar{u}(t), y(t))\| &\le \mu_t\|u(t) - \bar{u}(t)\| \\
&\le \mu_t(1 + \iota)\widetilde{D}(\tilde{T}_t(x(t)), \tilde{T}_t(\bar{x}(t))) \\
&\le \mu_t\kappa_t(1 + \iota)\|x(t) - \bar{x}(t)\|. \qquad (17)
\end{aligned}$$

Similarly, by the assumptions of $A_{2,t}, G_t, \tilde{T}_t$ and (16), we have

$$
\begin{aligned}
&\|a_2 - b_2\| \\
&= \|J^{N_t(\cdot,y(t))}_{\varrho_t,A_{2,t}}[A_{2,t}(y(t)) - \varrho_t G_t(u(t),y(t))] \\
&\quad - J^{N_t(\cdot,\bar{y}(t))}_{\varrho_t,A_{2,t}}[A_{2,t}(\bar{y}(t)) - \varrho_t G_t(\bar{u}(t),\bar{y}(t))]\| \\
&\le \|J^{N_t(\cdot,y(t))}_{\varrho_t,A_{2,t}}[A_{2,t}(y(t)) - \varrho_t G_t(u(t),y(t))] \\
&\quad - J^{N_t(\cdot,\bar{y}(t))}_{\varrho_t,A_{2,t}}[A_{2,t}(y(t)) - \varrho_t G_t(u(t),y(t))]\| \\
&\quad + \|J^{N_t(\cdot,\bar{y}(t))}_{\varrho_t,A_{2,t}}[A_{2,t}(y(t)) - \varrho_t G_t(u(t),y(t))] \\
&\quad - J^{N_t(\cdot,\bar{y}(t))}_{\varrho_t,A_{2,t}}[A_{2,t}(\bar{y}(t)) - \varrho_t G_t(\bar{u}(t),\bar{y}(t))]\| \\
&\le \upsilon_{2,t}\|y(t) - \bar{y}(t)\| \\
&\quad + \frac{\tau_{2,t}^{q-1}}{r_{2,t} - \varrho_t m_{2,t}}\|A_{2,t}(y(t)) - A_{2,t}(\bar{y}(t)) - \varrho_t(G_t(u(t),y(t)) - G_t(\bar{u}(t),\bar{y}(t)))\| \\
&\le \upsilon_{2,t}\|y(t) - \bar{y}(t)\| \\
&\quad + \frac{\tau_{2,t}^{q-1}}{r_{2,t} - \varrho_t m_{2,t}}\|A_{2,t}(y(t)) - A_{2,t}(\bar{y}(t)) - \varrho_t(G_t(u(t),y(t)) - G_t(u(t),\bar{y}(t)))\| \\
&\quad + \frac{\tau_{2,t}^{q-1}}{r_{2,t} - \varrho_t m_{2,t}}\varrho_t\|G_t(u(t),\bar{y}(t)) - G_t(\bar{u}(t),\bar{y}(t))\|.
\end{aligned} \tag{18}
$$

Since $A_{2,t}$ is randomly $s_{2,t}$-Lipschitz continuous and G_t is randomly $(\gamma_{2,t}, \alpha_{2,t})$-relaxed cocoercive with respect to $A_{2,t}$, from (16) and Lemma 2.8, we have

$$
\begin{aligned}
&\|A_{2,t}(y(t)) - A_{2,t}(\bar{y}(t)) - \varrho_t(G_t(u(t),y(t)) - G_t(u(t),\bar{y}(t)))\|^q \\
&\le \|A_{2,t}(y(t)) - A_{2,t}(\bar{y}(t))\|^q \\
&\quad - q\varrho_t\langle G_t(u(t),y(t)) - G_t(u(t),\bar{y}(t)), j_q(A_{2,t}(y(t)) - A_{2,t}(\bar{y}(t)))\rangle \\
&\quad + c_q\varrho_t^q\|A_{2,t}(y(t)) - A_{2,t}(\bar{y}(t))\|^q \\
&\le \|A_{2,t}(y(t)) - A_{2,t}(\bar{y}(t))\|^q - q\varrho_t(-\gamma_{2,t}\|G_t(u(t),y(t)) - G_t(u(t),\bar{y}(t))\|^q \\
&\quad + \alpha_{2,t}\|A_{2,t}(y(t)) - A_{2,t}(\bar{y}(t))\|^q) + c_q\varrho_t^q\|G_t(u(t),y(t)) - G_t(u(t),\bar{y}(t))\|^q \\
&\le s_{2,t}^q\|y(t) - \bar{y}(t)\|^q - q\varrho_t(-\gamma_{2,t}\zeta_t^q\|y(t) - \bar{y}(t)\|^q + \alpha_{2,t}s_{2,t}^q\|y(t) - \bar{y}(t)\|^q) \\
&\quad + c_q\varrho_t^q\zeta_t^q\|y(t) - \bar{y}(t)\|^q \\
&\le (s_{2,t}^q - q\varrho_t(-\gamma_{2,t}\zeta_t^q + \alpha_{2,t}s_{2,t}^q) + c_q\varrho_t^q\zeta_t^q)\|y(t) - \bar{y}(t)\|^q.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&\|A_{2,t}(y(t)) - A_{2,t}(\bar{y}(t)) - \varrho_t(G_t(u(t),y(t)) - G_t(u(t),\bar{y}(t)))\| \\
&\quad \le \sqrt[q]{s_{2,t}^q - q\varrho_t(-\gamma_{2,t}\zeta_t^q + \alpha_{2,t}s_{2,t}^q) + c_q\varrho_t^q\zeta_t^q}\,\|y(t) - \bar{y}(t)\|.
\end{aligned} \tag{19}
$$

Combining (18) and (19), we have

$$\|a_2 - b_2\| \le \theta_{2,t}(\iota)\|x(t) - \bar{x}(t)\| + \vartheta_{2,t}\|y(t) - \bar{y}(t)\|, \tag{20}$$

where

$$\vartheta_{2,t} = \upsilon_{2,t} + \frac{\tau_{2,t}^{q-1}}{r_{2,t} - \varrho_t m_{2,t}} \sqrt[q]{s_{2,t}^q - q\varrho_t(-\gamma_{2,t}\zeta_t^q + \alpha_{2,t}s_{2,t}^q) + c_q\varrho_t^q\zeta_t^q},$$

$$\theta_{2,t} = \frac{\tau_{2,t}^{q-1}\varrho_t\mu_t\kappa_t(1+\iota)}{r_{2,t} - \varrho_t m_{2,t}}.$$

It follows from (15) and (20) that

$$\begin{aligned} \|a_1 - b_1\| + \|a_2 - b_2\| &\le (\theta_{1,t} + \theta_{2,t}(1+\iota))\|x(t) - \bar{x}(t)\| \\ &\quad + (\vartheta_{1,t} + \vartheta_{2,t})\|y(t) - \bar{y}(t)\| \\ &\le \theta_t(\iota)(\|x(t) - \bar{x}(t)\| + \|y(t) - \bar{y}(t)\|), \end{aligned} \tag{21}$$

where

$$\theta_t(\iota) = \max\{\theta_{1,t} + \theta_{2,t}(\iota), \vartheta_{1,t} + \vartheta_{2,t}\}.$$

It follows from condition (6) that $\theta_t < 1$. Hence from (21), we get

$$\begin{aligned} d((a_1, a_2), Q_{\rho_t,\varrho_t}(\bar{x}(t), \bar{y}(t))) &= \inf_{(b_1,b_2)\in Q_{\rho_t,\varrho_t}(\bar{x}(t),\bar{y}(t))} \Big(\|a_1 - b_1\| + \|a_2 - b_2\|\Big) \\ &\le -\theta(\iota)\|(x(t), y(t)) - (\bar{x}(t), \bar{y}(t))\|. \end{aligned}$$

Since $(a_1, a_2) \in Q_{\rho_t,\varrho_t}(x(t), y(t))$ is arbitrary, we obtain

$$\begin{aligned} \sup_{(a_1,a_2)\in Q_{\rho_t,\varrho_t}(x(t),y(t))} & d((a_1, a_2), Q_{\rho_t,\varrho_t}(\bar{x}(t), \bar{y}(t))) \\ &\le -\theta_t(\iota)\|(x(t), y(t)) - (\bar{x}(t), \bar{y}(t))\|. \end{aligned}$$

By the same argument, we can prove

$$\begin{aligned} \sup_{(b_1,b_2)\in Q_{\rho_t,\varrho_t}(\bar{x}(t),\bar{y}(t))} & d((b_1, b_2), G_{\rho_t,\varrho_t}(x(t), y(t))) \\ &\le -\theta_t(\iota)\|(x(t), y(t)) - (\bar{x}(t), \bar{y}(t))\|. \end{aligned}$$

It follows from the definition of Hausdorff metric $\widetilde{D}$ on $CB(X_1 \times X_2)$ that

$$\widetilde{D}(Q_{\rho_t,\varrho_t}(x(t), y(t)), Q_{\rho_t,\varrho_t}(\bar{x}(t), \bar{y}(t))) \le -\theta_t(\iota)\|(x(t), y(t)) - (\bar{x}(t), \bar{y}(t))\|$$

for all $(x(t), \bar{x}(t)) \in X_1 \times X_1, (y(t), \bar{y}(t)) \in X_2 \times X_2, t \in \Omega$. Letting $\iota \to 0$, we get

$$\widetilde{D}(Q_{\rho_t,\varrho_t}(x(t), y(t)), Q_{\rho_t,\varrho_t}(\bar{x}(t), \bar{y}(t))) \le -\theta_t\|(x(t), y(t)) - (\bar{x}(t), \bar{y}(t))\|,$$

for all $(x(t), \bar{x}(t)) \in X_1 \times X_1, (y(t), \bar{y}(t)) \in X_2 \times X_2, t \in \Omega$, where $\vartheta_{2,t} : \Omega \to (0, 1)$ is measurable and

$$\theta_t = \max\{\theta_{1,t} + \theta_{2,t}, \vartheta_{1,t} + \vartheta_{2,t}\},$$

$$\theta_{1,t} = \upsilon_{1,t} + \frac{\tau_{1,t}^{q-1}}{r_{1,t} - \rho_t m_{1,t}} \sqrt[q]{s_{1,t}^q - q\rho_t(-\gamma_{1,t}\beta_t^q + \alpha_{1,t}s_t^q) + c_q\rho_t^q\beta_t^q},$$

$$\vartheta_{1,t} = \frac{\tau_{1,t}^{q-1}\rho_t\xi_t}{r_{1,t} - \rho_t m_{1,t}},$$

$$\vartheta_{2,t} = \upsilon_{2,t} + \frac{\tau_{2,t}^{q-1}}{r_{2,t} - \varrho_t m_{2,t}} \sqrt[q]{s_{2,t}^q - q\varrho_t(-\gamma_{2,t}\zeta_t^q + \alpha_{2,t}s_{2,t}^q) + c_q\varrho_t^q\zeta_t^q},$$

$$\theta_{2,t} = \frac{\tau_{2,t}^{q-1}\varrho_t\mu_t\kappa_t}{r_{2,t} - \varrho_t m_{2,t}}.$$

Let $x(t), \bar{x}(t) : \Omega \to X_1$ be measurable mappings, $y(t), \bar{y}(t) : \Omega \to X_2$ be also measurable mappings and $Q_{\rho_t,\varrho_t}(x(t), y(t))$ be a randomly multi-valued contractive mapping. By the fixed point Theorem of Nadler [26], $Q_{\rho_t,\varrho_t}(x(t), y(t))$ has a fixed point $(x(t), y(t)) \in X_1 \times X_2$ *i.e.,* $(x(t), y(t)) \in Q_{\rho_t,\varrho_t}(x(t), y(t))$. By the definition of Q_{ρ_t,ϱ_t}, we know that there exists $u(t) \in \tilde{T}_t(x(t)), T_{t,x(t)}(u(t)) \geq a(x(t))$ such that (2) holds. Thus it follows from Lemma 2.6 that there exist measurable mappings $x^*(t) : \Omega \to X_1$ and $y^*(t) : \Omega \to X_2$ such that $(x^*(t), y^*(t), u^*(t))$ with $u^*(t) \in \tilde{T}_t(x(t))$ is a solution of problem (1). □

4. Iterative Algorithms and Convergence Analysis

In this section, based on Lemma 2.6 and Nadler [26], we shall construct a new class of iterative algorithms for solving problems (1) and discuss the convergence analysis of the algorithms.

Algorithm 1. Assume that $X_i, A_i, \eta_i, M, N, E, G, T, \tilde{T}$ are same as in problem (1) for $i = 1, 2$ and $x_0 : \Omega \to X_1, y_0 : \Omega \to X_2, u_0 : \Omega \to 2^{X_1}$ are measurable mappings. For $a : X_2 \to (0,1)$ and the random element $(x(t), y(t), u(t)) \in X_1 \times X_2 \times X_1$, we define the random iterative sequences $\{x_n(t)\}, \{y_n(t)\}, \{u_n(t)\}$ by

$$\begin{cases} x_{n+1}(t) & = (1 - \lambda_n(t))x_n(t) \\ & \quad +\lambda_n(t)\left[J_{\rho_t,A_{1,t}}^{M_t(\cdot,x_n(t))}(A_{1,t}(x_n(t)) - \rho_t E_t(x_n(t), y_n(t)))\right] + p_n(t), \\ y_{n+1}(t) & = (1 - \lambda_n(t))y_n(t) \\ & \quad +\lambda_n(t)\left[J_{\varrho_t,A_{2,t}}^{N_t(\cdot,y_n(t))}(A_{2,t}(y_n(t)) - \varrho_t G_t(u_n(t), y_n(t)))\right] + q_n(t), \\ u(t) & \in \tilde{T}_t(x(t)), T_{t,x(t)}(u_n(t)) \geq a(x_n(t)), \\ \|u_n(t) & -u(t)\| \leq (1+\iota)\widetilde{\mathfrak{D}}(\tilde{T}_t(x_n(t)), \tilde{T}_t(x(t))), \end{cases} \tag{22}$$

where $\rho, \varrho : \Omega \to (0,1)$ are measurable mappings, $\{\lambda_n(t)\}$ is a measurable sequences in $(0,1]$, and $p_n(t), q_n(t)$ are two random errors sequences satisfying the same conditions 4.2 in X_1 and X_2, respectively.

Lemma 4.1. [27] *Let* $\{a_n\}, \{b_n\}$ *and* $\{c_n\}$ *be three real sequences of nonnegative numbers satisfying the following conditions:*

(i) $0 \leq b_n < 1, \ n = 0, 1, 2, \cdots$ *and* $\limsup_n b_n < 1$;
(ii) $\Sigma_{n=0}^{\infty} c_n < +\infty$;
(ii) $a_{n+1} \leq b_n a_n + c_n, \ n = 0, 1, 2, \cdots$.

Then, we have $\lim_{n\to\infty} a_n = 0$.

Theorem 4.2. *Let* $X_1, X_2, T_t, \tilde{T}_t, \eta_{1,t}, \eta_{2,t}, A_{1,t}, A_{2,t}, M_t, N_t, E_t, G_t$ *be the same as in Theorem 3.1. Assume that all the conditions of Theorem 3.1 hold and*

$$\limsup_n \lambda_n(t) < 1, \ \Sigma_{n=0}^{\infty} (\|p_n(t)\| + \|q_n(t)\|) < +\infty. \tag{23}$$

Then the random iterative sequences $(x_n(t), y_n(t), u_n(t))$ *with* $u_n(t) \in \tilde{T}_t(x(t))$ *defined by random Algorithm 1 converges strongly to the random solution* $(x^*(t), y^*(t), u^*(t))$ *with* $u^* \in \tilde{T}_t(x^*(t))$ *of* (1).

Proof. From Theorem 3.1, problem (1) admits a random solution $(x_n^*(t), y_n^*(t), u_n^*(t))$ with $u^* \in \tilde{T}_t(x^*(t))$. It follows from Lemma 2.6 that

$$\begin{cases} x^*(t) = (1-\lambda_n(t))x^*(t) + \lambda_n(t)\left[J_{\rho_t,A_{1,t}}^{M_t(\cdot,x^*(t))}(A_{1,t}(x^*(t)) - \rho_t E_t(x^*(t), y^*(t)))\right], \\ y^*(t) = (1-\lambda_n(t))y^*(t) + \lambda_n(t)\left[J_{\varrho_t,A_{2,t}}^{N_t(\cdot,y^*(t))}(A_{2,t}(y^*(t)) - \varrho_t G_t(u^*(t), y^*(t)))\right]. \end{cases} \tag{24}$$

It follows from (22) and (24) and the assumptions that

$$\begin{aligned} &\|x_{n+1}(t) - x^*(t)\| \\ &= (1-\lambda_n(t))\|x_n(t) - x^*(t)\| \\ &\quad + \lambda_n(t)\|J_{\rho_t,A_{1,t}}^{M_t(\cdot,x_n(t))}(A_{1,t}(x_n(t)) - \rho_t E_t(x_n(t), y_n(t))) \\ &\quad - J_{\rho_t,A_{1,t}}^{M_t(\cdot,x^*(t))}(A_{1,t}(x^*(t)) - \rho_t E_t(x^*(t), y^*(t)))\| + \|p_n(t)\| \\ &\leq (1-\lambda_n(t))\|x_n(t) - x^*(t)\| \\ &\quad + \lambda_n(t)\|J_{\rho_t,A_{1,t}}^{M_t(\cdot,x_n(t))}(A_{1,t}(x_n(t)) - \rho_t E_t(x_n(t), y_n(t))) \\ &\quad - J_{\rho_t,A_{1,t}}^{M_t(\cdot,x_n(t))}(A_{1,t}(x^*(t)) - \rho_t E_t(x^*(t), y^*(t)))\| + \|p_n(t)\| \\ &\quad + \lambda_n(t)\|J_{\rho_t,A_{1,t}}^{M_t(\cdot,x_n(t))}(A_{1,t}(x^*(t)) - \rho_t E_t(x^*(t), y^*(t))) \\ &\quad - J_{\rho_t,A_{1,t}}^{M_t(\cdot,x^*(t))}(A_{1,t}(x^*(t)) - \rho_t E_t(x^*(t), y^*(t)))\| \end{aligned}$$

$$
\begin{aligned}
&\leq (1-\lambda_n(t))\|x_n(t)-x^*(t)\| + \lambda_n(t)\upsilon_{1,t}\|x_n(t)-x^*(t)\| \\
&\quad + \lambda_n(t)\frac{\tau_{1,t}^{q-1}\rho_t}{r_{1,t}-\rho_t m_{1,t}}\|E_t(x^*(t),y_n(t)) - E_t(x^*(t),y^*(t)))\| + \|p_n(t)\| \\
&\quad + \lambda_n(t)\frac{\tau_{1,t}^{q-1}}{r_{1,t}-\rho_t m_{1,t}} \\
&\quad \times \|A_{1,t}(x_n(t)) - A_{1,t}(x^*(t)) - \rho_t(E_t(x_n(t),y_n(t)) - E_t(x^*(t),y_n(t)))\| \\
&\leq (1-\lambda_n(t))\|x_n(t)-x^*(t)\| + \lambda_n(t)\upsilon_{1,t}\|x_n(t)-x^*(t)\| \\
&\quad + \lambda_n(t)\frac{\tau_{1,t}^{q-1}\rho_t\xi_t}{r_{1,t}-\rho_t m_{1,t}}\|y_n(t)-y^*(t)\| + \|p_n(t)\| \\
&\quad + \lambda_n(t)\frac{\tau_{1,t}^{q-1}}{r_{1,t}-\rho_t m_{1,t}}\sqrt[q]{s_{1,t}^q q\rho_t(-\gamma_{1,t}\beta_t^q + \alpha_{1,t}s_{1,t}^q) + c_q\rho_t^q\beta_t^q}\|x_n(t)-x^*(t)\| \\
&\leq (1-\lambda_n(t))\|x_n(t)-x^*(t)\| + \lambda_n(t)\theta_{1,t}\|x_n(t)-x^*(t)\| \\
&\quad + \lambda_n(t)\vartheta_{1,t}\|y_n(t)-y^*(t)\| \quad + \|p_n(t)\|, \qquad (25)
\end{aligned}
$$

where

$$
\begin{aligned}
\theta_{1,t} &= \upsilon_{1,t} + \frac{\tau_{1,t}^{q-1}}{r_{1,t}-\rho_t m_{1,t}}\sqrt[q]{s_{1,t}^q - q\rho_t(-\gamma_{1,t}\beta_t^q + \alpha_{1,t}s_{1,t}^q) + c_q\rho_t^q\beta_t^q}, \\
\vartheta_{1,t} &= \frac{\tau_{1,t}^{q-1}\rho_t\xi_t}{r_{1,t}-\rho_t m_{1,t}}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\|y_{n+1}(t)-y^*(t)\| \\
&= (1-\lambda_n(t))\|y_n(t)-y^*(t)\| \\
&\quad + \lambda_n(t)\|J_{\varrho_t,A_{2,t}}^{N_t(\cdot,y_n(t))}(A_{2,t}(y_n(t)) - \varrho_t G_t(u_n(t),y_n(t))) \\
&\quad - J_{\varrho_t,A_{2,t}}^{N_t(\cdot,y^*(t))}(A_{2,t}(y^*(t)) - \varrho_t G_t(u^*(t),y^*(t)))\| + \|q_n(t)\| \\
&\leq (1-\lambda_n(t))\|y_n(t)-y^*(t)\| \\
&\quad + \lambda_n(t)\|J_{\varrho_t,A_{2,t}}^{N_t(\cdot,y_n(t))}(A_{2,t}(y_n(t)) - \varrho_t G_t(u_n(t),y_n(t))) \\
&\quad - J_{\varrho_t,A_{2,t}}^{N_t(\cdot,y_n(t))}(A_{2,t}(y^*(t)) - \varrho_t G_t(u^*(t),y^*(t)))\| + \|q_n(t)\| \\
&\quad + \lambda_n(t)\|J_{\varrho_t,A_{2,t}}^{N_t(\cdot,y_n(t))}(A_{2,t}(y^*(t)) - \varrho_t G_t(u^*(t),y^*(t))) \\
&\quad - J_{\varrho_t,A_{2,t}}^{N_t(\cdot,y^*(t))}(A_{2,t}(y^*(t)) - \varrho_t G_t(u^*(t),y^*(t)))\|
\end{aligned}
$$

$$
\begin{aligned}
&\le (1-\lambda_n(t))\|y_n(t)-y^*(t)\| + \lambda_n(t)\upsilon_{2,t}\|y_n(t)-y^*(t)\| \\
&\quad + \lambda_n(t)\frac{\tau_{2,t}^{q-1}\varrho_t}{r_{2,t}-\varrho_t m_{2,t}}\|G_t(u_n(t),y^*(t)) - G_t(u^*(t),y^*(t)))\| + \|q_n(t)\| \\
&\quad + \lambda_n(t)\frac{\tau_{2,t}^{q-1}}{r_{2,t}-\varrho_t m_{2,t}} \\
&\quad \times \|A_{2,t}(y_n(t)) - A_{2,t}(y^*(t)) - \varrho_t(G_t(u_n(t),y_n(t))) - G_t(u_n(t),y^*(t)))\| \\
&\le (1-\lambda_n(t))\|y_n(t)-y^*(t)\| + \lambda_n(t)\upsilon_{2,t}\|y_n(t)-y^*(t)\| \\
&\quad + \lambda_n(t)\frac{\tau_{2,t}^{q-1}\varrho_t\mu_t}{r_{2,t}-\varrho_t m_{2,t}}\|x_n(t)-x^*(t)\| + \|q_n(t)\| \\
&\quad + \lambda_n(t)\frac{\tau_{2,t}^{q-1}}{r_{2,t}-\varrho_t m_{2,t}}\sqrt[q]{s_{2,t}^q - q\varrho_t(-\gamma_{2,t}\zeta_t^q + \alpha_{2,t}s_{2,t}^q) + c_q\varrho_t^q\zeta_t^q}\|y_n(t)-y^*(t)\| \\
&\le (1-\lambda_n(t))\|y_n(t)-y^*(t)\| + \lambda_n(t)\theta_{2,t}(1+\iota)\|x_n(t)-x^*(t)\| \\
&\quad + \lambda_n(t)\vartheta_{2,t}\|y_n(t)-y^*(t)\| + \|q_n(t)\|, \qquad (26)
\end{aligned}
$$

where

$$
\begin{aligned}
\theta_{2,t} &= \frac{\tau_{2,t}^{q-1}\varrho_t\mu_t\kappa_t(1+\iota)}{r_{1,t}-\rho_t m_{1,t}}, \\
\vartheta_{2,t} &= \upsilon_{2,t} + \frac{\tau_{2,t}^{q-1}}{r_{2,t}-\varrho_t m_{2,t}}\sqrt[q]{s_{2,t}^q - q\varrho_t(-\gamma_{2,t}\zeta_t^q + \alpha_{2,t}s_{2,t}^q) + c_q\varrho_t^q\zeta_t^q}.
\end{aligned}
$$

From (25) and (26) we have

$$
\begin{aligned}
&\|x_{n+1}(t)-x^*(t)\| + \|y_{n+1}(t)-y^*(t)\| \\
&\le (1-\lambda_n(t)+\lambda_n(t)\theta_t(\iota))\left(\|x_n(t)-x^*(t)\| + \|y_n(t)-y^*(t)\|\right) \\
&\quad + \left(\|p_n(t)\| + \|q_n(t)\|\right), \qquad (27)
\end{aligned}
$$

where $\theta_t(\iota)$ is the same as in (21). Let $\iota \to 0$ and

$a_n = \|x_n(t)-x^*(t)\|+\|y_n(t)-y^*(t)\|, b_n = 1-\lambda_n(t)(1-\theta_t), c_n = \|p_n(t)\|+\|q_n(t)\|$

where θ_t is the same as in (21). Then (27) can be rewritten as

$$a_{n+1} \le b_n a_n + c_n,\ n = 0,1,2,\cdots.$$

From (23) we know that the $\limsup_n b_n < 1$ and $\sum_{n=0}^{\infty} c_n < +\infty$. It follows from Lemma 4.1 that

$$\|x_n(t)-x^*(t)\| + \|y_n(t)-y^*(t)\| \to 0 \text{ as } n \to \infty.$$

Therefore $(x_n(t), y_n(t), u_n(t))$ with $u_n(t) \in \tilde{T}_t(x(t)), t \in \Omega$ defined by Algorithm 1 converges strongly to the random solution $(x^*(t), y^*(t), u^*(t))$ with $u^*(t) \in \tilde{T}_t(x(t))$ and $T_{t,x(t)}(u(t)) \ge a(x(t))$. This completes the proof. □

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Jong Kyu Kim
Department of Mathematics Education,
Kyungnam University, Changwon, Gyeongnam, 51767, Korea
E-mail address: jongkyuk@kyungnam.ac.kr

Salahuddin
Department of Mathematics, Jazan University,
Jazan, Kingdom of Saudi Arabia
E-mail address: salahuddin12@mailcity.com