East Asian Math. J.
Vol. 34 (2018), No. 3, pp. 249–263
http://dx.doi.org/10.7858/eamj.2018.018

CERTAIN GRONWALL TYPE INEQUALITIES ASSOCIATED WITH RIEMANN-LIOUVILLE k- AND HADAMARD k-FRACTIONAL DERIVATIVES AND THEIR APPLICATIONS

Kottakkaran Sooppy Nisar, Gauhar Rahman, Junesang Choi*, Shahid Mubeen, and Muhammad Arshad

Abstract. We aim to establish certain Gronwall type inequalities associated with Riemann-Liouville k- and Hadamard k-fractional derivatives. The results presented here are sure to be new and potentially useful, in particular, in analyzing dependence solutions of certain k-fractional differential equations of arbitrary real order with initial conditions. Some interesting special cases of our main results are also considered.

1. Introduction and preliminaries

Fractional calculus which is calculus of integrals and derivatives of any arbitrary real or complex order has gained remarkable popularity and importance during the last four decades or so, due mainly to its demonstrated applications in diverse and widespread fields ranging from natural sciences to social sciences. (see, e.g., [2, 5, 9, 10, 12, 15, 17, 18] and references therein). Beginning with the classical Riemann-Liouville fractional integral and derivative operators, a large number of fractional integral and derivative operators and their generalizations have been presented. Also, many authors have established a variety of inequalities for those fractional integral and derivative operators, some of which have turned out to be useful in analyzing solutions of certain fractional integral and differential equations.

In this paper, we establish certain Gronwall type inequalities associated with Riemann-Liouville k- and Hadamard k-fractional derivatives. The results presented here are used in analyzing dependence solutions of certain k-fractional differential equations of arbitrary real order with initial conditions. Some interesting special cases of our main results are also considered. We recall the

Received January 31, 2018; Accepted February 19, 2018.
2010 *Mathematics Subject Classification.* 26A33, 26D10, 26D15.
Key words and phrases. fractional integral; fractional derivative; k-fractional integral; k-fractional derivative; Hadamard k-fractional derivative; fractional integral inequalities; Gronwall type inequalities.
*Corresponding author.

(pISSN 1226-6973, eISSN 2287-2833)

classical Gronwall inequality which is asserted by the following theorem (see, e.g., [4, pp. 14-15]).

Theorem 1.1. *If*

$$z(t) \leq y(t) + \int_{t_0}^{t} k(\rho)\, x(\rho)\, d\rho \quad (t \in [t_0, T)), \tag{1}$$

where the functions $z(t)$*,* $y(t)$*,* $k(t)$ *and* $x(t)$ *are continuous on* $[t_0, T)$*,* $T \leq +\infty$*, and* $k(t) \geq 0$*, then the function* $z(t)$ *satisfies the following inequality:*

$$z(t) \leq y(t) + \int_{t_0}^{t} k(\rho)\, x(\rho)\, \exp\left(\int_{\rho}^{t} k(\tau)\, d\tau \right) d\rho \quad (t \in [t_0, T)). \tag{2}$$

Moreover, if $x(t)$ *is a non-decreasing function on* $[t_0, T)$*, then the following inequality holds:*

$$z(t) \leq y(t) - x(t) + x(t)\, \exp\left(\int_{t_0}^{t} k(\tau)\, d\tau \right) \quad (t \in [t_0, T)). \tag{3}$$

The Gronwall inequality, which is often referred to as Gronwall-Bellman-Raid inequality, has been generalized and used in different contexts (see, e.g., [1, 14, 13]). We, also, recall some definitions of fractional integrals and derivatives by beginning with the Riemann-Liouville fractional integral operator, among various fractional integral operators, which has been extensively investigated. For more details, we refer the reader, for example, to the works [8, 6, 9, 15] and the references therein.

Definition 1. **(i)** The Riemann-Liouville fractional integral $I^{\alpha} f$ of order $\alpha \in \mathbb{C}$ ($\Re(\alpha) > 0$) is defined by

$$\left(I^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_{0}^{x} (x-\tau)^{\alpha-1} f(\tau)\, d\tau \quad (x > 0), \tag{4}$$

where $\Gamma(\alpha)$ is the familiar Gamma function (see, e.g., [19, Section 1.1]).

(ii) The Riemann-Liouville fractional derivative $D^{\alpha} f$ of $\alpha \in \mathbb{C}$ ($\Re(\alpha) \geq 0$) is defined by

$$\begin{aligned} \left(D^{\alpha} f\right)(x) &:= \left(\frac{d}{dx}\right)^{n} \left(I^{n-\alpha} f\right)(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^{n} \int_{0}^{x} \frac{f(\tau)\, d\tau}{(x-\tau)^{\alpha-n+1}} \\ &\quad (n = [\Re(\alpha)] + 1;\ x > 0). \end{aligned} \tag{5}$$

(iii) The Hadamard fractional integral ${}_HD^{\mu}_{1,x}f$ of order $\mu > 0$ is defined by

$$\left({}_HD^{\mu}_{1,x}f\right)(x) := \frac{1}{\Gamma(\mu)} \int\limits_1^x \left(\ln\frac{x}{\tau}\right)^{\mu-1} f(\tau)\frac{d\tau}{\tau} \quad (x > 1). \tag{6}$$

(iv) The Hadamard fractional derivative ${}_HD^{\mu}_{1,x}f$ of order $\mu > 0$ is defined by

$$\left({}_HD^{\mu}_{1,x}f\right)(x) = \frac{1}{\Gamma(n-\mu)} \left(x\frac{d}{dx}\right)^n \int\limits_1^x \left(\ln\frac{x}{\tau}\right)^{n-\mu-1} f(\tau)\,\frac{d\tau}{\tau} \tag{7}$$

$$(n = [\mu] + 1;\ x > 1).$$

Here and in the following, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}$, and $\mathbb{Z}_0^-$ be the sets of complex numbers, real numbers, positive real numbers, positive integers, and non-positive integers, respectively.

Díaz and Pariguan [7] introduced k-gamma function Γ_k defined by

$$\Gamma_k(z) = \int_0^\infty e^{-\frac{t^k}{k}}\, t^{z-1}\, dt \quad \left(\Re(z) > 0;\ k \in \mathbb{R}^+\right), \tag{8}$$

which has the following relationships:

$$\Gamma_k(z+k) = z\,\Gamma_k(z), \quad \Gamma_k(k) = 1 \tag{9}$$

and

$$\Gamma_k(\gamma) = k^{\frac{\gamma}{k}-1}\Gamma\left(\frac{\gamma}{k}\right). \tag{10}$$

Also, k-beta function $B_k(\alpha,\beta)$ is defined by

$$B_k(\alpha,\beta) = \begin{cases} \dfrac{1}{k}\displaystyle\int_0^1 t^{\frac{\alpha}{k}-1}\,(1-t)^{\frac{\beta}{k}-1}\,dt & (\min\{\Re(\alpha),\,\Re(\beta)\} > 0) \\ \dfrac{\Gamma_k(\alpha)\,\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)} & \left(\alpha,\,\beta \in \mathbb{C}\setminus k\mathbb{Z}_0^-\right), \end{cases} \tag{11}$$

where $k\mathbb{Z}_0^-$ denotes the set of k-multiples of the elements in $\mathbb{Z}_0^-$.

By using the k-gamma function Γ_k, Mubeen and Habibullah [11] introduced the following Riemann-Liouville k-fractional integral of order $\alpha \in \mathbb{R}^+$:

$$\left(I_k^\alpha f\right)(x) = \frac{1}{k\Gamma_k(\alpha)} \int\limits_0^x (x-\tau)^{\frac{\alpha}{k}-1} f(\tau)d\tau. \tag{12}$$

We define Hadamard k-fractional integral and derivative (see also [20]) as in the following definition.

Definition 2. (**i**) Hadamard k-fractional integral of order $\mu \in \mathbb{R}^+$ is defined by

$$\left({}_H^k I_{1,x}^{\mu} f\right)(x) = \frac{1}{k\Gamma_k(\mu)} \int_1^x \left(\ln \frac{x}{\tau}\right)^{\frac{\mu}{k}-1} f(\tau)\, \frac{d\tau}{\tau} \quad (x > 1). \tag{13}$$

(**ii**) Hadamard k-fractional derivative of order $\mu \in \mathbb{R}^+$ is defined by

$$\left({}_H^k D_{1,x}^{\mu} f\right)(x) = \frac{1}{k\Gamma_k(n-\mu)} \left(x\frac{d}{dx}\right)^n \int_1^x \left(\ln \frac{x}{\tau}\right)^{\frac{n-\mu}{k}-1} f(\tau)\, \frac{d\tau}{\tau} \tag{14}$$

$$(n = [\mu] + 1;\ x > 1).$$

2. Generalized Gronwall k-fractional integral inequalities

Here, we establish Gronwall type inequalities for the Riemann-Liouville k-fractional integral in (12) and the Hadamard k-fractional integral in (13) which are the generalized forms of the Gronwall inequality.

Theorem 2.1. *Let $k,\, \lambda \in \mathbb{R}^+$. Also, let h and u be non-negative and locally integrable functions defined on $[0,\, X)$ with $X \leq +\infty$. Further, let $\phi(x)$ be a non-negative, non-decreasing, and continuous function on $[0,\, X)$ which is bounded on $[0,\, X)$, that is, $\phi(x) \leq M$ for all $x \in [0,\, X)$ and some $M \in \mathbb{R}^+$. Suppose that the functions h, u, and ϕ satisfy the following inequality:*

$$u(x) \leq h(x) + k\,\phi(x) \int_0^x (x-\rho)^{\frac{\lambda}{k}-1}\, u(\rho)\, d\rho \quad (x \in [0,\, X)). \tag{15}$$

Then

$$u(x) \leq h(x) + \sum_{n=1}^{\infty} \frac{\{k\,\phi(x)\Gamma_k(\lambda)\}^n}{\Gamma_k(n\lambda)} \int_0^x (x-\rho)^{n\frac{\lambda}{k}-1} h(\rho)\, d\rho \quad (x \in [0,\, X)). \tag{16}$$

Proof. We can choose a function $\beta : [0, X) \to \mathbb{R}^+ \cup \{0\}$ satisfying

$$\beta(x)\, u(x) = k\,\phi(x) \int_0^x (x-\rho)^{\frac{\lambda}{k}-1}\, u(\rho)\, d\rho \quad (x \in [0,\, X)).$$

We find from (15) that

$$u(x) \leq h(x) + \beta(x)\, u(x),$$

which, upon repeating n times, yields

$$u(x) \leq \sum_{m=0}^{n-1} \{\beta(x)\}^m\, h(x) + \{\beta(x)\}^n\, u(x) \quad (n \in \mathbb{N}). \tag{17}$$

We claim that

$$\{\beta(x)\}^n\, u(x) \leq \frac{k^n\, \{\phi(x)\Gamma_k(\lambda)\}^n}{\Gamma_k(n\lambda)} \int\limits_0^x (x-\rho)^{n\frac{\lambda}{k}-1}\, u(\rho)\, d\rho \tag{18}$$

$$(x \in [0,\, X);\; n \in \mathbb{N})$$

for any non-negative and locally integrable function u on $[0,\, X)$.

We proceed to prove (18) by using mathematical induction on $n \in \mathbb{N}$. (18) holds trivially for $n = 1$. Assume that (18) holds for some $n \in \mathbb{N}$. Then we have

$$\begin{aligned}\{\beta(x)\}^{n+1}\, u(x) &= \beta(x)\, \{\beta(x)\}^n\, u(x) \\ &\leq \phi(x) \int_0^x (x-\rho)^{\frac{\lambda}{k}-1} \left\{ \int_0^\rho \frac{k^n\, \{\phi(\rho)\Gamma_k(\lambda)\}^n}{\Gamma_k(n\lambda)} (\rho-\tau)^{n\frac{\lambda}{k}-1}\, u(\tau)\, d\tau \right\} d\rho.\end{aligned}$$

Since ϕ is non-negative and non-decreasing on $[0, X)$, we obtain

$$\begin{aligned}&\{\beta(x)\}^{n+1}\, u(x) \\ &\quad\leq \{\phi(x)\}^{n+1} \int_0^x (x-\rho)^{\frac{\lambda}{k}-1} \left\{ \int_0^\rho \frac{\{\Gamma_k(\lambda)\}^n}{k\Gamma_k(n\lambda)} (\rho-\tau)^{n\frac{\lambda}{k}-1}\, u(\tau)\, d\tau \right\} d\rho.\end{aligned}$$

Changing the order of integration, we get

$$\{\beta(x)\}^{n+1}\, u(x) \leq \{\phi(x)\}^{n+1}\; k^{n+1}\, \frac{\{\Gamma_k(\lambda)\}^n}{\Gamma_k(n\lambda)} \int_0^x I(x,\tau)\, u(\tau)\, d\tau, \tag{19}$$

where

$$I(x,\tau) := \frac{1}{k} \int_\tau^x (x-\rho)^{\frac{\lambda}{k}-1} (\rho-\tau)^{n\frac{\lambda}{k}-1}\, d\rho.$$

Changing the variable by letting

$$w = \frac{\rho-\tau}{x-\tau},$$

we find

$$I(x,\tau) = (x-\tau)^{(n+1)\frac{\lambda}{k}-1}\, \frac{1}{k} \int_0^1 (1-w)^{\frac{\lambda}{k}-1}\, w^{n\frac{\lambda}{k}-1}\, dw.$$

Using (11), we have

$$I(x,\tau) = (x-\tau)^{(n+1)\frac{\lambda}{k}-1}\, \frac{\Gamma_k(\lambda)\,\Gamma_k(n\lambda)}{\Gamma_k((n+1)\lambda)}. \tag{20}$$

Applying (20) to (19), we obtain

$$\{\beta(x)\}^{n+1}\, u(x) \leq \{\phi(x)\}^{n+1}\; k^{n+1}\, \frac{\{\Gamma_k(\lambda)\}^{n+1}}{\Gamma_k((n+1)\lambda)} \int_0^x (x-\tau)^{(n+1)\frac{\lambda}{k}-1}\, u(\tau)\, d\tau,$$

which proves (18) for $n+1$. Therefore, by the principle of mathematical induction, (18) holds for all $n \in \mathbb{N}$.

Also, we claim that

$$\lim_{n\to\infty} \{\beta(x)\}^n\, u(x) = 0 \tag{21}$$

for each $x \in [0,\, X)$. Indeed, since $n \to \infty$, we assume that $n\,\frac{\lambda}{k} - 1 > 0$. Then we have $(x-\rho)^{n\frac{\lambda}{k}-1} \le x^{n\frac{\lambda}{k}-1}$ for $\rho \in [0, x]$. Since u is non-negative and locally integrable on $[0, X)$, u is integrable on $[0, x]$ $(x \in [0, X))$. So, u is bounded on $[0, x]$, say, $u(\rho) \le L$ for some $L \in \mathbb{R}^+$ and all $\rho \in [0, x]$. Considering all these accounts into the right-hand side of (18), we have

$$\{\beta(x)\}^n\, u(x) \le \frac{L\, \left(k\, M\, x^{\frac{\lambda}{k}}\right)^n\, \{\Gamma_k(\lambda)\}^n}{\Gamma_k(n\lambda)}.$$

By using (10), we get

$$\{\beta(x)\}^n\, u(x) \le k\, L\, \left\{M\, x^{\frac{\lambda}{k}}\, \Gamma\left(\frac{\lambda}{k}\right)\right\}^n\, \frac{1}{\Gamma\left(\frac{n\,\lambda}{k}\right)}. \tag{22}$$

Applying the following asymptotic formula (see, e.g., [19, p. 6, Eq.(33)]):

$$\Gamma(x+1) \sim \left(\frac{x}{e}\right)^x\, \sqrt{2\pi x} \quad \left(x \to \infty;\ x \in \mathbb{R}^+\right) \tag{23}$$

to the right-hand side of (22), we obtain

$$k\, L\, \left\{M\, x^{\frac{\lambda}{k}}\, \Gamma\left(\frac{\lambda}{k}\right)\right\}^n\, \frac{1}{\Gamma\left(\frac{n\,\lambda}{k}\right)} \sim L \cdot \left(\frac{k\lambda}{2\pi}\right)^{\frac{1}{2}}\, \frac{p^n}{n^{n\frac{\lambda}{k}-\frac{1}{2}}}, \tag{24}$$

where

$$p := M\left(\frac{xek}{\lambda}\right)^{\frac{\lambda}{k}}\, \Gamma\left(\frac{\lambda}{k}\right).$$

Since p, $\lambda/k \in \mathbb{R}^+$ are fixed, it is easy to see that

$$L \cdot \left(\frac{k\lambda}{2\pi}\right)^{\frac{1}{2}}\, \frac{p^n}{n^{n\frac{\lambda}{k}-\frac{1}{2}}} \to 0 \quad (n \to \infty). \tag{25}$$

Considering (24) and (25) in (22), we prove (21).

Finally, taking the limit on both sides of (17) as $n \to \infty$ with the aid of (21) and applying (18) with u replaced by h to the resulting inequality, we obtain (16). This completes the proof. □

We consider two special cases of the result in Theorem 2.1. First, by setting $\phi(x) = b$, where $b \in \mathbb{R}^+ \cup \{0\}$ is a constant, in Theorem 2.1, we have the following assertion in Corollary 2.2.

Corollary 2.2. *Let k, $\lambda \in \mathbb{R}^+$ and $b \in \mathbb{R}^+ \cup \{0\}$. Also, let h and u be non-negative and locally integrable functions defined on $[0,\, X)$ with $X \le +\infty$. Suppose that the functions h and u satisfy the following inequality:*

$$u(x) \le h(x) + k\, b \int_0^x (x-\rho)^{\frac{\lambda}{k}-1}\, u(\rho)\, d\rho \quad (x \in [0,\, X)). \tag{26}$$

Then

$$u(x) \le h(x) + \sum_{n=1}^{\infty} \frac{\{k\, b\, \Gamma_k(\lambda)\}^n}{\Gamma_k(n\lambda)} \int_0^x (x-\rho)^{n\frac{\lambda}{k}-1} h(\rho)\, d\rho \quad (x \in [0,\, X)). \tag{27}$$

Among many generalizations of the Mittag-Leffler function, one of them is recalled (see [21, 22]):

$$E_{\lambda,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\lambda n + \beta)} \quad (\lambda,\, \beta \in \mathbb{C};\, \Re(\lambda) > 0), \tag{28}$$

which is further generalized and called k-Mittag-Leffler function as follows:

$$E_{k,\lambda,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_k(\lambda n + \beta)} \quad \left(\lambda,\, \beta \in \mathbb{C};\, \Re(\lambda) > 0;\, k \in \mathbb{R}^+\right). \tag{29}$$

Corollary 2.3. *Let $k,\, \lambda \in \mathbb{R}^+$. Also, let u be non-negative and locally integrable function defined on $[0,\, X)$ with $X \le +\infty$ and h be non-negative, non-decreasing and locally integrable function on $[0,\, X)$. Further, let $\phi(x)$ be a non-negative, non-decreasing, and continuous function on $[0,\, X)$ which is bounded on $[0,\, X)$, that is, $\phi(x) \le M$ for all $x \in [0,\, X)$ and some $M \in \mathbb{R}^+$. Suppose that the functions h, u, and ϕ satisfy the following inequality:*

$$u(x) \le h(x) + k\, \phi(x) \int_0^x (x-\rho)^{\frac{\lambda}{k}-1}\, u(\rho)\, d\rho \quad (x \in [0,\, X)). \tag{30}$$

Then

$$u(x) \le \left\{1 - k + k\, E_{k,\lambda,k}\left(k\, \Gamma_k(\lambda)\, \phi(x)\, x^{\frac{\lambda}{k}}\right)\right\}\, h(x). \tag{31}$$

Proof. Since h is non-decreasing on $[0,\, X)$, $h(\rho) \le h(x)$ for all $\rho \in [0, x] \subseteq [0,\, X)$. Then we have

$$\int_0^x (x-\rho)^{n\frac{\lambda}{k}-1} h(\rho)\, d\rho \le h(x) \int_0^x (x-\rho)^{n\frac{\lambda}{k}-1}\, d\rho = \frac{k}{n\lambda}\, h(x)\, x^{\frac{n\lambda}{k}}. \tag{32}$$

Applying (32) to (16) and using (9), we obtain the desired inequality (31). □

Remark 1. The particular case of the result in Corollary 2.3 when $k = 1$ is easily seen to reduce to the inequality in [23, Corollary 2].

Theorem 2.4. *Let $k,\, \lambda \in \mathbb{R}^+$. Also, let h and u be non-negative and locally integrable functions defined on $[1,\, X)$ with $X \le +\infty$. Further, let $\phi(x)$ be a non-negative, non-decreasing, and continuous function on $[0,\, X)$ which is bounded* *on* $[1, X)$*, that is,* $\phi(x) \leq M$ *for all* $x \in [1, X)$ *and some* $M \in \mathbb{R}^+$*. Suppose that the functions* h*,* u*, and* ϕ *satisfy the following inequality:*

$$u(x) \leq h(x) + k\,\phi(x) \int_1^x \left(\ln \frac{x}{\rho}\right)^{\frac{\lambda}{k}-1} u(\rho) \frac{d\rho}{\rho} \quad (x \in [1, X)). \tag{33}$$

Then

$$u(x) \leq h(x) + \sum_{n=1}^{\infty} \frac{\{k\,\phi(x)\Gamma_k(\lambda)\}^n}{\Gamma_k(n\lambda)} \int_1^x \left(\ln \frac{x}{\rho}\right)^{n\frac{\lambda}{k}-1} h(\rho)\, \frac{d\rho}{\rho} \quad (x \in [1, X)). \tag{34}$$

Proof. The proof would run parallel to that of Theorem 2.4. We omit the details.

□

Corollary 2.5. *Let* $k, \lambda \in \mathbb{R}^+$*. Also, let* u *be non-negative and locally integrable functions defined on* $[1, X)$ *with* $X \leq +\infty$ *and* h *be non-negative, non-decreasing and locally integrable functions defined on* $[1, X)$*. Further, let* $\phi(x)$ *be a non-negative, non-decreasing, and continuous function on* $[1, X)$ *which is bounded on* $[1, X)$*, that is,* $\phi(x) \leq M$ *for all* $x \in [1, X)$ *and some* $M \in \mathbb{R}^+$*. Suppose that the functions* h*,* u*, and* ϕ *satisfy the following inequality:*

$$u(x) \leq h(x) + k\,\phi(x) \int_1^x \left(\ln \frac{x}{\rho}\right)^{\frac{\lambda}{k}-1} u(\rho) \frac{d\rho}{\rho} \quad (x \in [1, X)).$$

Then

$$u(x) \leq \left\{1 - k + k\,E_{k,\lambda,k}\left(k\,\Gamma_k(\lambda)\phi(x)(\ln x)^{\frac{\lambda}{k}}\right)\right\} h(x). \tag{35}$$

Proof. A similar argument as in the proof of Corollary 2.3 will establish the result here. We omit the details.

□

3. Application to some dependence solutions of k-fractional differential equations

Many researchers have been devoted to study dependence solutions of Riemann-Liouville type or Caputo type fractional differential equations of arbitrary order with some initial conditions, while a few have studied similar problems with Hadamard type fractional derivatives. In this sequel, we show that our results presented in the previous section are useful in analyzing dependence solutions of certain k-fractional differential equations of arbitrary real order with initial conditions. Here, we consider the following system of initial value problem (see [15]):

$$D_k^{\alpha} y(x) = f(x, y(x)) \tag{36}$$

with

$$D_k^{\alpha-1} y(x)\big|_{t=0} = \eta, \tag{37}$$

where $0 < \alpha < 1$, $0 \le x < X$ $(X \le +\infty)$, $f : [0, X) \times \mathbb{R} \to \mathbb{R}$ is a function, and D_k^{α} denotes Riemann-Liouville k-fractional derivative operator of order α. Also, we consider the following system of initial value problem (see [9]):

$${}_{H}^{k}D_{1,x}^{\alpha} y(x) = f(x, y(x)), \tag{38}$$

with

$${}_{H}^{k}D_{1,x}^{\alpha-1} y(x)\big|_{x=1} = \mu, \tag{39}$$

where $0 < \alpha < 1$, $1 \le x < X$ $(X \le +\infty)$, $f : [1, X) \times \mathbb{R} \to \mathbb{R}$ and ${}_{H}^{k}D_{1,x}^{\alpha}$ denotes Hadamard k-fractional derivative operator of order α.

The existence and uniqueness of the initial value problems given as in (36) and (38) with, respectively, Riemann-Liouville and Hadamard fractional derivative operators of order $\alpha \in \mathbb{R}^+$ have been investigated (see, e.g., [9, 15]). We take a function

$$y(x) = \frac{\eta}{k\Gamma_k(\alpha)} x^{\frac{\alpha}{k}-1} + \frac{1}{k\Gamma_k(\alpha)} \int_0^x (x-\rho)^{\frac{\alpha}{k}-1} f(\rho, y(\rho))\, d\rho, \tag{40}$$

which is a solution of the system (36) with (37).

It is noted that, the case $k = 1$ in (40) is equivalent to the initial value problem (36) with (37) given as in [15, pp. 127-128].

Now, we give our results as in the following theorems and corollaries.

Theorem 3.1. *Let k, α, $\delta \in \mathbb{R}^+$ with $0 < \alpha - \delta < \alpha \le 1$. Also, let f be a continuous function satisfying the following Lipschitz condition:*

$$|f(x, y(x)) - f(x, z(x))| \le L\, |y(x) - z(x)| \quad (0 \le x < X), \tag{41}$$

where $L \in \mathbb{R}^+$ is a constant which is independent of the variables x, $y(x)$, $z(x) \in \mathbb{R}$. Further, let y and z be the solutions of (36) with (37) and

$$D_k^{\alpha-\delta} z(x) = f(x, z(x)) \tag{42}$$

with

$$D_k^{\alpha-\delta-1} z(x)\big|_{x=0} = \zeta, \tag{43}$$

respectively. Then, for $0 < x \le X$, we have

$$\begin{aligned} |z(x) - y(x)| \le \mathcal{A}(x; k, \alpha, \delta) &+ \sum_{n=1}^{\infty} \left\{ \frac{L\,\Gamma_k(\alpha-\delta)}{k\,\Gamma_k(\alpha)} \right\}^n \\ &\times \frac{1}{\Gamma_k(n(\alpha-\delta))} \int_0^x (x-\rho)^{\frac{n(\alpha-\delta)}{k}-1} \mathcal{A}(\rho; k, \alpha, \delta)\, d\rho, \end{aligned} \tag{44}$$

where

$$\begin{aligned}\mathcal{A}(x;k,\alpha,\delta) :=& \frac{1}{k}\left|\frac{\zeta}{\Gamma_k(\alpha-\delta)}x^{\frac{\alpha-\delta}{k}-1} - \frac{\eta}{\Gamma_k(\alpha)}x^{\frac{\alpha}{k}-1}\right| \\ &+ \|f\|_\infty \left|\frac{1}{\Gamma_k(\alpha-\delta)} - \frac{1}{\Gamma_k(\alpha)}\right| x^{\frac{\alpha-\delta}{k}} + \frac{\|f\|_\infty}{\Gamma_k(\alpha)}\left(\frac{x^{\frac{\alpha-\delta}{k}}}{\alpha-\delta} + \frac{x^{\frac{\alpha}{k}}}{\alpha}\right)\end{aligned} \tag{45}$$

and

$$\|f\|_\infty := \sup_{0\le\rho\le x} |f(\rho, y(\rho))|.$$

Proof. In view of (40), the solutions of the initial value problems (36) with (37) and (42) with (43) are given, respectively, by

$$y(x) = \frac{\eta}{k\Gamma_k(\alpha)}x^{\frac{\alpha}{k}-1} + \frac{1}{k\Gamma_k(\alpha)}\int_0^x (x-\rho)^{\frac{\alpha}{k}-1} f(\rho, y(\rho))\, d\rho \tag{46}$$

and

$$z(x) = \frac{\zeta}{k\Gamma_k(\alpha-\delta)}x^{\frac{\alpha-\delta}{k}-1} + \frac{1}{k\Gamma_k(\alpha-\delta)}\int_0^x (x-\rho)^{\frac{\alpha-\delta}{k}-1} f(\rho, z(\rho))\, d\rho. \tag{47}$$

We find from (46) and (47) that

$$\begin{aligned}|z(x)-y(x)| \le& \frac{1}{k}\left|\frac{\zeta}{\Gamma_k(\alpha-\delta)}x^{\frac{\alpha-\delta}{k}-1} - \frac{\eta}{\Gamma_k(\alpha)}x^{\frac{\alpha}{k}-1}\right| \\ &+ |\mathcal{R}_1(x;k,\alpha,\delta)| + |\mathcal{R}_2(x;k,\alpha,\delta)| + |\mathcal{R}_3(x;k,\alpha,\delta)|,\end{aligned} \tag{48}$$

where

$$\begin{aligned}\mathcal{R}_1(x;k,\alpha,\delta) :=& \frac{1}{k\Gamma_k(\alpha-\delta)}\int_0^x (x-\rho)^{\frac{\alpha-\delta}{k}-1} f(\rho, z(\rho))\, d\rho \\ &- \frac{1}{k\Gamma_k(\alpha)}\int_0^x (x-\rho)^{\frac{\alpha-\delta}{k}-1} f(\rho, z(\rho))\, d\rho,\end{aligned}$$

$$\begin{aligned}\mathcal{R}_2(x;k,\alpha,\delta) :=& \frac{1}{k\Gamma_k(\alpha)}\int_0^x (x-\rho)^{\frac{\alpha-\delta}{k}-1} f(\rho, z(\rho))\, d\rho \\ &- \frac{1}{k\Gamma_k(\alpha)}\int_0^x (x-\rho)^{\frac{\alpha-\delta}{k}-1} f(\rho, y(\rho)) d\rho\end{aligned}$$

and

$$\mathcal{R}_3(x;k,\alpha,\delta) := \frac{1}{k\Gamma_k(\alpha)} \int_0^x (x-\rho)^{\frac{\alpha-\delta}{k}-1} f(\rho, y(\rho))\, d\rho - \frac{1}{k\Gamma_k(\alpha)} \int_0^x (x-\rho)^{\frac{\alpha}{k}-1} f(\rho, y(\rho))\, d\rho.$$

Using the conditions given in this theorem, we obtain

$$|\mathcal{R}_1(x;k,\alpha,\delta)| \le \left| \frac{1}{\Gamma_k(\alpha-\delta)} - \frac{1}{\Gamma_k(\alpha)} \right| \|f\|_\infty\, x^{\frac{\alpha-\delta}{k}}, \tag{49}$$

$$|\mathcal{R}_2(x;k,\alpha,\delta)| \le \frac{L}{k\,\Gamma_k(\alpha)} \int_0^x (x-\rho)^{\frac{\alpha-\delta}{k}-1} |z(\rho)-y(\rho)|\, d\rho \tag{50}$$

and

$$|\mathcal{R}_3(x;k,\alpha,\delta)| \le \frac{\|f\|_\infty}{\Gamma_k(\alpha)} \left(\frac{x^{\frac{\alpha-\delta}{k}}}{\alpha-\delta} + \frac{x^{\frac{\alpha}{k}}}{\alpha} \right). \tag{51}$$

Applying (49), (50) and (51) to the inequality (49), we get

$$|z(x)-y(x)| \le \mathcal{A}(x;k,\alpha,\delta) + \frac{L}{k\,\Gamma_k(\alpha)} \int_0^x (x-\rho)^{\frac{\alpha-\delta}{k}-1} |z(\rho)-y(\rho)|\, d\rho. \tag{52}$$

Using the result in Theorem 2.1 in the inequality (52), we get the desired result (44). □

Corollary 3.2. *Let k, $\alpha \in \mathbb{R}^+$ with $0 < \alpha \le 1$. Also, let f be a continuous function satisfying the following Lipschitz condition:*

$$|f(x,y(x)) - f(x,z(x))| \le L\,|y(x)-z(x)| \quad (0 \le x < X), \tag{53}$$

where $L \in \mathbb{R}^+$ is a constant which is independent of the variables x, $y(x)$, $z(x) \in \mathbb{R}$. Further, let y and z be the solutions of (36) with (37) and

$$D_k^\alpha z(x) = f(x, z(x)) \tag{54}$$

with

$$D_k^{\alpha-1} z(x)\big|_{x=0} = \zeta, \tag{55}$$

respectively. Then, for $0 < x \le X$, we have

$$\begin{aligned} |z(x)-y(x)| \le{} & \left(\frac{1}{k}-1\right) \frac{|\zeta-\eta|}{\Gamma_k(\alpha)}\, x^{\frac{\alpha}{k}-1} + (1-k)\,\frac{2\,\|f\|_\infty\, x^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \\ & + |\zeta-\eta|\, x^{\frac{\alpha}{k}-1}\, \frac{\Gamma_k\left(\frac{\alpha}{k}\right)}{\Gamma_k(\alpha)}\, E_{k,\frac{\alpha}{k},\frac{\alpha}{k}}\left(\frac{L\,x^{\frac{\alpha}{k}}}{k}\right) \\ & + 2\,\|f\|_\infty\, x^{\frac{\alpha}{k}}\, \frac{k\,\Gamma_k\left(\frac{\alpha}{k}+1\right)}{\Gamma_k(\alpha+k)}\, E_{k,\frac{\alpha}{k},\frac{\alpha}{k}+1}\left(\frac{L\,x^{\frac{\alpha}{k}}}{k}\right), \end{aligned} \tag{56}$$

where $E_{k,\alpha,\beta}(\cdot)$ *is the k-Mittag-Leffler function in* (29) *and*

$$\|f\|_{\infty} := \sup_{0\le\rho\le x} |f(\rho, y(\rho))|.$$

In particular, for $k=1$ *and* $0 < x \le X$*, we have*

$$|z(x) - y(x)| \le |\zeta - \eta|\, x^{\alpha-1}\, E_{\alpha,\alpha}\left(L\, x^{\alpha}\right) + 2\,\|f\|_{\infty}\, x^{\alpha}\, E_{\alpha,\alpha+1}\left(L\, x^{\alpha}\right), \tag{57}$$

where $E_{\alpha,\beta}(\cdot)$ *is the Mittag-Leffler function in* (28)*.*

Proof. Setting $\delta = 0$ in Theorem 3.1, after a simplification, we get the desired result. We omit the details. □

Consider the fractional system as given in (38) with (39) in terms of Hadamard k-fractional derivatives. In this regard, we define the following Volterra-type integral which satisfies (38) with (39):

$$y(x) = \frac{\mu}{k\Gamma_k(\alpha)}\,(\ln x)^{\frac{\alpha}{k}-1} + \frac{1}{k\Gamma_k(\alpha)} \int_1^x \left(\ln\frac{x}{\rho}\right)^{\frac{\alpha}{k}-1} f(\rho, y(\rho))\,\frac{d\rho}{\rho}, \tag{58}$$

which, upon letting $k \to 1$, yields the Volterra-type integral satisfying the system with $k=1$ to the initial value problem in [16].

Theorem 3.3. *Let* k, α, $\delta \in \mathbb{R}^+$ *with* $0 < \alpha - \delta < \alpha \le 1$*. Also, let* f *be a continuous function satisfying the following Lipschitz condition:*

$$|f(x, y(x)) - f(x, z(x))| \le L\,|y(x) - z(x)| \quad (1 \le x < X), \tag{59}$$

where $L \in \mathbb{R}^+$ *is a constant which is independent of the variables* x, $y(x)$, $z(x) \in \mathbb{R}$*. Further, let* y *and* z *be the solutions of* (38) *with* (39) *and*

$${}^{k}_{H}D^{\alpha-\delta}_{1,x}\, z(x) = f(x, z(x)) \tag{60}$$

with

$${}^{k}_{H}D^{\alpha-\delta-1}_{1,x}\, z(x)\Big|_{x=1} = \nu, \tag{61}$$

respectively. Then, for $1 \le x < X$*, we have*

$$\begin{aligned}|z(x) - y(x)| \le\,& \mathcal{B}(x; k, \alpha, \delta) + \sum_{n=1}^{\infty} \frac{\left\{\frac{L\,\Gamma_k(\alpha-\delta)}{k\,\Gamma_k(\alpha)}\right\}^n}{\Gamma_k(n(\alpha-\delta))} \\ &\times \int_1^x \left(\ln\frac{x}{\rho}\right)^{\frac{n(\alpha-\delta)}{k}-1} \mathcal{B}(\rho; k, \alpha, \delta))\,\frac{d\rho}{\rho},\end{aligned} \tag{62}$$

where

$$\begin{aligned}\mathcal{B}(x;k,\alpha,\delta) :=&\frac{1}{k}\left|\frac{\nu}{\Gamma_k(\alpha-\delta)}(\ln x)^{\frac{\alpha-\delta}{k}-1}-\frac{\mu}{\Gamma_k(\alpha)}(\ln x)^{\frac{\alpha}{k}-1}\right|\\ &+\frac{1}{\alpha-\delta}\left|\frac{1}{\Gamma_k(\alpha-\delta)}-\frac{1}{\Gamma_k(\alpha)}\right|(\ln x)^{\frac{\alpha-\delta}{k}}\\ &+\frac{\|f\|_\infty}{\Gamma_k(\alpha)}\left\{\frac{1}{\alpha-\delta}(\ln x)^{\frac{\alpha-\delta}{k}}+\frac{1}{\alpha}(\ln x)^{\frac{\alpha}{k}}\right\}\end{aligned}$$

and

$$\|f\|_\infty := \sup_{0\le\rho\le x}|f(\rho,y(\rho))|.$$

Proof. The solutions of the initial value problems (38) with (39) and (60) with (61) are given by

$$y(x)=\frac{\mu}{k\Gamma_k(\alpha)}(\ln x)^{\frac{\alpha}{k}-1}+\frac{1}{k\Gamma_k(\alpha)}\int_1^x\left(\ln\frac{x}{\rho}\right)^{\frac{\alpha}{k}-1}f(\rho,y(\rho))\frac{d\rho}{\rho}$$

and

$$z(x)=\frac{\nu}{k\Gamma_k(\alpha-\delta)}(\ln x)^{\frac{\alpha-\delta}{k}-1}+\frac{1}{k\Gamma_k(\alpha-\delta)}\int_1^x\left(\ln\frac{x}{\rho}\right)^{\frac{\alpha-\delta}{k}-1}f(\rho,z(\rho))\frac{d\rho}{\rho},$$

respectively. Similarly as in the proof of Theorem 3.1, we obtain

$$|z(x)-y(x)|\le\mathcal{B}(x;k,\alpha,\delta)+\frac{L}{k\,\Gamma_k(\alpha)}\int_1^x\left(\ln\frac{x}{\rho}\right)^{\frac{\alpha-\delta}{k}-1}|z(\rho)-y(\rho)|\,\frac{d\rho}{\rho}.$$

Finally, applying Theorem 2.4, we get the desired result. □

4. Concluding remarks

The main results presented here are further generalizations of the generalized Gronwall type inequalities associated with Riemann-Liouville k- and Hadamard k-fractional derivatives. Also, they are sure to be new and potentially useful. Their special cases when $k = 1$ are seen to yield certain results similar to those known Gronwall type inequalities for the Riemann-Liouville and Hadamard fractional derivatives (cf., [16, 23]).

Acknowledgments

The authors would like to express their deep felt thanks for the reviewers' helpful comments.

References

[1] R. P. Agarwal, S. Deng and W. Zhang, Generalization of a retarded Gronwall-like inequality and its applications, *Appl. Math. Comput.* **165** (2005), 599–612.

[2] S. Belarbi and Z. Dahmani, On some new fractional integral inequalities, *J. Inequal. Pure Appl. Math.* **10**(3) (2009), Article 86.

[3] P. L. Chebyshev, Sur les expressions approximatives des integrales definies par les autres prises entre les mmes limites, *Proc. Math. Soc. Charkov* **2** (1882), 93–98.

[4] C. Corduneanu, *Principles of Differential and Integral Equations*, **55**, Allyn and Bacon, Boston, 1971

[5] Z. Dahmani and L. Tabharit, On weighted Grüss type inequalities via fractional integration, *J. Adv. Res. Pure Math.* **2** (2010), 31–38.

[6] D. Delbosco and L. Rodino, Existence and uniqueness for a nonlinear fractional differential equation, *J. Math. Anal. Appl.* **204** (1996), 609–625.

[7] R. Díaz and E. Pariguan, On hypergeometric functions and k-Pochhammer symbol, *Divulg. Mat.* **15**(2) (2007), 179–192.

[8] K. Diethelm and N. J. Ford, Analysis of fractional differential equations, *J. Math. Anal. Appl.* **265** (2002), 229–248.

[9] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematical Studies, Vol. **204**, Elsevier (North-Holland) Science Publishers, Amsterdam, London and New York, 2006.

[10] V. S. Kiryakova, *Generalized Fractional Calculus and Applications*, Pitman Res Notes Math. **301**, Longman Scientific & Technical; Harlow, Co-published with John Wiley, New York, 1994.

[11] S. Mubeen and G. M. Habibullah, k-fractional integrals and application, *Int. J. Contem. Math. Sci.* **7** (2012), 89–94.

[12] K. S. Ntouyas, P. Agarwal and J. Tariboon, On Polya-Szego and Chebyshev types inequalities involving the Riemann-Liouville fractional integral operators, *J. Math. Inequal.* **10**(2) (2016), 491–504.

[13] B. G. Pachpatte, *Inequalities for Differential and Integral Equations*, Academic Press, New York, 1998.

[14] B. G. Pachpatte, On some generalizations of Bellman's lemma, *J. Math. Anal. Appl.* **5** (1975), 141–150.

[15] I. Podlubny, *Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and Some of their Applications*, Academic Press, San Diego, CA, 1999.

[16] D. Qian, Z. Gong and C. Li, A generalized Gronwall inequality and its application to fractional differential equations with Hadamard derivatives, http://nsc10.cankaya.edu.tr/proceedings/PAPERS/Symp2-FractionalCalculus.

[17] M. Z. Sarikaya, Z. Dahmani, M. E. Kiris and F. Ahmad, $(k;s)$-Riemann-Liouville fractional integral and applications, *Hacet. J. Math. Stat.* **45**(1) (2016), 77–89.

[18] E. Set, M. Tomar and M. Z. Sarikaya, On generalized Grüss type inequalities for k-fractional integrals, *Appl. Math. Comput.* **269** (2015), 29–34.

[19] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London and New York, 2012.

[20] M. Tomar, S. Mubeen and J. Choi, Certain inequalities associated with Hadamard k-fractional integral operators, *J. Inequal. Appl.* **2016** (2016), Article ID 234.

[21] A. Wiman, Über den fundamentalsatz in der theorie der funktionen $E_\alpha(x)$, *Acta Math.* **29** (1905), 191–201.

[22] A. Wiman, Über die Nullstellen der funktionen $E_\alpha(x)$, *Acta Math.* **29** (1905), 217–234.

[23] H. Ye, J. Gao and Y. Ding, A generalized Gronwall inequality and its application to a fractional differential equation, *J. Math. Anal. Appl.* **328** (2007), 1075–1081.

Kottakkaran Sooppy Nisar
Department of Mathematics, College of Arts and Science-Wadi Aldawaser, 11991, Prince Sattam bin Abdulaziz University, Saudi Arabia
E-mail address: n.sooppy@psau.edu.sa

Gauhar Rahman
Department of Mathematics, International Islamic University, Islamabad, Pakistan
E-mail address: gauhar55uom@gmail.com

Junesang Choi
Department of Mathematics, Dongguk University, Gyeongju 38066, Republic of Korea
E-mail address: junesang@mail.dongguk.ac.kr

Shahid Mubeen
Department of Mathematics, University of Sargodha, Sargodha, Pakistan
E-mail address: smjhanda@gmail.com

Muhammad Arshad
Department of Mathematics, International Islamic University, Islamabad, Pakistan
E-mail address: marshad_zia@yahoo.com