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SOLVING SECOND ORDER SINGULARLY PERTURBED DELAY DIFFERENTIAL EQUATIONS WITH LAYER BEHAVIOR VIA INITIAL VALUE METHOD

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Abstract. In this paper, an initial value method for solving a class of singularly perturbed delay differential equations with layer behavior is proposed. In this approach, first the given problem is modified in to an equivalent singularly perturbed problem by approximating the term containing the delay using Taylor series expansion. Then from the modified problem, two explicit Initial Value Problems which are independent of the perturbation parameter, ε, are produced: the reduced problem and boundary layer correction problem. Finally, these problems are solved analytically and combined to give an approximate asymptotic solution to the original problem. To demonstrate the efficiency and applicability of the proposed method three linear and one nonlinear test problems are considered. The effect of the delay on the layer behavior of the solution is also examined. It is observed that for very small ε the present method approximates the exact solution very well.

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1. Introduction

Singularly perturbed differential equations containing a small parameter, ε, multiplying the highest derivative term arise in many fields, such as fluid mechanics, fluid dynamics, chemical reactor theory and elasticity, which have received significant attention. The solution of these types of problems shows a multi-scale character, with a narrow region called the boundary layer, in which their solution changes rapidly, and an outer region in which the solution changes smoothly. Thus, the treatment of such problems is not trivial because of the boundary layer behavior of their solutions[14, 23].

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Any system involving a feedback control will almost involve time delays. These arise because a finite time is required to sense the information and then react to it. A general way for describing this process is to formulate a delay differential equation. If we restrict the class of delay differential equations in which the highest derivative is multiplied by a small parameter, then it is said to be a singularly perturbed delay differential equation (SPDDE) [13].
In the past less attention had been given for the solutions of SPDDEs. However, in recent years, there has been a growing interest in the treatment of such problems. This is due to the versatility of such type of differential equations in the mathematical modeling of processes in various application fields: such as, in the first exit-time problem in the modeling of the activation of neuronal variability [16], in micro-scale heat transfer [21], in optical bi-stable devices [3], in the hydrodynamics of liquid helium [10], to describe the human pupil-light reflex [18], in a variety of models for physiological processes or diseases [19, 22], and variational problems in control theory [9, 17].
Due to this, many researchers have been trying to develop asymptotic and numerical methods for solving such problems. Lange and Miura [15, 16] gave an asymptotic approach for a class of boundary value problems for linear second-order differential-difference equations and initiated the study of such problems. The same author's, have also shown that the effect of very small shifts (of order ε) on the solution and pointed out that they drastically affect the solution and therefore cannot be neglected. On the other hand, Kadalbajoo and Sharma [12, 13] presented numerical approaches to solve singularly perturbed delay differential equations which contains negative shift in the convention term. In recent years, further studies on the numerical and asymptotic treatments of such problems have been considered by different researchers, such as [1, 2, 5, 6, 7, 8]. Therefore, it is natural to raise the question that whether there may be other better techniques that can be developed to solve such problems.
In this paper, an initial value method for solving a class of singularly perturbed delay differential equations with layer behavior is proposed. In this approach, first the given problem is modified in to an equivalent singularly perturbed problem by approximating the term containing the delay using Taylor series expansion. Then from the modified problem, two explicit Initial Value Problems which are independent of the perturbation parameter, ε, are produced; namely, the reduced problem and boundary layer correction problem. Finally, these problems are solved analytically and combined to give an approximate asymptotic solution to the original problem. To demonstrate the efficiency and applicability of the proposed method three linear and one nonlinear test problems are considered. Furthermore, The effect of the delay on the layer behavior of the solution is also examined.

2. Statement of The Problem

Consider a linear singularly perturbed two point boundary value problem with delay on the convection term

$$\varepsilon y''(x) + a(x)y'(x-\delta) + b(x)y(x) = f(x), \tag{1}$$

on $0 < x < 1, \quad 0 < \varepsilon << 1$, subject to the interval and boundary conditions

$$y(x) = \phi(x), \ on \ -\delta \leq x \leq 0, \tag{2a}$$

$$y(1) = \beta, \tag{2b}$$

where $a(x)$, $b(x)$, $f(x)$, and $\phi(x)$ are smooth functions, β is a known constant and δ is the delay parameter. For a function $y(x)$ to constitute a "smooth" solution of this problem, it must satisfy the boundary value problem (1)-(2), be continuous in $[0,1]$, and be continuously differentiable on $(0,1)$ [16].
For $\delta = 0$, the problem becomes a boundary value problem for a singularly perturbed differential equation and as the singular perturbation parameter tends to zero, the order of the corresponding reduced problem is decreased by one, and hence there will be one layer. It may be a boundary layer or an interior layer depending on the nature of the coefficient of the convection term, $a(x)$, i.e., if the sign of $a(x)$ changes sign throughout the interval [0,1], the layer will be an interior layer otherwise it will be a boundary layer at either of the end points of the interval $[0,1]$ [13].
The layer behavior of the problem under consideration is maintained only for $\delta \neq 0$ but sufficiently small (i.e., δ is of $o(\varepsilon)$). When the delay parameter δ exceeds the perturbation parameter, ε(i.e., δ is of $O(\varepsilon)$), the layer behavior is no longer maintained; rather, the solution exhibits an oscillatory behavior or is diminished [13]. Here we consider only the case when δ is of $o(\varepsilon)$ (i.e., $\delta < \varepsilon$).
Since δ is of $o(\varepsilon)$ and the solution $y(x)$ of (1)-(2) is sufficiently differentiable, by using Taylors series expansion, we obtain

$$y'(x-\delta) = y'(x) - \delta y''(x) + O(\delta^2) \tag{3}$$

Substituting (3) into (1) and simplifying, gives the following equivalent two-point boundary value problem

$$\begin{gathered}(\varepsilon - \delta a(x))y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \\ y(0) = \phi(0), \quad y(1) = \beta.\end{gathered} \tag{4}$$

Consider a function $\hat{a}(x) \geq M > 0$ for $\forall x \in [0,1]$, where M is a positive constant, depending on the nature of the function $a(x)$ three different cases can occur for $\delta a(x) < \varepsilon$:

$$\begin{cases} a(x) = \hat{a}(x) \implies boundary\ layer\ at\ left\ end, \\ a(x) = -\hat{a}(x) \implies boundary\ layer\ at\ right\ end, \\ a(x)\ changes\ sign\ in\ [0,1] \implies interior\ layer. \end{cases} \tag{5}$$

In this paper, the first two cases mentioned in (5) are considered. In addition, the effect of δ on the boundary layer behavior of the solution is also examined.

2.1. Layer on the left side. .
For an appropriate choice of δ such that $0 \leq \gamma = \varepsilon - \delta\zeta \ll 1$, where $\zeta = \min_{0\leq x\leq 1}\{a(x)\}$ equation (4) can be written as an equivalent singularly perturbed differential equation to the original problem (1)-(2), in the form of:

$$\gamma y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \tag{6}$$

with the boundary conditions

$$y(0) = \phi_0 = \alpha, \;\; y(1) = \beta, \tag{7}$$

where γ is a small positive perturbation parameter $0 < \gamma \ll 1$, α and β are given constants, $a(x)$, $b(x)$ and $f(x)$ are assumed to be sufficiently smooth functions. Further, if we assume that $a(x) \geq M > 0 \; \forall x \in [0,1]$, M being a positive constant. From the theory of singular perturbation it is known that, under the above assumptions (6)-(7), has a unique solution which in general, display a left boundary layer of width $O(\gamma)$ at $x = 0$ [23].
Equation (6) can be written as

$$\gamma\frac{d^2y}{dx^2} + \frac{d}{dx}(a(x)y(x)) = F(x,y), \; x \in [0,1], \tag{8}$$

where

$$F(x,y) = f(x) + a'(x)y(x) - b(x)y(x).$$

Let $u(x)$ be the solution of the reduced problem i.e., when $\gamma = 0$, So (6) is reduced to an IVP of the form

$$a(x)u'(x) + b(x)u(x) = f(x), \;\; u(1) = \beta. \tag{9}$$

Since $u(x)$ is the solution of the problem (6)-(7) in most part of the interval $[0,1]$, by replacing the solution $y(x)$ by $u(x)$ we obtain an asymptotically equivalent approximation to (8) as:

$$\begin{gathered}\gamma\frac{d^2y}{dx^2} + \frac{d}{dx}(a(x)y(x)) = F(x,u) + O(\gamma), \; x \in [0,1] \\ \text{with the boundary conditions, } \;\; y(0) = \alpha, \;\; y(1) = \beta,\end{gathered} \tag{10}$$

where

$$F(x,u) = f(x) + a'(x)u(x) - b(x)u(x).$$

Integrating (10) with respect to x, results

$$\gamma\frac{dy}{dx} + a(x)y(x) = \int F(x,u)dx + O(\gamma), \;\; x \in [0,1] \tag{11}$$

where

$$\int F(x,u)dx = \int (f(x) + a'(x)u(x) - b(x)u(x))dx.$$

Using (9) in the above integral equation, we obtain

$$\int F(x,u)dx = \int (f(x) + a'(x)u(x) - f(x) + a(x)u'(x))dx$$
$$= \int \frac{d}{dx}(a(x)u(x))dx = a(x)u(x) + k$$

Then substituting this in to (11), gives

$$\gamma \frac{dy}{dx} + a(x)y(x) = a(x)u(x) + k + O(\gamma), \tag{12}$$

where k is an integration constant. In order to determine k, we introduce the condition that the reduced equation of (12) should satisfy the boundary condition at $x = 1$; i.e., when $\gamma = 0$, (12) becomes

$$a(x)y(x) = a(x)u(x) + k$$

using the boundary condition at $x = 1$, i.e.,

$$a(1)y(1) = a(1)u(1) + k, \ \ where \ \ y(1) = u(1) = \beta,$$

Thus, we get $k = 0$.
Hence, by substituting the value of k in (12), a first order initial value problem which is asymptotically equivalent to the second order BVP (6)-(7) is obtained, and written as:

$$\gamma \frac{dw}{dx} + a(x)w(x) = a(x)u(x), \quad \text{with an initial condition,} \ \ w(0) = \alpha. \tag{13}$$

Over most of the domain $[0,1]$, the solution $u(x)$ of the reduced problem (9) behaves like the solution of (13). However, in the neighborhood of $x = 0$, there is a region in which this solution varies greatly from the solution of (13). To compensate the solution over this region(inner layer), a new inner variable is introduced by stretching the spatial coordinate x, as

$$t = \frac{x}{\gamma} \ \ \Rightarrow \ \ x = \gamma t \ \ and \ \ \frac{dt}{dx} = \frac{1}{\gamma}.$$

Using this stretching transformation in to (13), gives us

$$\frac{dw}{dt} + a(\gamma t)w = a(\gamma t)u(\gamma t). \tag{14}$$

In spite of the simplification, this equation remains first order differential equation and also regularly perturbed. For $\gamma = 0$, the equation becomes

$$\frac{dw}{dt} + a(0)w = a(0)u(0). \tag{15}$$

This is a differential equation for the solution of the layer region. Moreover, the solution of (15) is supposed to counter act for the fact that the solution of the reduced problem do not satisfy the boundary condition at $x = 0$ and that this solution satisfies

$$\lim_{t\to\infty} w(t) = 0. \tag{16}$$

Furthermore, using the substitution $W(t) = w(t) - u(0)$ in to (15), we obtain the following boundary layer correction problem

$$\frac{dW}{dt} + a(0)W = 0, \ with \ W(0) = \alpha - u(0). \tag{17}$$

Since this equation is a linear IVP with constant coefficient which can be solved analytically and gives,

$$W(\frac{x}{\gamma}) = (\alpha - u(0))e^{\frac{-a(0)x}{\gamma}}. \tag{18}$$

Finally, from standard singular perturbation theory it follows that the solution of the IVP (13) admits the representation in terms of the solutions of the reduced problem (9) and the boundary layer correction problem (18); that is,

$$\begin{aligned} w(x) &= u(x) + W(\frac{x}{\gamma}) + O(\gamma), \\ w(x) &= u(x) + (\alpha - u(0))e^{\frac{-a(0)x}{\gamma}} + O(\gamma). \end{aligned} \tag{19}$$

which approximates the exact solution $y(x)$ of the original problem (1)-(2), and it is similar to the the solution obtained by using the well known WKB method [20], rather applying an alternatively new and simple approach than the asymptotic expansion technique. Further, the following theorems gives the error bounds for the solution and their derivatives:

Theorem 2.1 ([4])**.** *The approximate solution $w(x)$ satisfies the inequality*

$$|y(x) - w(x)| \leq C\gamma, \forall x \in [0, 1],$$

where $y(x)$ is the solution of the original problem (1)-(2) and C is a positive constant independent of γ.

Proof. See (Doolan et. al. [4]). □

Theorem 2.2. *(*[4]*) Let $w(x)$ be the approximate solution of BVP (6)-(7), then*

$$|w^{(k)}(x)| \leq C[1 + \gamma^{-k}e^{-Mx/\gamma}], \forall x \in [0, 1], \ \ and \ \ k = 1, 2, ...$$

Proof. See (Doolan et. al. [4]). □

Remark 2.1. The reduced problem (9) and the resulting boundary layer correction problem (17) are both independent of the perturbation parameter, γ, and both can easily be solved by using one of the standard analytical or numerical methods for first order linear ordinary differential equations.

2.2. Layer on the right side. .

In order to apply the proposed method in this case, first let's rewrite the modified problem (4) equivalently as:

$$-(\varepsilon+\delta\hat{a}(x))y''(x)+\hat{a}(x)y'(x)+\hat{b}(x)y(x)=\hat{f}(x), \tag{20}$$

where $\hat{a}(x)=-a(x)$, $\hat{b}(x)=-b(x)$, and $\hat{f}(x)=-f(x)$.

For an appropriate choice of δ such that $0\leq\gamma=\varepsilon+\delta\zeta\ll 1$, where $\zeta=\min_{0\leq x\leq 1}\{\hat{a}(x)\}$ equation (20) written as an equivalent singularly perturbed differential equation to the original problem (1)-(2), in the form of:

$$-\gamma y''(x)+\hat{a}(x)y'(x)+\hat{b}(x)y(x)=\hat{f}(x), \tag{21}$$

with the boundary conditions

$$y(0)=\phi_0=\alpha,\ \ y(1)=\beta, \tag{22}$$

where γ is a small positive perturbation parameter $0<\gamma\ll 1$, α and β are given constants, $\hat{a}(x)$, $\hat{b}(x)$ and $\hat{f}(x)$ are assumed to be sufficiently smooth functions. Further, if we assume that $\hat{a}(x)\geq M>0$ $\forall x\in[0,1]$, M being a positive constant. From the theory of singular perturbation it is known that, under the above assumptions (21)-(22) has a unique solution which, in general, display a right boundary layer of width $O(\gamma)$ at $x=1$.

Equation (21) can be written equivalently as

$$-\gamma\frac{d^2y}{dx^2}+\frac{d}{dx}(\hat{a}(x)y(x))=F(x,y),\ x\in[0,1] \tag{23}$$

$$\text{where,}\ \ F(x,y)=\hat{f}(x)+\hat{a}'(x)y(x)-\hat{b}(x)y(x).$$

Let $u(x)$ be the solution of the reduced problem

$$\hat{a}(x)u'(x)+\hat{b}(x)u(x)=\hat{f}(x),\ \ u(0)=\alpha. \tag{24}$$

Since $u(x)$ is the solution of the problem (21)-(22)) in most part of the interval $[0,1]$, by replacing the solution $y(x)$ by $u(x)$ we obtain an asymptotically equivalent approximation to (23) as:

$$\begin{gathered}-\gamma\frac{d^2y}{dx^2}+\frac{d}{dx}(\hat{a}(x)y(x))=F(x,u)+O(\gamma),\ \ x\in[0,1]\\ \text{with the boundary conditions,}\ \ y(0)=\alpha,\ \ y(1)=\beta,\end{gathered} \tag{25}$$

$$\text{where}\ \ F(x,u)=\hat{f}(x)+\hat{a}'(x)u(x)-\hat{b}(x)u(x).$$

Integrating (25) with respect to x, results

$$-\gamma\frac{dy}{dx}+\hat{a}(x)y(x)=\int F(x,u)dx+O(\gamma),\ x\in[0,1], \tag{26}$$

$$\text{where}\ \int F(x,u)dx=\int(\hat{f}(x)+\hat{a}'(x)u(x)-\hat{b}(x)u(x))dx.$$

Using (24) in the above equation, we obtain

$$\int F(x,u)dx \ \ = \ \ \int(\hat{f}(x)+\hat{a}'(x)u(x)-\hat{b}(x)u(x))dx=\hat{a}(x)u(x)+k$$

Then substituting this in to (26), gives

$$-\gamma\frac{dy}{dx}+\hat{a}(x)y(x)=\hat{a}(x)u(x)+k+O(\gamma), \tag{27}$$

where k is an integration constant. In order to determine k, we introduce the condition that the reduced equation of (27) should satisfy the boundary condition at $x=0$. Thus, we get $k=0$.
Substituting the value of k in (27), a first order initial value problem which is asymptotically equivalent to the second order BVP (21)-(22) is obtained, and written as

$$\begin{aligned}&-\gamma\frac{dw}{dx}+\hat{a}(x)w(x)=\hat{a}(x)u(x),\\ &\text{with an initial condition,}\quad w(1)=\beta.\end{aligned} \tag{28}$$

Over most of the domain $[0,1]$, the solution $u(x)$ of the reduced problem (24) behaves like the solution of (28). However, in the vicinity of $x=1$, there is a region in which the solution varies greatly from the solution of (28). To portray the solution over this region, a new inner variable is introduced by stretching the spatial coordinate x as

$$t=\frac{1-x}{\gamma}\quad\Rightarrow\quad x=1-\gamma t\ \ and\ \ \frac{dt}{dx}=-\frac{1}{\gamma}.$$

Using this stretching transformation in to (28), gives

$$\frac{dw}{dt}+\hat{a}(1-\gamma t)w=\hat{a}(1-\gamma t)u(1-\gamma t). \tag{29}$$

In spite of the simplification, this equation remains first order differential equation and also regularly perturbed. For $\gamma=0$, the equation becomes

$$\frac{dw}{dt}+\hat{a}(1)w=\hat{a}(1)u(1). \tag{30}$$

This is a differential equation for the solution of the layer region. Moreover, the solution of (30) is supposed to counter act for the fact that the solution of the reduced problem do not satisfy the boundary condition at $x=1$ and that this solution satisfies

$$\lim_{t\to\infty}w(t)=0. \tag{31}$$

Furthermore, using the substitution $W(t)=w(t)-u(1)$ in to (30), we obtain the following boundary layer correction problem

$$\frac{dW}{dt}+\hat{a}(1)W=0,\ with\ W(0)=\beta-u(1). \tag{32}$$

Finally, from standard singular perturbation theory it follows that the solution of the IVP (28) admits the representation in terms of the solutions of the reduced and boundary layer correction problems; that is,

$$y(x)=u(x)+W(\frac{1-x}{\gamma})+O(\gamma). \tag{33}$$

which approximates the exact solution $y(x)$ of the original problem (1)-(2).

3. Test Problems and Numerical Results

To demonstrate the applicability and efficiency of the method, four test problems which commonly occur in the literature of SPDDEs are considered, three linear problems (two with left layer and one with right layer) and the remaining one on nonlinear problem.
In the case where exact solution of the test problem is available, for comparison the maximum absolute error is calculated and displayed using tables. In addition, the comparison of the method with other numerical methods using maximum errors is also accomplished. For each test problems, the effect of the delay term is examined by taking different values of δ and the results are displayed using figures.

3.1. Linear Problems.

Example 3.1. Consider the following BVP with left boundary layer:

$$\varepsilon y''(x) + y'(x-\delta) - y(x) = 0,$$

under the interval and boundary conditions

$$y(x) = 1,\ -\delta \leq x \leq 0\ , \quad \text{and} \quad y(1) = 1.$$

The exact solution is given by:

$$y(x) = \frac{(1-e^{m_2})e^{m_1 x} - (e^{m_1}-1)e^{m_2 x}}{e^{m_1} - e^{m_2}},$$

where

$$m_1 = \frac{-1-\sqrt{1+4(\varepsilon-\delta)}}{2(\varepsilon-\delta)} \ \textit{and}\ m_2 = \frac{-1+\sqrt{1+4(\varepsilon-\delta)}}{2(\varepsilon-\delta)}.$$

Solution:
Using an appropriate choice of δ such that, $0 \leq \gamma = \varepsilon - \delta \ll 1$, the given problem is expressed as an asymptotically equivalent singularly perturbed problem as

$$\gamma y''(x) + y'(x) - y(x) = 0,$$

$$\text{with the boundary conditions } \ y(0) = 1,\ \ y(1) = 1.$$

Let $u(x)$ be the solution of the reduced problem

$$u'(x) - u(x) = 0, \ \text{ with } \ u(1) = 1.$$

Since this equation is separable, it can be solved analytically and gives

$$u(x) = e^{x-1}.$$

The boundary layer correction problem becomes,

$$\frac{dW}{dt} + a(0)W = 0, \ \text{ with } \ W(0) = y(0) - u(0) = 1 - e^{-1},$$

since $a(0) = 1$, this equation becomes $\frac{dW}{dt} + W = 0$, which is also separable with solution

$$W(t) = (1-e^{-1})e^{-t}, \ \text{ where } \ t = \frac{x}{\gamma}.$$

Finally, the solution of the given problem is approximated by

$$y(x) = u(x) \ + \ W(\frac{x}{\gamma}) \ + \ O(\gamma),$$

$$y(x) = e^{-(1-x)} \ + \ (1-e^{-1})e^{\frac{-x}{\varepsilon-\delta}} \ + \ O(\varepsilon-\delta).$$

This is an asymptotic approximate solution for the given problem. As it can be seen from this example, the present method offers a relatively simple and easy tool for obtaining asymptotic approximate solution for such problems.

TABLE 1. Maximum errors for Example 3.1 for different values of ε and δ.

$\varepsilon_{\downarrow}/\delta \rightarrow$	$\delta = 0.0 * \varepsilon$	$\delta = 0.1 * \varepsilon$	$\delta = 0.3 * \varepsilon$	$\delta = 0.6 * \varepsilon$	$\delta = 0.9 * \varepsilon$
0.50	$8.55E-02$	$6.85E-02$	$3.63E-02$	$2.49E-02$	$1.57E-02$
0.10	$2.46E-02$	$2.33E-02$	$2.00E-02$	$1.30E-02$	$3.60E-03$
0.05	$1.57E-02$	$1.44E-02$	$1.16E-02$	$7.00E-03$	$1.82E-03$
0.01	$3.60E-03$	$3.25E-03$	$2.54E-03$	$1.46E-03$	$3.68E-04$
0.001	$3.67E-04$	$3.31E-04$	$2.57E-04$	$1.47E-04$	$3.68E-05$
0.0001	$3.67E-05$	$3.31E-05$	$2.57E-05$	$1.47E-05$	$3.68E-06$
0.00001	$3.67E-06$	$3.31E-06$	$2.57E-06$	$1.47E-06$	$3.68E-07$

TABLE 2. Maximum errors for Example 3.1 with $\delta = 0.1 * \varepsilon$ and step length $h = 0.01$.

$\varepsilon_{\downarrow}$	Method of [1]	Method of [6]	Method of [7]	Present Method
0.1	$1.93E-02$	$2.25E-00$	$1.92E-00$	$2.33E-02$
0.01	$2.03E-03$	$2.08E-01$	$2.53E-01$	$3.25E-03$
0.001	$4.56E-04$	$1.31E-02$	$5.74E-04$	$3.31E-04$
0.0001	$7.54E-04$	$5.61E-03$	$2.11E-03$	$3.31E-05$
0.00001	$7.83E-04$	$1.93E-03$	$1.60E-03$	$3.31E-06$
0.000001	$7.86E-04$	$6.22E-03$	$2.25E-03$	$3.31E-07$

Example 3.2. Consider the following BVP with left boundary layer:

$$\varepsilon y''(x) + e^{-0.5x}y'(x-\delta) - y(x) = 0, \ \ x \in [0,1]$$

with $$y(x) = 1, \ -\delta \leq x \leq 0 \ , \qquad y(1) = 1.$$

The exact solution of this problem is not known.

Solution:

Using an appropriate choice of δ such that $0 \leq \gamma = \varepsilon - \eta\delta \ll 1$, where

TABLE 3. Maximum errors for Example 3.1 with $\delta = 0.6 * \varepsilon$ and step length $h = 0.01$.

$\varepsilon_\downarrow$	Method of [1]	Method of [6]	Method of [7]	Present Method
0.1	$9.00E-03$	$1.59E-00$	$1.48E-00$	$2.00E-02$
0.01	$1.14E-03$	$3.10E-01$	$3.04E-01$	$1.46E-03$
0.001	$6.39E-04$	$7.64E-04$	$3.24E-02$	$1.47E-04$
0.0001	$7.72E-04$	$4.36E-03$	$1.03E-03$	$1.47E-05$
0.00001	$7.85E-04$	$1.80E-03$	$1.29E-03$	$1.47E-06$
0.000001	$7.87E-04$	$6.10E-04$	$2.22E-03$	$1.47E-07$

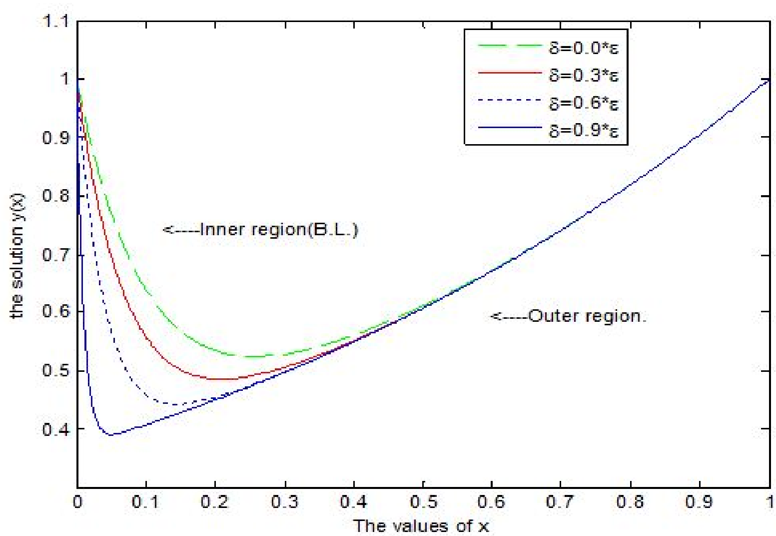

FIGURE 1. *Plot of the solution of Example 3.1 for* $\varepsilon = 0.1$ *and different values of* δ.

$\eta = \min_{0\le x\le 1}\{e^{-0.5x}\}$, the given problem is expressed as an asymptotically equivalent singularly perturbed problem as

$$\gamma y''(x) + e^{-0.5x} y'(x) - y(x) = 0,$$

$$\text{with the boundary conditions,}\quad y(0) = 1,\ \ y(1) = 1.$$

Preceding like example 3.1, the proposed method gives the following asymptotic approximate solution:

$$y(x) = e^{2(e^{0.5x} - e^{0.5})} + (1 - e^{2(1-e^{0.5})})e^{\frac{-x}{\varepsilon - \eta\delta}} + O(\varepsilon).$$

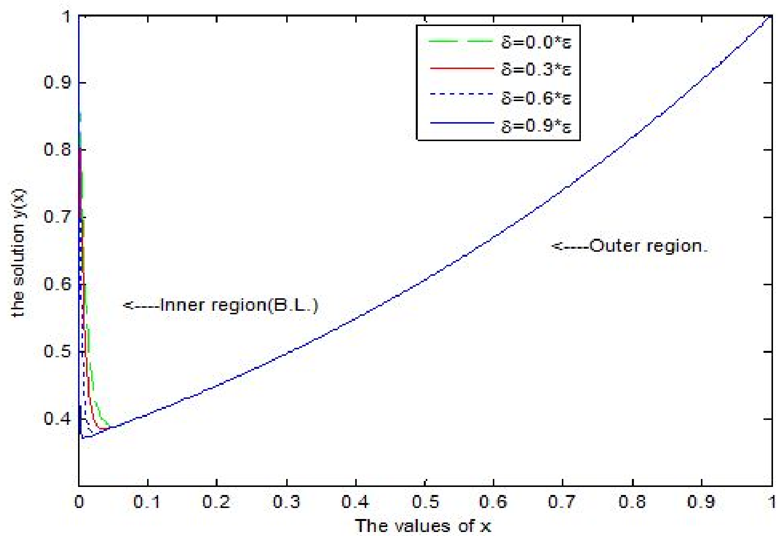

FIGURE 2. *Plot of the solution of Example 3.1 for* $\varepsilon = 0.01$ *and different values of* δ.

Example 3.3. Consider the following BVP with right boundary layer:

$$\varepsilon y''(x) - y'(x-\delta) - y(x) = 0, \;\; x \in [0,1],$$

under the interval and boundary conditions

$$y(x) = 1, \; -\delta \leq x \leq 0 \;, \quad \text{and} \quad y(1) = -1,.$$

The exact solution is given by:

$$y(x) = \frac{(1+e^{m_2})e^{m_1 x} - (e^{m_1}+1)e^{m_2 x}}{e^{m_2} - e^{m_1}},$$

where

$$m_1 = \frac{1-\sqrt{1+4(\varepsilon+\delta)}}{2(\varepsilon+\delta)} \; and \; m_2 = \frac{1+\sqrt{1+4(\varepsilon+\delta)}}{2(\varepsilon+\delta)}.$$

Solution:

Using an appropriate choice of δ such that $0 \leq \gamma = \varepsilon + \delta \ll 1$, the given problem is expressed as an asymptotically equivalent singularly perturbed problem as

$$-\gamma y''(x) + y'(x) + y(x) = 0,$$

$$\text{with the boundary conditions,} \;\; y(0) = 1, \;\; y(1) = -1.$$

This is a singularly perturbed problem with boundary layer at $x = 1$ (i.e., with right boundary layer). Then the reduced equation and the boundary layer correction problem for this BVP respectively becomes

$$u'(x) + u(x) = 0, \;\; with \;\; u(0) = 1.$$

$$\frac{dW}{dt} + W = 0, \;\; with \;\; W(0) = -1 - e^{-1},$$

Since both the IVPs are separable, they can be solved analytically. Thus, the approximate asymptotic solution of the given problem becomes

$$y(x) = e^{-x} + (1 + e^{-1})e^{\frac{x-1}{\varepsilon+\delta}} \; + \; O(\varepsilon + \delta).$$

TABLE 4. Numerical results of Example 3.2 for $\varepsilon = 0.01$, $h = 0.01$, and different values of δ.

x	$\delta = 0.0 * \varepsilon$	$\delta = 0.3 * \varepsilon$	$\delta = 0.6 * \varepsilon$	$\delta = 0.9 * \varepsilon$
0.00	1.0000000	1.0000000	1.0000000	1.0000000
0.01	0.5433465	0.4900287	0.4268695	0.3563569
0.02	0.3771349	0.3418174	0.3101033	0.2876659
0.09	0.2996638	0.2995882	0.2995766	0.2995761
0.10	0.3027671	0.3027377	0.3027342	0.3027341
0.40	0.4254376	0.4254376	0.4254376	0.4254376
0.95	0.9218118	0.9218118	0.9218118	0.9218118
1.00	1.0000000	1.0000000	1.0000000	1.0000000

TABLE 5. Maximum errors for Example 3.3 for different values of ε and δ.

$\varepsilon_{\downarrow}/\delta \rightarrow$	$\delta = 0.0 * \varepsilon$	$\delta = 0.1 * \varepsilon$	$\delta = 0.3 * \varepsilon$	$\delta = 0.6 * \varepsilon$	$\delta = 0.9 * \varepsilon$
0.1	$6.59E-02$	$7.17E-02$	$8.28E-02$	$9.87E-02$	$1.14E-01$
0.05	$3.51E-02$	$3.84E-02$	$4.48E-02$	$5.41E-02$	$6.31E-02$
0.01	$7.36E-03$	$7.82E-03$	$9.45E-03$	$1.18E-02$	$1.38E-02$
0.005	$3.41E-03$	$3.88E-03$	$4.77E-03$	$5.95E-03$	$6.96E-03$
0.0015	$5.67E-04$	$6.36E-04$	$7.92E-04$	$1.08E-03$	$1.42E-03$

TABLE 6. Maximum errors for Example 3.3 with $\delta = 0.2 * \varepsilon$, $\delta = 0.6 * \varepsilon$ and step length $h = 0.01$.

$\varepsilon_{\downarrow}$	For $\delta = 0.2 * \varepsilon$		For $\delta = 0.8 * \varepsilon$	
	Method of [6]	Present Method	Method of [6]	Present Method
0.1	$6.37E-01$	$7.73E-02$	$3.12E-01$	$1.09E-01$
0.05	$5.79E-01$	$4.16E-02$	$2.97E-01$	$6.01E-02$
0.01	$2.73E-01$	$8.59E-03$	$1.37E-01$	$1.31E-02$
0.005	$5.62E-02$	$4.33E-03$	$9.88E-03$	$6.64E-03$
0.0015	$8.27E-02$	$7.10E-04$	$1.53E-01$	$1.30E-03$

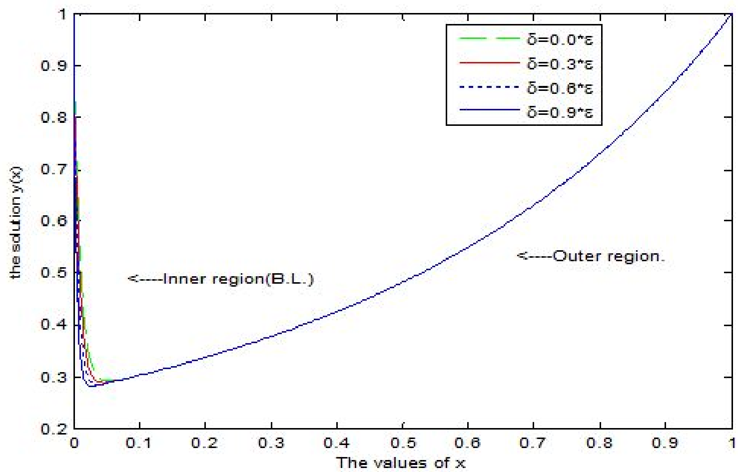

FIGURE 3. *Plot of the solution of Example 3.2 for* $\varepsilon = 0.01$ *and different values of* δ.

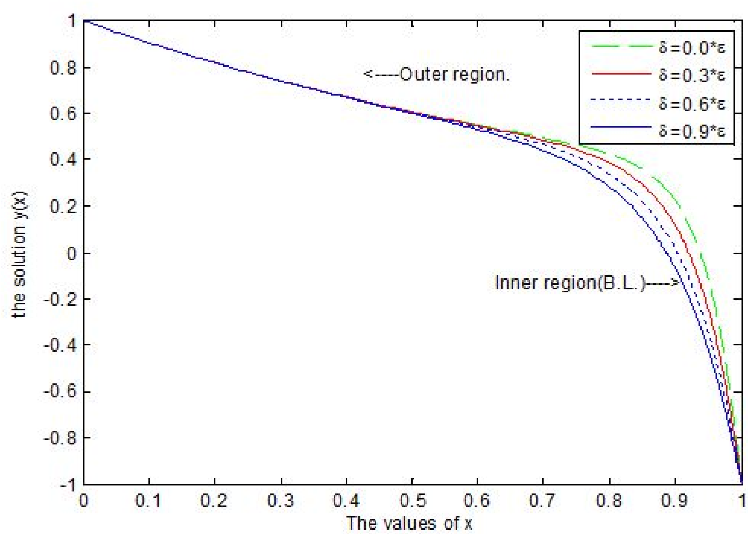

FIGURE 4. *Plot of the solution of Example 3.3 for* $\varepsilon = 0.05$ *and different values of* δ.

3.2. Nonlinear Problems. .
To apply this method on nonlinear problems first we approximate the given problem by a linear one using the method of quasi-linearization [11], and then applying the procedure in section 2.1 or section 2.2 depending on the position of the boundary layer to obtain the approximate solution.

Example 3.4. Consider the following nonlinear BVP:

$$\varepsilon y''(x) + 2y(x)y'(x-\delta) + e^{y(x)} = 0, \;\; x \in [0,1]$$

with $\quad y(x) = 0, \;\; \delta \leq x \leq 0 \;, \quad$ and $\quad y(1) = 0$.
By using the method of quasi-linearization [11], we obtain the corresponding linear BVP as

$$\varepsilon y''(x) + 2y'(x-\delta) + y(x) = -1, \;\; x \in [0,1]$$
$$\text{with the boundary condition,} \;\; y(0) = 0, \;\; y(1) = 0.$$

Solution:
For appropriate choice of δ which is $o(\varepsilon)$, let $\gamma = \varepsilon - 2\delta$ such that $0 \leq \gamma \ll 1$ then the given problem is expressed as an asymptotically equivalent singularly perturbed problem as

$$\gamma y''(x) + 2y'(x) + y(x) = -1, \;\; x \in [0,1]$$
$$\text{with the boundary condition,} \;\; y(0) = 0, \;\; y(1) = 0.$$

This is the corresponding singularly perturbed problem with boundary layer at $x = 0$ (i.e., with left boundary layer). Next, the reduced problem of this equation becomes

$$2u'(x) + u(x) = -1, \;\; with \;\; u(1) = 0.$$

Since this equation is separable, it can be solved analytically and the solution becomes,

$$u(x) = e^{\frac{1-x}{2}} - 1.$$

and the boundary layer correction problem becomes,

$$\frac{dW}{dt} + a(0)W = 0, \;\; with \;\; W(0) = 1 - e^{1/2}.$$

Since $a(0) = 2$, this equation becomes $\frac{dW}{dt} + 2W = 0$, which is also separable with solution

$$W(t) = (1 - e^{1/2})e^{-2t}, \;\; where \;\; t = \frac{x}{\gamma}.$$

Hence, the solution of the given problem is approximated by

$$y(x) = e^{\frac{1-x}{2}} - 1 + (1 - e^{1/2})e^{\frac{-2x}{\varepsilon - 2\delta}} + O(\varepsilon - 2\delta).$$

TABLE 7. Numerical results of Example 3.4 for $\varepsilon = 0.01$, $h = 0.01$, and different values of δ.

x	$\delta = 0.0 * \varepsilon$	$\delta = 0.1 * \varepsilon$	$\delta = 0.3 * \varepsilon$	$\delta = 0.45 * \varepsilon$
0.00	0.0000000	0.0000000	0.0000000	0.0000000
0.01	0.5527034	0.5872479	0.6361272	0.6404982
0.02	0.6204345	0.6279452	0.6322868	0.6323162
0.09	0.5761734	0.5761734	0.5761734	0.5761734
0.10	0.5683122	0.5683122	0.5683122	0.5683122
0.40	0.3498588	0.3498588	0.3498588	0.3498588
0.90	0.0512711	0.0512711	0.0512711	0.0512711
0.95	0.0253151	0.0253151	0.0253151	0.0253151
1.00	0.0000000	0.0000000	0.0000000	0.0000000

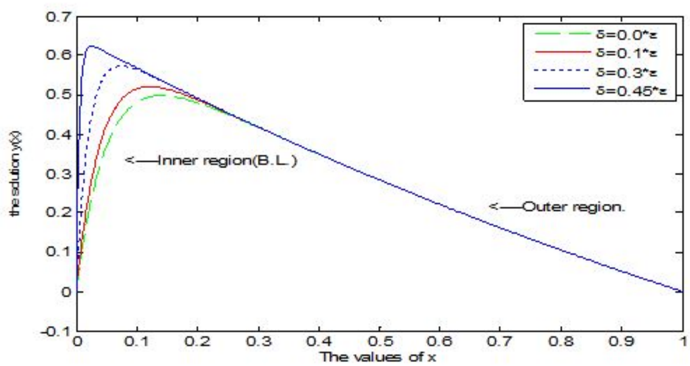

FIGURE 5. *Plot of the solution of Example 3.4 for* $\varepsilon = 0.1$ *and different values of* δ.

4. Discussion

In this article, an initial value method for solving a class of second order singularly perturbed delay differential equations when δ is of $o(\varepsilon)$ is considered. First, by applying two term Taylor's series expansion on the term containing the delay and taking an appropriate choice of δ, the given problem is modified to an asymptotically equivalent singularly perturbed problem with a new perturbation parameter γ. Then the solution of the modified problem is computed analytically by solving two initial value problems, namely the reduced problem and the boundary layer correction problem, which are independent of the perturbation parameter γ. The method is simple to apply, very easy to implement on a computer and offers a relatively simple tool for obtaining approximate asymptotic solution.

To show the efficiency and applicability of the proposed method, four test problems are considered. For problems with exact solution, the maximum absolute error is calculated and tabulated in Tables 1 and 5 for different values of ε and δ. Also comparison with existing three numerical methods is accomplished and presented in Tables 2, 3 and 6. On the other hand, for the problems with no exact solution, the solutions are computed at all the mesh points of step length h, but only few values have been reported in Tables 4 and 7. To examine the effect of delay δ on the boundary layer behavior of the solutions, graphs of the solutions of the examples considered are plotted in Figure 1- Figure 5 for different values of δ.

From the results it is observed that, the present method approximates the solution of the problems very well. In addition, for very small ε the present method is superior to the numerical methods in [1], [6] and [7]. For $\delta = o(\varepsilon)$, the layer behavior is maintained in both the cases (for the left layer and right layer cases). Further, as δ increases, the thickness of the left layer decreases as shown in Figure 1, 2, 3, and 5, where as in the case of right layer it increases as shown in Figure 4.

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