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LIE IDEALS IN THE UPPER TRIANGULAR OPERATOR ALGEBRA ALG$\mathcal{L}^{\dagger}$

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Abstract. Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space with a fixed orthonormal base $\{e_1, e_2, \cdots\}$. Let $\mathcal{L}$ be the subspace lattice generated by the subspaces $\{[e_1], [e_1, e_2], [e_1, e_2, e_3], \cdots\}$ and let Alg$\mathcal{L}$ be the algebra of bounded operators which leave invariant all projections in $\mathcal{L}$. Let p and q be natural numbers ($p < q$). Let $\mathcal{A}$ be a linear manifold in Alg$\mathcal{L}$ such that $T_{(p,q)} = 0$ for all T in $\mathcal{A}$. If $\mathcal{A}$ is a Lie ideal, then $T_{(p,p)} = T_{(p+1,p+1)} = \cdots = T_{(q,q)}$ and $T_{(i,j)} = 0, p \leqslant i \leqslant q$ and $i < j \leqslant q$ for all T in $\mathcal{A}$.

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1. Introduction

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space with a fixed orthonormal base $\{e_1, e_2, \cdots\}$ and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded operators on $\mathcal{H}$. If $x_1, x_2, \cdots, x_k$ are vectors in $\mathcal{H}$, we denote by $[x_1, x_2, \cdots, x_k]$ the closed subspace spanned by the vectors $x_1, x_2, \cdots, x_k$. A subspace lattice is a strongly closed lattice of orthogonal projections acting on $\mathcal{H}$. In this paper, we denote by $\mathcal{L}$ the subspace lattice generated by the subspaces $\{[e_1], [e_1, e_2][e_1, e_2, e_3], \cdots\}$. By Alg$\mathcal{L}$, we mean the algebra of bounded operators which leave invariant all projections in $\mathcal{L}$. It is easy to see that all such operators have the following matrix form

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$$\begin{pmatrix} * & * & * & * & * & & & \\ & * & * & * & \ddots & \ddots & & \\ & & * & * & * & \ddots & \ddots & \\ & & & * & * & * & \ddots & \ddots \\ & & & & * & * & * & \ddots \\ & & & & & \ddots & \ddots & \ddots \end{pmatrix}$$

where all non-starred entries are zero. We call the algebra Alg$\mathcal{L}$ by the upper triangular operator algebra.

The Lie product in the algebra Alg$\mathcal{L}$ is defined by

$$[A, B] = AB - BA$$

for operators A and B in Alg$\mathcal{L}$. A linear manifold $\mathcal{A}$ in Alg$\mathcal{L}$ is called *a Lie ideal* in Alg$\mathcal{L}$ if $[A, X]$ is in $\mathcal{A}$ for all A in Alg$\mathcal{L}$ and all X in $\mathcal{A}$.

In this paper we find examples of Lie ideals and investigate relationships between Lie ideals in the algebra Alg$\mathcal{L}$. Let I be the identity operator on $\mathcal{H}$ in this paper. Let $\mathbb{C}$ be the set of all complex numbers and let $\mathbb{N} = \{1, 2, \cdots\}$.

2. Examples of Lie ideals in Alg$\mathcal{L}$

If we know the following facts, then we can easily prove the following examples.

Let $A = (a_{ij})$ and $T = (t_{ij})$ be operators in Alg$\mathcal{L}$. Then

(1) the (p, p)-entry of AT is $a_{pp}t_{pp}$ for all $p = 1, 2, \cdots$

(2) the (p, p)-entry of TA is $t_{pp}a_{pp}$ for all $p = 1, 2, \cdots$

(3) the (p, q)-entry of AT is $a_{pp}t_{pq} + a_{p\ p+1}t_{p+1\ q} + \cdots + a_{pq}t_{qq} (p < q)$

(4) the (p, q)-entry of TA is $t_{pp}a_{pq} + t_{p\ p+1}a_{p+1\ q} + \cdots + t_{pq}a_{qq} (p < q)$

(5) the (p, p)-entry of $AT - TA$ is 0 for all $p = 1, 2, \cdots$

(6) the (p, q)-entry of $AT - TA$ is $a_{pp}t_{pq} + a_{p\ p+1}t_{p+1\ q} + \cdots + a_{pq}t_{qq}$ - $(t_{pp}a_{pq} + t_{p\ p+1}a_{p+1\ q} + \cdots + t_{pq}a_{qq})$

We denote the (i, j)-component of T by $T_{(i,j)}$.

Example 2.1. i)Let $\mathcal{A}_0 = \{\ T \in \text{Alg}\mathcal{L} \mid T_{(i,i)} = 0,\ i \in \mathbb{N}\ \}$. Then $\mathcal{A}_0$ is a Lie ideal in Alg$\mathcal{L}$.

ii)Let Γ be a nonempty subset of $\mathbb{N}$ and let $\mathcal{A}_\Gamma = \{\ T \in \text{Alg}\mathcal{L} \mid T_{(i,i)} = 0, i \in \Gamma\}$. Then $\mathcal{A}_\Gamma$ is a Lie ideal in Alg$\mathcal{L}$.

If $\Gamma = \emptyset$, then $\mathcal{A}_\Gamma = \text{Alg}\mathcal{L}$. If $\Gamma = \mathbb{N}$, then $\mathcal{A}_\Gamma = \mathcal{A}_0$.

Example 2.2. i)Let I be the identity operator on $\mathcal{H}$ and let $\mathcal{A}_1 = \{\ \alpha I \mid \alpha \in \mathbb{C}\ \}$. Then $\mathcal{A}_1$ is a Lie ideal in Alg$\mathcal{L}$.

ii)Let $\mathcal{A}_2 = \{\ \alpha I + T \mid T \in \mathcal{A}_0, \alpha \in \mathbb{C}\ \}$. Then $\mathcal{A}_2$ is a Lie ideal in Alg$\mathcal{L}$.

Example 2.3. i)Let p be a natural number and let $\mathcal{A}_{0,p} = \{\ T \in \mathcal{A}_0 \mid T_{(p,p+1)} = 0\ \}$. Then $\mathcal{A}_{0,p}$ is a Lie ideal in Alg$\mathcal{L}$.

ii)Let $\Lambda = \{p_1, p_2, \cdots\}$ be a subset of $\mathbb{N}$. Let $\mathcal{A}_{0,\Lambda} = \{\ T \in \mathcal{A}_0 \mid T_{(p_i,p_i+1)} = 0, i = 1, 2, \cdots \}$. Then $\mathcal{A}_{0,\Lambda}$ is a Lie ideal in Alg$\mathcal{L}$.

iii)Let p be a natural number and let $\mathcal{A}_{2,p} = \{\ T \in \mathcal{A}_2 \mid T_{(p,p+1)} = 0\ \}$. Then $\mathcal{A}_{2,p}$ is a Lie ideal in Alg$\mathcal{L}$.

Example 2.4. Let p and q be natural numbers such that $p < q$.

i)Let $\mathcal{B}_{p,q} = \{\ T \in \text{Alg}\mathcal{L}\ \mid T_{(p,q)} = 0\ \}$. Then $\mathcal{B}_{p,q}$ is not a Lie ideal in Alg$\mathcal{L}$.

ii)Let $\mathcal{B}^{(1)}{}_{p,q} = \{\ T \in \text{Alg}\mathcal{L}\ \mid T_{(p,k)} = 0, k = p, p+1, \cdots, q\ \}$. Then $\mathcal{B}^{(1)}{}_{p,q}$ is not a Lie ideal in Alg$\mathcal{L}$.

iii)Let $\mathcal{B}^{(2)}{}_{p,q} = \{\ T \in \text{Alg}\mathcal{L}\ \mid T_{(k,q)} = 0, k = p, p+1, \cdots, q\ \}$. Then $\mathcal{B}^{(2)}{}_{p,q}$ is not a Lie ideal in Alg$\mathcal{L}$.

iv)Let $\mathcal{C}_{p,q} = \{\ T \in \text{Alg}\mathcal{L}\ \mid T_{(p,i)} = 0 = T_{(j,q)}, p \leqslant i, j \leqslant q\ \}$. If $q = p+1$, then $\mathcal{C}_{p,q}$ is a Lie ideal in Alg$\mathcal{L}$. If $p+1 < q$, then $\mathcal{C}_{p,q}$ is not a Lie ideal in Alg$\mathcal{L}$.

Example 2.5. Let p and q be natural numbers such that $p < q$.

i)Let $\mathcal{D}_{p,q} = \{\ T \in \text{Alg}\mathcal{L}\ \mid T_{(p,p)} = T_{(p+1,p+1)} = \cdots = T_{(q,q)}\ \}$. Then $\mathcal{D}_{p,q}$ is a Lie ideal in Alg$\mathcal{L}$.

ii)Let $\mathcal{D}_{p,\infty} = \{\ T \in \text{Alg}\mathcal{L}\ \mid T_{(p,p)} = T_{(p+1,p+1)} = \cdots \}$. Then $\mathcal{D}_{p,\infty}$ is a Lie ideal in Alg$\mathcal{L}$.

Example 2.6. Let p and q be natural numbers such that $p < q$.

i)Let $\mathcal{A}_{p,q} = \{\ T \in \mathcal{D}_{p,q} \mid\ T_{(i,j)} = 0, p \leqslant i \leqslant q-1 \text{ and } i < j \leqslant q\ \}$. Then $\mathcal{A}_{p,q}$ is a Lie ideal in Alg$\mathcal{L}$.

ii)Let $\mathcal{A}^{(0)}_{p,q} = \{\ T \in \text{Alg}\mathcal{L}\ \mid\ T_{(i,j)} = 0, p \leqslant i \leqslant q \text{ and } i \leqslant j \leqslant q\ \}$. Then $\mathcal{A}^{(0)}_{p,q}$ is a Lie ideal in Alg$\mathcal{L}$.

Proof. i) and ii) $\mathcal{A}_{p,q}$ is clearly a linear manifold in Alg$\mathcal{L}$. Let $T = (t_{ij}) \in \mathcal{A}_{p,q}$ and let $A = (a_{ij}) \in \text{Alg}\mathcal{L}$. Then $(AT - TA)_{(p,p)} = \cdots = (AT - TA)_{(q,q)} = 0$ and so $AT - TA \in \mathcal{D}_{p,q}$. For $p \leqslant i \leqslant q-1$ and $i < j \leqslant q$, $(AT - TA)_{(i,j)} = a_{ii}t_{ij} + a_{i\ i+1}t_{i+1\ j} + \cdots + a_{ij}t_{jj}$ - $(t_{ii}a_{ij} + t_{i\ i+1}a_{i+1\ j} + \cdots + t_{ij}a_{jj}) = a_{ij}t_{jj} - t_{ii}a_{ij} = a_{ij}(t_{jj} - t_{ii}) = 0$ because $t_{jj} = t_{ii}$. So $AT - TA \in \mathcal{A}_{p,q}$. Hence $\mathcal{A}_{p,q}$ is a Lie ideal in Alg$\mathcal{L}$. □

Example 2.7. Let p and q be natural numbers such that $p < q$.

i)Let $\mathcal{A}^{(1)}_{p,q} = \{\ T \in \mathcal{D}_{p,q} \mid\ T_{(p,i)} = 0 \text{ and } T_{(k,j)} = 0, i = p+1, \cdots, q-1, p+1 \leqslant k \leqslant q-1, k < j \leqslant q\ \}$. Then $\mathcal{A}^{(1)}_{p,q}$ is a Lie ideal in Alg$\mathcal{L}$.

ii)Let $\mathcal{A}^{(2)}_{p,q} = \{\ T \in \text{Alg}\mathcal{L}\ \mid\ T_{(p,i)} = 0 \text{ and } T_{(k,j)} = 0, p \leqslant i \leqslant q-1, p+1 \leqslant k \leqslant q-1, k \leqslant j \leqslant q\ \}$. Then $\mathcal{A}^{(2)}_{p,q}$ is a Lie ideal in Alg$\mathcal{L}$.

Proof. i) and ii) $\mathcal{A}^{(1)}_{p,q}$ and $\mathcal{A}^{(2)}_{p,q}$ are linear manifolds in Alg$\mathcal{L}$. Let $T = (t_{ij}) \in \mathcal{A}^{(1)}_{p,q}$ and let $A = (a_{ij}) \in \text{Alg}\mathcal{L}$. Then $(AT - TA)_{(p,p)} = \cdots = (AT - TA)_{(q,q)} = 0$.

$(AT - TA)_{(p,i)} = a_{pp}t_{pi} + a_{p\ p+1}t_{p+1\ i} + \cdots + a_{pi}t_{ii}$ - $(t_{pp}a_{pi} + t_{p\ p+1}a_{p+1\ i} + \cdots + t_{pi}a_{ii}) = 0$ because $t_{ii} = t_{pp}$ for $i = p+1, \cdots, q-1$.

$(AT - TA)_{(k,j)} = a_{kk}t_{kj} + a_{k\ k+1}t_{k+1\ j} + \cdots + a_{kj}t_{jj}$ - $(t_{kk}a_{kj} + t_{k\ k+1}a_{k+1\ j} + \cdots + t_{kj}a_{jj}) = 0$ because $t_{kk} = t_{jj}$ for $p+1 \leqslant k < q$ and $k \leqslant j \leqslant q$. □

3. Main results

Theorem 3.1. *Let p and q be natural numbers* $(p < q)$. *Let* $\mathcal{A}$ *be a linear manifold in Alg*$\mathcal{L}$ *such that* $\{0\} \subsetneq \mathcal{A} \subset \mathcal{B}_{p,q}$. *If* $\mathcal{A}$ *is a Lie ideal in Alg*$\mathcal{L}$, *then* $T_{(p,p)} = T_{(p+1,p+1)} = \cdots = T_{(q,q)}$ *and* $T_{(i,j)} = 0, p \leqslant i \leqslant q$ *and* $i < j \leqslant q$ *for all* T *in* $\mathcal{A}$ *i.e.* $\mathcal{A} \subset \mathcal{A}_{p,q}$.

Proof. Let $\mathcal{A}$ be a Lie ideal in Alg$\mathcal{L}$. Let $T = (t_{ij}) \in \mathcal{A}$ and let $A = (a_{ij}) \in$ Alg$\mathcal{L}$. Then $t_{pq} = 0$ and

$0 = (AT - TA)_{(p,q)} = a_{pp}t_{pq} + a_{p\ p+1}t_{p+1\ q} + \cdots + a_{pq}t_{qq}$ - $(t_{pp}a_{pq} + t_{p\ p+1}a_{p+1\ q} + \cdots + t_{pq}a_{qq}) \cdots\cdots\cdots (*_1)$.

Since $(*_1)$ holds for all A in Alg$\mathcal{L}$, $t_{pp} = t_{qq}, t_{p+1\ q} = 0, \cdots, t_{q-1\ q} = 0$ and $t_{p\ p+1} = 0, t_{p\ p+2} = 0, \cdots, t_{pq} = 0$.

$0 = (AT - TA)_{(p+1,q)} = a_{p+1\ p+1}t_{p+1\ q} + a_{p+1\ p+2}t_{p+2\ q} + \cdots + a_{p+1\ q}t_{qq}$ - $(t_{p+1\ p+1}a_{p+1\ q} + t_{p+1\ p+2}a_{p+2\ q} + \cdots + t_{p+1\ q}a_{qq}) \cdots\cdots\cdots (*_2)$.

Since $(*_2)$ holds for all A in Alg$\mathcal{L}$, $t_{p+1\ p+1} = t_{qq}$, $t_{p+1\ p+2} = 0, \cdots$ and $t_{p+1\ q} = 0$.

$0 = (AT - TA)_{(q-1,q)} = a_{q-1\ q-1}t_{q-1\ q} + a_{q-1\ q}t_{qq}$ - $(t_{q-1\ q-1}a_{q-1\ q} + t_{q-1\ q}a_{qq})$, $t_{q-1\ q-1} = t_{qq}, t_{q-1\ q} = 0$. Hence $t_{pp} = \cdots = t_{qq}, t_{ij} = 0$, where $p \leqslant i \leqslant q$ and $i < j \leqslant q$. i.e. $\mathcal{A} \subset \mathcal{A}_{p,q}$. □

We can prove Thoerem 3.2, Theorem 3.3, Theorem 3.4, Theorem 3.5 by the same way with the proof of Theorem 3.1.

Theorem 3.2. *Let p and q be natural numbers* $(p < q)$. *Let* $\mathcal{A}$ *be a linear manifold in Alg*$\mathcal{L}$ *such that* $\mathcal{A}_{p,q} \subset \mathcal{A} \subset \mathcal{B}_{p,q}$. *Then* $\mathcal{A}$ *is a Lie ideal in Alg*$\mathcal{L}$ *if and only if* $\mathcal{A} = \mathcal{A}_{p,q}$.

Theorem 3.3. *Let p and q be natural numbers* $(p < q)$. *Let* $\mathcal{A}$ *be a linear manifold in Alg*$\mathcal{L}$ *such that* $\mathcal{A}_{p,q} \subset \mathcal{A} \subset \mathcal{B}^{(1)}{}_{p,q}$. *Then* $\mathcal{A}$ *is a Lie ideal in Alg*$\mathcal{L}$ *if and only if* $\mathcal{A} = \mathcal{A}_{p,q}$.

Theorem 3.4. *Let p and q be natural numbers* $(p < q)$. *Let* $\mathcal{A}$ *be a linear manifold in Alg*$\mathcal{L}$ *such that* $\mathcal{A}_{p,q} \subset \mathcal{A} \subset \mathcal{B}^{(2)}{}_{p,q}$. *Then* $\mathcal{A}$ *is a Lie ideal in Alg*$\mathcal{L}$ *if and only if* $\mathcal{A} = \mathcal{A}_{p,q}$.

Theorem 3.5. *Let p and q be natural numbers* $(p + 1 < q)$. *Let* $\mathcal{A}$ *be a linear manifold in Alg*$\mathcal{L}$ *such that* $\mathcal{A}_{p,q} \subset \mathcal{A} \subset \mathcal{C}_{p,q}$. *Then* $\mathcal{A}$ *is a Lie ideal in Alg*$\mathcal{L}$ *if and only if* $\mathcal{A} = \mathcal{A}_{p,q}$.

Theorem 3.6. *Let p and q be natural numbers such that* $p < q$. *Let* $\mathcal{A}$ *be a linear manifold in Alg*$\mathcal{L}$ *such that* $\mathcal{A}^{(0)}_{p,q} \subset \mathcal{A} \subset \mathcal{A}_{p,q}$. *Then* $\mathcal{A}$ *is a Lie ideal in Alg*$\mathcal{L}$ *if and only if* $\mathcal{A} = \mathcal{A}^{(0)}_{p,q}$ *or* $\mathcal{A} = \mathcal{A}_{p,q}$.

Proof. Let $\mathcal{A}$ be a Lie ideal in Alg$\mathcal{L}$. Let $\mathcal{A} \neq \mathcal{A}^{(0)}_{p,q}$. Then there exists an operaror T in $\mathcal{A}$ and $T \notin \mathcal{A}^{(0)}_{p,q}$. Since T is in $\mathcal{A}$, T is in $\mathcal{A}_{p,q}$. So there is α in $\mathbb{C}$ such that $T_{(p,p)} = T_{(p+1,p+1)} = \cdots = T_{(q,q)} = \alpha \neq 0$ and $T_{(i,j)} = 0, p \leqslant i \leqslant q$

and $i < j \leqslant q$. Let $A \in \mathcal{A}_{p,q}$. Then there is a complex number β such that $A_{(p,p)} = A_{(p+1,p+1)} = \cdots = A_{(q,q)} = \beta$. If $\beta = 0$, then $A \in \mathcal{A}_{p,q}^{(0)}$ and so $A \in \mathcal{A}$. If $\beta = \alpha$, then $T - A \in \mathcal{A}_{p,q}^{(0)}$. So $T - A \in \mathcal{A}$. Since $\mathcal{A}$ is a linear manifold in $\mathcal{A}$, $A - T + T = A \in \mathcal{A}$. Let $\beta \neq 0$ and $\beta \neq \alpha$. Then $\frac{\alpha}{\beta} A \in \mathcal{A}$ by the above $\beta = \alpha$ case. So $A \in \mathcal{A}$. Hence $\mathcal{A} = \mathcal{A}_{p,q}$. □

Theorem 3.7. *Let p a fixed natural number. Let $\mathcal{A}$ be a linear manifold in Alg$\mathcal{L}$ such that $\mathcal{A}_{0,p} \subset \mathcal{A} \subset \mathcal{A}_0$. Then $\mathcal{A}$ is a Lie ideal in Alg$\mathcal{L}$ if and only if $\mathcal{A} = \mathcal{A}_{0,p}$ or $\mathcal{A} = \mathcal{A}_0$.*

Proof. Let $\mathcal{A}$ be a Lie ideal in Alg$\mathcal{L}$ and let $\mathcal{A} \neq \mathcal{A}_{0,p}$. Then there exists an operaror T in $\mathcal{A}$ such that $T \notin \mathcal{A}_{0,p}$, i.e. $T_{(p,p+1)} \neq 0$. Let $A = (a_{ij}) \in \mathcal{A}_0$. If $a_{p\ p+1} = 0$, then $A \in \mathcal{A}_{0,p}$ and so $A \in \mathcal{A}$. Let $a_{p\ p+1} \neq 0$. Let A_1 be an operator defined by

$$\begin{cases} A_{1(p,p+1)} = 0 \\ A_{1(i,j)} = a_{ij} \text{ otherwise.} \end{cases}$$

Then $A_1 \in \mathcal{A}_{0,p}$. Let T_1 be an operator defined by

$$\begin{cases} T_{1(p,p+1)} = 0 \\ T_{1(i,j)} = -T_{(i,j)} \text{ otherwise.} \end{cases}$$

Then $T_1 \in \mathcal{A}_{0,p} \subset \mathcal{A}$. Let $T_2 = T + T_1$. Then $T_2 \in \mathcal{A}$ and $T_{2(p,p+1)} = T_{(p,p+1)}$. Let $\alpha = \frac{a_{p\ p+1}}{T_{(p,p+1)}}$. Then $A = \alpha T_2 + A_1$ and $A \in \mathcal{A}$. Hence $\mathcal{A} = \mathcal{A}_0$. □

We omit the proof of the following Theorem because it can be given easily by modifying the proof of Theorem 7.

Theorem 3.8. *1) Let $k_1, k_2, \cdots$ be natural numbers such that $k_i < k_{i+1}$. Let $\Omega_1 = \{k_1\}$, $\Omega_2 = \{k_1, k_2\}, \cdots$, $\Omega_n = \{k_1, k_2, \cdots, k_n\}, \cdots, \Omega = \{k_1, k_2, \cdots\}$. Then*

$\mathcal{A}_{0,\Omega} \subset \cdots \subset \mathcal{A}_{0,\Omega_n} \subset \mathcal{A}_{0,\Omega_{n-1}} \subset \cdots \subset \mathcal{A}_{0,\Omega_2} \subset \mathcal{A}_{0,\Omega_1} = \mathcal{A}_{0,k_1}$.

2)Let p be a natural number. Then $\mathcal{D}_{p,\infty} \subset \cdots \subset \mathcal{D}_{p,n} \subset \cdots \subset \mathcal{D}_{p,2} \subset \mathcal{D}_{p,1}$.

3)Let p and q be natural numbers such that $1 < p < q$. Then

i) $\mathcal{A}_{p,q} \supset \mathcal{A}_{p-1,q} \supset \mathcal{A}_{p-2,q} \supset \cdots \supset \mathcal{A}_{1,q}$

ii) $\mathcal{A}_{p,q} \supset \mathcal{A}_{p-1,q+1} \supset \mathcal{A}_{p-2,q+2} \supset \cdots \supset \mathcal{A}_{1,q+p-1}$

iii)$\mathcal{A}_{p,q} \supset \mathcal{A}_{p,q+1} \supset \mathcal{A}_{p,q+2} \supset \cdots \supset \mathcal{A}_{p,q+n} \supset \cdots$

Theorem 3.9. *Let p and q be natural numbers such that $p < q$. Let $\mathcal{A}$ be a linear manifold in Alg$\mathcal{L}$ such that $\mathcal{D}_{p,q+1} \subset \mathcal{A} \subset \mathcal{D}_{p,q}$. Then $\mathcal{A}$ is a Lie ideal in Alg$\mathcal{L}$ if and only if $\mathcal{A} = \mathcal{D}_{p,q}$ or $\mathcal{A} = \mathcal{D}_{p,q+1}$.*

Proof. Let $\mathcal{A}$ be a Lie ideal in Alg$\mathcal{L}$ and let $\mathcal{A} \neq \mathcal{D}_{p,q+1}$. Then there exists an operaror T in $\mathcal{A}$ and $T \notin \mathcal{D}_{p,q+1}$. Then $T_{(p,p)} = T_{(p+1,p+1)} = \cdots = T_{(q,q)}$ } and $T_{(q,q)} \neq T_{(q+1,q+1)}$. Let $A = (a_{ij}) \in \mathcal{D}_{p,q}$. Then $a_{pp} = a_{p+1\ p+1} = \cdots = a_{q\ q}$. If

$a_{q+1\ q+1} = a_{q\ q}$, then $A \in \mathcal{D}_{p,q+1} \subset \mathcal{A}$. Let $a_{q+1\ q+1} \neq a_{q\ q}$. Define an operator A_1 by

$$\begin{cases} A_{1(q+1,q+1)} = a_{q\ q} \\ A_{1(i,j)} = a_{ij} \text{ otherwise.} \end{cases}$$

Then $A_1 \in \mathcal{D}_{p,q+1} \subset \mathcal{A}$. Let T_1 be an operator defined by

$$\begin{cases} T_{1(q+1,q+1)} = -T_{(q,q)} \\ T_{1(i,j)} = -T_{(i,j)} \text{ otherwise.} \end{cases}$$

Then $T_1 \in \mathcal{D}_{p,q+1} \subset \mathcal{A}$. Put $T_2 = T + T_1$. Then $T_2 \in \mathcal{A}$.

Let $\alpha = \frac{a_{q+1\ q+1} - a_{q\ q}}{T_{(q+1,q+1)} - T_{(q,q)}}$. Then $A = \alpha T_2 + A_1$ and $A \in \mathcal{A}$. Hence $\mathcal{A} = \mathcal{D}_{p,q}$. □

Theorem 3.10. *Let p be a natural number. Let $\mathcal{A}$ be a linear manifold in Alg$\mathcal{L}$ such that $\mathcal{C}_{p,p+1} \subset \mathcal{A} \subset \mathcal{E}$, where $\mathcal{E} = \{\ T \in \text{Alg}\mathcal{L} \mid T_{(p,p)} = 0 = T_{(p+1,p+1)}\ \}$. Then $\mathcal{A}$ is a Lie ideal in Alg$\mathcal{L}$ if and only if $\mathcal{A} = \mathcal{C}_{p,p+1}$ or $\mathcal{A} = \mathcal{E}$.*

Proof. Let $\mathcal{A}$ be a Lie ideal in Alg$\mathcal{L}$ and let $\mathcal{A} \neq \mathcal{C}_{p,p+1}$. Then there exists an operaror T in $\mathcal{A}$ and $T \notin \mathcal{C}_{p,p+1}$, i.e. $T_{(p,p+1)} \neq 0$. Let $A = (a_{ij}) \in \mathcal{E}$. If $a_{p\ p+1} = 0$, then $A \in \mathcal{C}_{p,p+1}$ and so $A \in \mathcal{A}$. Let $a_{p\ p+1} \neq 0$. Let A_1 be an operator defined by

$$\begin{cases} A_{1(p,p+1)} = 0 \\ A_{1(i,j)} = a_{ij} \text{ otherwise.} \end{cases}$$

Then $A_1 \in \mathcal{C}_{p,p+1} \subset \mathcal{A}$ and so $A_1 \in \mathcal{A}$. Let T_1 be an operator defined by

$$\begin{cases} T_{1(p,p+1)} = 0 \\ T_{1(i,j)} = -T_{(i,j)} \text{ otherwise.} \end{cases}$$

Since $T_1 \in \mathcal{C}_{p,p+1}$, $T_1 \in \mathcal{A}$. Put $T_2 = T + T_1$. Then $T_2 \in \mathcal{A}$ and $T_{2(p,p+1)} = T_{(p,p+1)}$. Let $\beta = \frac{a_{p\ p+1}}{T_{(p,p+1)}}$. Then $\beta T_2 + A_1 = A$ and $A \in \mathcal{A}$. Hence $\mathcal{E} = \mathcal{A}$. □

Theorem 3.11. *Let $\mathcal{A}$ be a linear manifold in Alg$\mathcal{L}$ such that $\mathcal{A}_0 \subset \mathcal{A} \subset \mathcal{A}_2$. Then $\mathcal{A}$ is a Lie ideal in Alg$\mathcal{L}$ if and only if $\mathcal{A} = \mathcal{A}_0$ or $\mathcal{A} = \mathcal{A}_2$.*

Proof. Let $\mathcal{A}$ be a Lie ideal in Alg$\mathcal{L}$. Let $\mathcal{A}_0 \neq \mathcal{A}$. Then there exists an operaror T in $\mathcal{A}$ and $T \notin \mathcal{A}_0$. Since $\mathcal{A} \subset \mathcal{A}_2$, $T_{(i,i)} = \alpha \neq 0 (i = 1, 2, \cdots)$ for some α in $\mathbb{C}$ and $T - \alpha I \in \mathcal{A}_0$. Since $\alpha \neq 0$ and $\mathcal{A}$ is a linear manifold in Alg$\mathcal{L}$, $T - (T - \alpha I) = \alpha I \in \mathcal{A}$. Since $\alpha \neq 0$, $I \in \mathcal{A}$. Let $A \in \mathcal{A}_2$. If $A_{(i,i)} = 0$ for all $i \in \mathbb{N}$, then $A \in \mathcal{A}_0$ and $A \in \mathcal{A}$. Let $A_{(i,i)} = \beta \neq 0$. Then $A - \beta I \in \mathcal{A}_0 \subset \mathcal{A}$. Since $\mathcal{A}$ is a linear manifold in Alg$\mathcal{L}$, $(A - \beta I) + \beta I = A \in \mathcal{A}$. Hence $\mathcal{A} = \mathcal{A}_2$. □

Theorem 3.12. *Let p and q be natural numbers such that $p < q$. Let $\mathcal{A}$ be a linear manifold in Alg$\mathcal{L}$ such that $\mathcal{A}_{p,q} \subset \mathcal{A} \subset \mathcal{A}^{(1)}_{p,q}$. Then $\mathcal{A}$ is a Lie ideal in Alg$\mathcal{L}$ if and only if $\mathcal{A} = \mathcal{A}_{p,q}$ or $\mathcal{A} = \mathcal{A}^{(1)}_{p,q}$.*

Proof. Let $\mathcal{A}$ be a Lie ideal in Alg$\mathcal{L}$. Let $\mathcal{A} \neq \mathcal{A}_{p,q}$. Then there exists an operaror T in $\mathcal{A}$ and $T \notin \mathcal{A}_{p,q}$. i.e. $T_{(p,q)} = \alpha \neq 0$. Let $A = (a_{ij}) \in \mathcal{A}^{(1)}_{p,q}$. If $a_{pq} = 0$, then $A \in \mathcal{A}_{p,q}$ and so $A \in \mathcal{A}$. Let $a_{pq} = \beta \neq 0$. If $\beta = \alpha$, then $T - A \in \mathcal{A}_{p,q}$ and so $T - A \in \mathcal{A}$. Since $\mathcal{A}$ is a linear manifold in Alg$\mathcal{L}$, $T - (T - A) = A \in \mathcal{A}$. Let $\beta \neq \alpha$. Then $\frac{\alpha}{\beta}A \in \mathcal{A}$ by the above case $\beta = \alpha$. So $A \in \mathcal{A}$. Hence $\mathcal{A} = \mathcal{A}^{(1)}_{p,q}$. □

Theorem 3.13. *Let p and q be natural numbers such that $p < q$. Let $\mathcal{A}$ be a linear manifold in Alg$\mathcal{L}$ such that $\mathcal{A}^{(0)}_{p,q} \subset \mathcal{A} \subset \mathcal{A}^{(2)}_{p,q}$. Then $\mathcal{A}$ is a Lie ideal in Alg$\mathcal{L}$ if and only if $\mathcal{A} = \mathcal{A}^{(0)}_{p,q}$ or $\mathcal{A} = \mathcal{A}^{(2)}_{p,q}$.*

Proof. Let $\mathcal{A}$ be a Lie ideal in Alg$\mathcal{L}$. Let $\mathcal{A} \neq \mathcal{A}^{(0)}_{p,q}$. Then there exists an operaror $T = (t_{ij})$ in $\mathcal{A}$ and $T \notin \mathcal{A}^{(0)}_{p,q}$. i.e. $t_{(pq)} \neq 0$. Let $A = (a_{ij}) \in \mathcal{A}^{(2)}_{p,q}$. If $a_{pq} = 0$, then $A \in \mathcal{A}^{(0)}_{p,q}$ and so $A \in \mathcal{A}$. Let $a_{pq} \neq 0$. If $a_{pq} = t_{pq}$, then $T - A \in \mathcal{A}^{(0)}_{p,q}$ and so $T - A \in \mathcal{A}$. Since $\mathcal{A}$ is a linear manifold in Alg$\mathcal{L}$, $T - (T - A) = A \in \mathcal{A}$. Let $a_{pq} \neq t_{pq}$. Then $\frac{t_{pq}}{a_{pq}}A \in \mathcal{A}$ by the above case $a_{pq} = t_{pq}$. So $A \in \mathcal{A}$. Hence $\mathcal{A} = \mathcal{A}^{(2)}_{p,q}$. □

Theorem 3.14. *Let p and q be natural numbers such that $p < q$. Let $\mathcal{A}$ be a linear manifold in Alg$\mathcal{L}$ such that $\mathcal{A}^{(2)}_{p,q} \subset \mathcal{A} \subset \mathcal{A}^{(1)}_{p,q}$. Then $\mathcal{A}$ is a Lie ideal in Alg$\mathcal{L}$ if and only if $\mathcal{A} = \mathcal{A}^{(1)}_{p,q}$ or $\mathcal{A} = \mathcal{A}^{(2)}_{p,q}$.*

Proof. Let $\mathcal{A}$ be a Lie ideal in Alg$\mathcal{L}$. Suppose that $\mathcal{A} \neq \mathcal{A}^{(2)}_{p,q}$. Then there exists an operaror $T = (t_{ij})$ in $\mathcal{A}$ and $T \notin \mathcal{A}^{(2)}_{p,q}$. i.e. $t_{pp} = t_{p+1\ p+1} = \cdots = t_{qq} \neq 0$. Put $t_{pp} = \alpha$. Let $A = (a_{ij}) \in \mathcal{A}^{(1)}_{p,q}$. If $a_{pp} = a_{p+1\ p+1} = \cdots = a_{qq} = 0$, then $A \in \mathcal{A}^{(2)}_{p,q} \subset \mathcal{A}$. Let $a_{pp} = \beta \neq 0$. Then $T - (\frac{\alpha}{\beta}A)$ is an element of $\mathcal{A}^{(2)}_{p,q}$. Since $T \in \mathcal{A}$, and $\mathcal{A}$ is a linear manifold in Alg$\mathcal{L}$, $T - (T - (\frac{\alpha}{\beta}A)) = (\frac{\alpha}{\beta}A) \in \mathcal{A}$ and $A \in \mathcal{A}$. Hence $\mathcal{A} = \mathcal{A}^{(1)}_{p,q}$. □

Theorem 3.15. *Let $\mathcal{A}$ be a linear manifold in Alg$\mathcal{L}$ such that $\mathcal{A}_0 \subset \mathcal{A} \subset \mathcal{A}_2$. Then $\mathcal{A}$ is a Lie ideal in Alg$\mathcal{L}$ if and only if $\mathcal{A} = \mathcal{A}_0$ or $\mathcal{A} = \mathcal{A}_2$.*

Proof. Let $\mathcal{A}$ be a Lie ideal in Alg$\mathcal{L}$. Let $\mathcal{A}_0 \neq \mathcal{A}$. Then there exists an operaror T in $\mathcal{A}$ and $T \notin \mathcal{A}_0$. Since $\mathcal{A} \subset \mathcal{A}_2$, $T_{(i,i)} = \alpha \neq 0 (i = 1, 2, \cdots)$ for some α in $\mathbb{C}$ and $T - \alpha I \in \mathcal{A}_0$. Since $\alpha \neq 0$ and $\mathcal{A}$ is a linear manifold in Alg$\mathcal{L}$, $T - (T - \alpha I) = \alpha I \in \mathcal{A}$. Since $\alpha \neq 0$, $I \in \mathcal{A}$. Let $A \in \mathcal{A}_2$. If $A_{(i,i)} = 0$ for all $i \in \mathbb{N}$, then $A \in \mathcal{A}_0$ and $A \in \mathcal{A}$. Let $A_{(i,i)} = \beta \neq 0$. Then $A - \beta I \in \mathcal{A}_0 \subset \mathcal{A}$. Since $\mathcal{A}$ is a linear manifold in Alg$\mathcal{L}$, $(A - \beta I) + \beta I = A \in \mathcal{A}$. Hence $\mathcal{A} = \mathcal{A}_2$. □

Let $\mathcal{A}$ be a Lie ideal in Alg$\mathcal{L}$. Let $X = \{ (p,q) \mid T_{(p,q)} = 0 \text{ for all } T \in \mathcal{A} \}$. Let i and j be natural numbers and let E_{ij} be the operator whose (i,j)- component is 1 and all other entries are 0. Let k be a natural number. Put $E^{(k)}_n = \sum_{i=1}^{n} E_{i\ i+k}$, $E^{(k)} = \sum_{i=1}^{\infty} E_{i\ i+k}$. Then $E^{(k)}_n \longrightarrow E^{(k)}$(strongly).

Theorem 3.16. *Let $\mathcal{A}$ be a strongly closed Lie ideal in Alg$\mathcal{L}$ and let k be a natural number. Assume that $X = \emptyset$. Then $E^{(k)} \in \mathcal{A}$.*

Proof. Since $X = \emptyset$, for each $(i, i+k) \in \mathbb{N} \times \mathbb{N}$ there exists $T^{(i,i+k)} \in \mathcal{A}$ such that $T^{(i,i+k)}_{(i,i+k)} \neq 0$. Let $T^{(i,i+k)'} = E_{i\ i} T^{(i,i+k)} - T^{(i,i+k)} E_{i\ i}$. Then $T^{(i,i+k)'} \in \mathcal{A}$ for all $i \in \mathbb{N}$. $E_{i+k\ i+k} T^{(i,i+k)'} - T^{(i,i+k)'} E_{i+k\ i+k} = T^{(i,i+k)}_{(i,i+k)} E_{i\ i+k}$. Since $T^{(i,i+k)}_{(i,i+k)} \neq 0$, $E_{i\ i+k} \in \mathcal{A}$. Since $\mathcal{A}$ is a linear manifold in Alg$\mathcal{L}$, $E^{(k)}_n \in \mathcal{A}$. Since $E^{(k)}_n \longrightarrow E^{(k)}$(strongly), $E^{(k)} \in \mathcal{A}$. □

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