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ON STABILITY OF A POLYNOMIAL

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ABSTRACT. A polynomial, $p(z) = a_0z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n$, with real coefficients is called a stable or a Hurwitz polynomial if all its zeros have negative real parts. We show that if a polynomial is a Hurwitz polynomial then so is the polynomial $q(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$ (with coefficients in reversed order). As consequences, we give simple ratio checking inequalities that would determine unstability of a polynomial of degree 5 or more and extend conditions to get some previously known results.

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1. Introduction

A polynomial

$$p(z) = a_0z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n \tag{1}$$

with real coefficients is called a stable or a Hurwitz polynomial if all its zeros have negative real parts. It is well known that all the coefficients of a stable polynomial have the same sign. In this paper, we consider the polynomial with positive coefficients.

The criteria for deciding a Hurwitz polynomial is related to the stability of a linear system. If the characteristic polynomial of a matrix A is a Hurwitz polynomial, then the system $\dot{x} = Ax$ is stable. (For more details, we refer the readers to the Introduction of [7]). There are several criteria to verify the stability where among them Routh-Hurwitz and Liénard-Chipart criterion are most known [3], [5], [6]. These involve calculation of principal minors of the Hurwitz matrix which is defined below. Recent results examine relationships between the coefficients of the polynomial to determine stability [1], [4], [8].

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For the Routh-Hurwitz criterion, consider a polynimial $p(z)$ as in (1) with positive real coefficients. Then, the Hurwitz matrix of $p(z)$ is defined as an $n \times n$ matrix

$$\begin{bmatrix} a_1 & a_3 & a_5 & \cdots & \\ a_0 & a_2 & a_4 & \cdots & \\ 0 & a_1 & a_3 & \cdots & \\ 0 & a_0 & a_2 & \cdots & \\ 0 & 0 & a_1 & \cdots & \\ \cdots & & & & \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix}$$

with the criterion stated below as a theorem.

Theorem 1.1 (Routh-Hurwitz, [3])**.** *A polynomial $p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n$ is a Hurwitz polynomial if and only if each leading principal minor Δ_j $(1 \leq j \leq n)$ is positive.*

Whereas, the Liénard-Chipart criterion take less computations compared to Routh-Hurwitz.

Theorem 1.2 (Liénard-Chipart, [5])**.** *A polynomial $p(z)$ with positive real coefficients is Hurwitz stable if and only if each leading principal minor Δ_{2j} $(1 \leq j < \frac{n+1}{2})$ is positive.*

We note that the Liénard-Chipart criterion can be stated using odd terms Δ_{2j-1} instead of even terms Δ_{2j} as well.

The aim of this paper is therefore to decide unstability just by checking the coefficients' simple ratio relations (instead of calculating principal minors; so it's computationally less costly) by examining some characteristics of a Hurwitz polynomial. The paper is composed as follows. In section 2, we present the main results showing that the stability of a new polynomial obtained from the original by reversing the order of coefficients is stable if and only if the original is so. This is proved using a complex analysis idea. In section 3, we give some applications of the result. Finally, a conclusion and future work are mentioned in section 4.

2. Main results

We find it interesting to see that the stability of a polynomial is not affected by reversing the order of coefficients.

Theorem 2.1. *If a polynomial*

$$p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n \tag{2}$$

is a Hurwitz polynomial, then so is the polynomial

$$q(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0. \tag{3}$$

Proof. Since we assumed all coefficients a_j of $p(z)$ are real, we can obtain the polynomial $q(z)$ from $p(z)$ by defining

$$q(z) := z^n \overline{p(1/\overline{z})}.$$

Let $z_j (j = 1, 2, \cdots, n)$ be the roots of $p(z)$ which can be written as

$$p(z) = a_0 \prod_{j=1}^{n} (z - z_j).$$

Note that $q(z)$ then becomes $q(z) = a_0 \prod_{j=1}^{n} (1 - z\overline{z}_j)$. Since all the roots z_j of $p(z)$ have negative real parts, all the roots of $q(z)$, namely,

$$\frac{1}{\overline{z_j}} = \frac{z_j}{|z_j|^2}$$

also have negative real parts. □

Then, we notice the following relations among the coefficients of a Hurwitz polynomial.

Theorem 2.2. *If* $p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$ *is a Hurwitz polynomial of degree* $n \geq 4$*, then*

$$\frac{a_1}{a_0} > \frac{a_3}{a_2} \quad \textit{and} \quad \frac{a_{n-2}}{a_{n-3}} > \frac{a_n}{a_{n-1}}. \tag{4}$$

Proof. Let $q(z)$ be the polynomial formed by reversing the coefficients of $p(z)$. Then both of $p(z)$ and $q(z)$ are Hurwitz polynomials. So the inequalities come from

$$\Delta_2^p = a_1 a_2 - a_0 a_3 > 0, \quad \text{and}$$
$$\Delta_2^q = a_{n-1} a_{n-2} - a_{n-3} a_n > 0,$$

where Δ_j^p denotes the j-th leading principal minor of Hurwitz matrix for $p(z)$, etc. □

Hence, if in particular, $p(z)$ is a Hurwitz polynomial of degree five, we get the following simple relations among the coefficients.

Corollary 2.3. *If* $p(z)$ *is a Hurwitz polynomial of degree five, then the coefficients of* $p(z)$ *satisfy the inequalities*

$$\frac{a_1}{a_0} > \frac{a_3}{a_2} > \frac{a_5}{a_4}.$$

Note that this Corollary provides a way of determining unstability in a simpler way than calculating principal minors of the Hurwitz matrix of $p(z)$. For example, the polynomial

$$p(z) = z^5 + 2z^4 + 3z^3 + 3z^2 + 2z + 4$$

is unstable, since $\frac{a_3}{a_2} < \frac{a_5}{a_4}$. This polynomial has roots $z \approx 0.435027 \pm 0.982555i$.

On the other hand, in [9], Z., Zahreddine showed that

Theorem 2.4 (Zahreddine, [9])**.** *If $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ is stable, then*

$$h(z) = a_0 + a_2 z + a_4 z^2 + \cdots , \tag{5}$$

and

$$k(z) = a_1 + a_3 z + a_5 z^2 + \cdots \tag{6}$$

have negative real zeros only.

Hence, Zahreddine's Theorem implies that a polynomial formed by taking alternate coefficients of a stable polynomial is also stable. From this, we have the following

Corollary 2.5. *If a polynomial $p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$ $(n \geq 7)$ is stable, then*

$$\frac{a_2}{a_0} > \frac{a_6}{a_4}, \quad \frac{a_3}{a_1} > \frac{a_7}{a_5}, \quad \frac{a_{n-4}}{a_{n-6}} > \frac{a_n}{a_{n-2}}, \quad \text{and} \quad \frac{a_{n-5}}{a_{n-7}} > \frac{a_{n-1}}{a_{n-3}}. \tag{7}$$

Proof. Since $p(z)$ is stable, the polynomials $p_e(z)$ and $p_o(z)$ in Theorem 2.4. are stable. By applying Theorem 2.2. to $p_e(z)$ and $p_o(z)$, we get the stated inequalities. □

Moreover, by observing Δ_4 of the Hurwitz matrix, we have inequalities for the unstability of a polynomial of degree $n \geq 5$. The result is stated below where we assume $a_j = 0$ if $j > n$. We note that the last part of the following theorem (where we assume the leading coefficient $a_0 = 1$) imply Corollary 1 of R. Bortolatto [2].

Theorem 2.6. *Let $p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$ $(n \geq 5)$. Suppose that $\Delta_2 > 0$ and $a_0 a_7 - a_1 a_6 \geq 0$. (Here, we assume that $a_k = 0$ for $k > n$, i.e., if $n = 5$, then $a_6 = 0 = a_7$, etc.) If*

$$a_2 a_5 - a_3 a_4 \geq 0, \tag{8}$$

then $p(z)$ is unstable. And if for polynomial with leading coefficient $a_0 = 1$

$$a_5 - a_1 a_4 \geq 0, \tag{9}$$

then $p(z)$ is unstable.

Proof. Consider the 4×4 principal submatrix of the Hurwitz matrix of the given polynomial $p(z)$, namely,

$$\begin{bmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 & a_4 \end{bmatrix}.$$

Through cofactor expansion on the first column and rearrangement of the terms, the determinant of the above matrix can be written as

$$\begin{aligned}\Delta_4 =& - a_2(a_0a_5 - a_1a_4)\Delta_2 - a_4\Delta_2^2 - (a_0a_7 - a_1a_6)\Delta_2 - (a_1a_4 - a_0a_5)^2 \\ &+ (a_0 - 1)(a_2a_5 - a_3a_4)\Delta_2. \end{aligned} \tag{10}$$

Since we assumed all coefficients are positive, the case $a_2a_5 - a_3a_4 \geq 0$ implies $\Delta_4 < 0$ by (6). When $a_0 = 1$, the case $a_5 - a_1a_4 \geq 0$ similarly implies $\Delta_4 < 0$ by (7). Hence, the polynomial is unstable in both cases. □

Remark 2.1. Note that, due to Theorem 2.1., either one of the following inequalities

$$a_2a_5 - a_3a_4 \geq 0, \quad \text{or} \quad a_{n-2}a_{n-5} - a_{n-3}a_{n-4} \geq 0. \tag{11}$$

will yield the same conclusion. Hence, we have extended conditions in Corollary 1 of R. Bortolatto [2] (see the statement right before Theorem 2.6).

3. Applications

As a special case, let us consider a polynomial of degree 6 with real positive coefficients,

$$p(z) = a_0z^6 + a_1z^5 + a_2z^4 + a_3z^3 + a_4z^2 + a_5z + a_6. \tag{12}$$

Suppose $p(z)$ is stable, then the coefficients would satisfy all of the following:

$$\frac{a_1}{a_0} > \frac{a_3}{a_2}, \quad \frac{a_4}{a_3} > \frac{a_6}{a_5}, \qquad \text{(by Theorem 2.2.)} \tag{13}$$

$$\frac{a_2}{a_0} > \frac{a_6}{a_4}. \qquad \text{(by Corollary 2.5.)} \tag{14}$$

For example, both of the polynomials

$$\begin{aligned} g_1(z) =& z^6 + 3z^5 + 2z^4 + 4z^3 + 3z^2 + 3z + 4, \quad \text{and} \\ g_2(z) =& 3z^6 + 7z^5 + 2z^4 + 3z^3 + 4z^2 + 6z + 5 \end{aligned}$$

are unstable because $g_1(z)$ violates (13), and $g_2(z)$ violates (14).

4. Conclusion

In this paper, by proving the polynomial with coefficients in reversed order does not affect the stability of the original, we were able to obtain conditions on a_{n-k} with the information on a_k. Hence, simple conditions on ratios of coefficients to determine unstability were drawn. As a future work, we try to obtain precise relations among coefficients of a polynomial that will be sufficient and necessary for stablilty. For example, the polynomials

$$\begin{aligned} h_1(z) =& z^6 + 5z^5 + 11z^4 + 12z^3 + 7z^2 + 2z + 1, \quad \text{and} \\ h_2(z) =& z^6 + 5z^5 + 11z^4 + 12z^3 + 7z^2 + 2z + \frac{1}{2} \end{aligned}$$

both satisfy inequalities (13) and (14), yet $h_1(z)$ is unstable and $h_2(z)$ is stable! This fact prompts to ask what differentiates h_1 from h_2.

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