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STABILITY RESULTS OF POSITIVE WEAK SOLUTION FOR SINGULAR p-LAPLACIAN NONLINEAR SYSTEM

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ABSTRACT. In this paper, we investigate the stability of positive weak solution for the singular p-Laplacian nonlinear system $-div[|x|^{-ap}|\nabla u|^{p-2}\nabla u]+m(x)|u|^{p-2}u=\lambda|x|^{-(a+1)p+c}b(x)f(u)$ in Ω, $Bu=0$ on $\partial\Omega$, where $\Omega\subset R^n$ is a bounded domain with smooth boundary $Bu=\delta h(x)u+(1-\delta)\frac{\partial u}{\partial n}$ where $\delta\in[0,1]$, $h:\partial\Omega\to R^+$ with $h=1$ when $\delta=1$, $0\in\Omega$, $1<p<n$, $0\leq a<\frac{n-p}{p}$, $m(x)$ is a weight function, the continuous function $b(x)$: $\Omega\to R$ satisfies either $b(x)>0$ or $b(x)<0$ for all $x\in\Omega$, λ is a positive parameter and $f:[0,\infty)\to R$ is a continuous function. We provide a simple proof to establish that every positive solution is unstable under certain conditions.

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1. Introduction

Elliptic problems involving more general operator, such as the degenerate quasi-linear elliptic operator given by $-div[|x|^{-ap}|\nabla u|^{p-2}\nabla u]$, were motivated by Caaffarelli, Kohn and Nirenberg's inequality (see [6],[19]). The study of this type of problem is motivated by its various applications, for example, in fluid mechanics, in Newtonian fluids, in flow through porous media and in glaciology ([5, 7]).

Many authors are interested in the study of the stability and instability of nonnegative solutions of linear [3] , semilinear ([9, 13, 17]), semipositone ([4, 8, 16]) and nonlinear ([1, 2, 12]) systems, due to the great number of applications in reaction-diffusion problems, in fluid mechanics, in Newtonian fluids, glaciology, population dynamics, etc.; see [5] and references therein. Also, in the recent past, many authors devoted their attention to study the singular p-Laplacian nonlinear systems ([10, 11, 15, 18, 19]).

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In this paper we consider the stability and instability of positive weak solution for the singular p-Laplacian nonlinear system

$$\left.\begin{array}{c} -div[|x|^{-ap}|\nabla u|^{p-2}\nabla u]+m(x)|u|^{p-2}u=\lambda|x|^{-(a+1)p+c}b(x)f(u) \text{ in } \Omega, \\ B\phi=0, \text{ on } \partial\Omega, \end{array}\right\} \quad (1)$$

where $\Omega\subset R^n$ is a bounded domain with smooth boundary $Bu=\delta h(x)u+(1-\delta)\frac{\partial u}{\partial n}$ where $\delta\in[0,1]$, $h:\partial\Omega\to R^+$ with $h=1$ when $\delta=1$, $0\in\Omega$, $1<p<n$, $0\leq a<\frac{n-p}{p}$, $m(x)$ is a weight function, the continuous function $b(x):\Omega\to R$ satisfies either $b(x)>0$ or $b(x)<0$ for all $x\in\Omega$, λ is a positive parameter and $f:[0,\infty)\to R$ is a continuous function.

Tertikas in [16] have been proved the stability and instability results of positive solutions for the semilinear system

$$-\Delta u=\lambda f(u) \text{ in } \Omega,\ Bu=0 \text{ on } \partial\Omega,$$

under various choices of the function f. In [4], the authors have been studied the uniqueness and stability of nonnegative solutions for classes of nonlinear elliptic Dirichlet problems in a ball, when the nonlinearity is monotone, negative at the origin, and either concave or convex.

In the case $a=0$, $p=2$, $m(x)=0$, $c=p$ and $b(x)=1$, system (1) have been studied by several authors (see [8, 9, 13, 14])

Finally, let us explain the plan of the paper. In section 2, we study the stability and instability of the positive weak solution of (1). In section 3, we introduce some applications regarding the stability properties of the positive weak solution of (1).

We recall that, if u be any positive weak solution of (1), then the linearized equation about u is given by

$$\left.\begin{array}{c} (p-1)[-\text{div}[|x|^{-ap}|\nabla u|^{p-2}\nabla\phi]+m(x)|u|^{p-2}\phi] \\ -\lambda|x|^{-(a+1)p+c}b(x)f_u(u)\phi=\mu\phi \text{ in } \Omega, \\ B\phi=0 \text{ on } \partial\Omega, \end{array}\right\} \quad (2)$$

where $f_u\ (u)$ denotes the derivative of $f(u)$ with respect to u, μ is the eigenvalue corresponding to the eigenfunction ϕ.

Definition 1.1. We call a solution u of (1) a linearly stable solution if all eigenvalues of (2) are strictly positive, which can be implied if the principal eigenvalue $\mu_1>0$. Otherwise u is linearly unstable.

2. Main results

In this section, we shall prove the stability and instability of the positive weak solution u of (1) under some certain conditions. Our main results are formulate in the following theorems.

Theorem 2.1. *If $f(u)/u^{p-1}$ is strictly increasing and $b(x)>0$ for all $x\in\Omega$, then every positive weak solution of (1) is linearly unstable.*

Proof. Let u_0 be any positive weak solution of (1), then the linearized equation about u_0 is given by

$$\left.\begin{array}{c}(p-1)[-\mathrm{div}[|x|^{-ap}|\nabla u_0|^{p-2}\nabla\phi]+m(x)|u_0|^{p-2}\phi]\\ -\lambda|x|^{-(a+1)p+c}b(x)f_{u_0}(u_0)\phi=\mu\phi \text{ in } \Omega,\\ B\phi=0 \text{ on } \partial\Omega.\end{array}\right\} \tag{3}$$

Let μ_1 be the first eigenvalue of (3) and let $\psi(x)\geq 0$ be the corresponding eigenfunction. Multiplying (1) by ψ and integrating over Ω, we have

$$\begin{aligned}&-\int_\Omega \psi\mathrm{div}[|x|^{-ap}|\nabla u_0|^{p-2}\nabla u_0]dx+\int_\Omega m(x)|u_0|^{p-2}u_0\psi\\ =\;&\lambda\int_\Omega \psi|x|^{-(a+1)p+c}b(x)f(u_0)dx.\end{aligned} \tag{4}$$

The first term of the L.H.S. of (4) may be written in the form

$$\begin{aligned}\int_\Omega \psi\mathrm{div}[|x|^{-ap}|\nabla u_0|^{p-2}\nabla u_0]dx &= \int_\Omega \psi\nabla u_0\nabla[|x|^{-ap}|\nabla u_0|^{p-2}]dx\\ &+\int_\Omega \psi[|x|^{-ap}|\nabla u_0|^{p-2}]\mathrm{div}(\nabla u_0)dx.\end{aligned}$$

Applying Green's first identity, we have

$$\begin{aligned}\int_\Omega \psi\mathrm{div}[|x|^{-ap}|\nabla u_0|^{p-2}\nabla u_0]dx &= -\int_\Omega [|x|^{-ap}|\nabla u_0|^{p-2}]\nabla u_0\nabla\psi dx\\ &+\int_{\partial\Omega} \psi[|x|^{-ap}|\nabla u_0|^{p-2}]\frac{\partial u_0}{\partial n}ds.\end{aligned} \tag{5}$$

From (5) in (4), we have

$$\begin{aligned}&\int_\Omega [|x|^{-ap}|\nabla u_0|^{p-2}]\nabla u_0\nabla\psi dx+\int_\Omega m(x)|u_0|^{p-2}u_0\psi\\ =\;&\lambda\int_\Omega \psi|x|^{-(a+1)p+c}b(x)f(u_0)dx+\int_{\partial\Omega}\psi[|x|^{-ap}|\nabla u_0|^{p-2}]\frac{\partial u_0}{\partial n}ds.\end{aligned} \tag{6}$$

Also, Multiplying (3) by $(-u_0)$ and integrating over Ω, we have

$$\begin{aligned}&(p-1)\int_\Omega u_0\mathrm{div}[|x|^{-ap}|\nabla u_0|^{p-2}\nabla\psi]dx-(p-1)\int_\Omega m(x)u_0[|u_0|^{p-2}\psi]dx\\ &+\lambda\int_\Omega u_0|x|^{-(a+1)p+c}b(x)f_{u_0}(u_0)\psi]dx=-\mu_1\int_\Omega u_0\psi dx.\end{aligned} \tag{7}$$

The first term of the L.H.S. of (7) may be written in the form

$$\int_\Omega u_0 \mathrm{div}[|x|^{-ap}|\nabla u_0|^{p-2}\nabla\psi]dx = \int_\Omega u_0[|x|^{-ap}|\nabla u_0|^{p-2}]\nabla\cdot\nabla\psi dx + \int_\Omega u_0\nabla\psi\nabla[|x|^{-ap}|\nabla u_0|^{p-2}]dx.$$

Using Green's first identity, one have

$$\int_\Omega u_0 \mathrm{div}[|x|^{-ap}|\nabla u_0|^{p-2}\nabla\psi]dx = -\int_\Omega [|x|^{-ap}|\nabla u_0|^{p-2}]\nabla u_0\nabla\psi + \int_{\partial\Omega} u_0[|x|^{-ap}|\nabla u_0|^{p-2}]\frac{\partial\psi}{\partial n}ds. \quad (8)$$

From (8) in (7) we have

$$(p-1)[\int_{\partial\Omega} u_0[|x|^{-ap}|\nabla u_0|^{p-2}]\frac{\partial\psi}{\partial n}ds - \int_\Omega [|x|^{-ap}|\nabla u_0|^{p-2}]\nabla u_0\nabla\psi] -(p-1)\int_\Omega m(x)u_0[|u_0|^{p-2}\psi]dx + \lambda\int_\Omega u_0|x|^{-(a+1)p+c}b(x)f_{u_0}(u_0)\psi]dx = -\mu_1\int_\Omega u_0\psi dx. \quad (9)$$

Multiplying (6) by $(p-1)$ and adding with (9), we have

$$-\mu_1\int_\Omega u_0\psi dx = (p-1)\int_{\partial\Omega}[|x|^{-ap}|\nabla u_0|^{p-2}][u_0\frac{\partial\psi}{\partial n} - \psi\frac{\partial u_0}{\partial n}]ds + \lambda\int_\Omega |x|^{-(a+1)p+c}b(x)\psi[u_0 f_{u_0}(u_0) - (p-1)f(u_0)]. \quad (10)$$

Now, when $\delta = 1$, we have $Bu_0 = u_0 = 0$ for $s \in \partial\Omega$ and also we have $\psi = 0$ for $s \in \partial\Omega$. Then

$$\int_{\partial\Omega}[|x|^{-ap}|\nabla u_0|^{p-2}][u_0\frac{\partial\psi}{\partial n} - \psi\frac{\partial u_0}{\partial n}]ds = 0. \quad (11)$$

Also, when $\delta \neq 1$, we have

$$\frac{\partial u_0}{\partial n} = -\frac{\delta h u_0}{1-\delta} \quad \text{and} \quad \frac{\partial\psi}{\partial n} = -\frac{\delta h\psi}{1-\delta},$$

which implies again the result given by (11).
Hence

$$-\mu_1\int_\Omega u_0\psi dx = \lambda\int_\Omega |x|^{-(a+1)p+c}b(x)\psi[u_0 f_{u_0}(u_0) - (p-1)f(u_0)]dx. \quad (12)$$

Since $\frac{f(u_0)}{u_0^{p-1}}$ is strictly increasing and $b(x) > 0$ for all $x \in \Omega$, we have

$$b(x)[u_0 f_{u_0}(u_0) - (p-1)f(u_0)] > 0, \tag{13}$$

and hence (12) becomes

$$-\mu_1 \int_\Omega u_0 \psi dx > 0, \tag{14}$$

so $\mu_1 < 0$ and the result follows. □

Theorem 2.2. *If $f(u)/u^{p-1}$ is strictly decreasing and $b(x) > 0$ for all $x \in \Omega$, then every positive weak solution of (1) is linearly stable.*

Proof. As in the proof of Theorem 2.1., we can easily obtain that

$$-\mu_1 \int_\Omega u_0 \psi dx < 0, \tag{15}$$

so $\mu_1 > 0$ and the result follows. □

Corollary 2.3. *If $f(u)/u^{p-1}$ is strictly increasing (decreasing) and $b(x) < 0$ for all $x \in \Omega$, then every positive weak solution of (1) is linearly stable (unstable).*

Proof. The proof is similar to that of Theorems 2.1. and 2.2. □

Now, we generalize Theorem 2.1. and 2.2. for the following singular p-Laplacian nonlinear system

$$\left.\begin{array}{lr} -div[|x|^{-ap}|\nabla u|^{p-2}\nabla u] = \lambda |x|^{-(a+1)p+c} g(x,u) & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega, \end{array}\right\} \tag{16}$$

where $g : \Omega \times R^+ \to R$ is a continuous function, Our results are the following

Theorem 2.4. *If $g(x,u)/u^{p-1}$ is strictly increasing for $u > 0$ at each fixed $x \in \Omega$, then every positive weak solution of (16) is linearly unstable.*

Proof. The proof proceeds in the same way as for Theorem 2.1. but instead of (12), we have

$$-\mu_1 \int_\Omega u_0 \psi dx = \lambda \int_\Omega |x|^{-(a+1)p+c} \psi [u_0 g_{u_0}(x,u_0) - (p-1)g(x,u_0)] dx. \tag{17}$$

Since $\frac{g(x,u_0)}{u_0^{p-1}}$ is strictly increasing, we have

$$u_0 g_{u_0}(x,u_0) - (p-1)g(x,u_0) > 0, \tag{18}$$

and hence (17) becomes

$$-\mu_1 \int_\Omega u_0 \psi dx > 0,$$

so $\mu_1 < 0$ and the result follows. □

Theorem 2.5. *If $g(x,u)/u^{p-1}$ is strictly decreasing for $u > 0$ at each fixed $x \in \Omega$, then every positive weak solution of (16) is linearly stable.*

Proof. The proof proceeds in the same way as for Theorem 2.4. □

3. Applications

Here we introduce some examples

Example 1.

Consider the singular p-Laplacian nonlinear system

$$\left.\begin{array}{c} -div[|x|^{-ap}|\nabla u|^{p-2}\nabla u] + m(x)|u|^{p-2}u = \lambda |x|^{-(a+1)p+c} b(x) u^{\alpha} \text{ in } \Omega, \\ B\phi = 0 \text{ on } \partial\Omega, \end{array}\right\} \quad (19)$$

where $b(x)$ is a weight function and $0 < \alpha < p-1$. It is easy to see that $g(x,u)/u^{p-1} = b(x)u^{\alpha-p+1}$ is strictly decreasing function and hence according to Theorem 2.5., the positive weak solution of (19) is linearly stable.

Example 2.

Consider the nonlinear system

$$\left.\begin{array}{c} -div[|x|^{-ap}|\nabla u|^{p-2}\nabla u] + m(x)|u|^{p-2}u = \lambda |x|^{-(a+1)p+c} b(x)[u^{\alpha} + u^{\beta}] \text{ in } \Omega, \\ B\phi = 0 \text{ on } \partial\Omega, \end{array}\right\} \quad (20)$$

where $b(x)$ is a weight function and $0 < \beta \leq \alpha < p-1$. It is easy to see that $g(x,u)/u^{p-1} = b(x)[u^{\alpha-p+1} + u^{\beta-p+1}]$ is strictly decreasing function and hence according to Theorem 2.5., the positive weak solution of (20) is linearly stable.

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