

CERTAIN FRACTIONAL INTEGRALS AND IMAGE FORMULAS OF GENERALIZED \mathbf{k} -BESSEL FUNCTION

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ABSTRACT. We aim to establish certain Saigo hypergeometric fractional integral formulas for a finite product of the generalized \mathbf{k} -Bessel functions, which are also used to present image formulas of several integral transforms including beta transform, Laplace transform, and Whittaker transform. The results presented here are potentially useful, and, being very general, can yield a large number of special cases, only two of which are explicitly demonstrated.

1. Introduction and preliminaries

Díaz and Pariguan [4] introduced the \mathbf{k} -Pochhammer symbol as follows:

$$(1.1) \quad (\gamma)_{n,\mathbf{k}} := \begin{cases} \frac{\Gamma_{\mathbf{k}}(\gamma + n\mathbf{k})}{\Gamma_{\mathbf{k}}(\gamma)} & (n \in \mathbb{N}; \mathbf{k} \in \mathbb{R}^+; \gamma \in \mathbb{C} \setminus \{0\}), \\ \gamma(\gamma + \mathbf{k}) \cdots (\gamma + (n-1)\mathbf{k}) & (n \in \mathbb{N}; \gamma \in \mathbb{C}), \end{cases}$$

where $\Gamma_{\mathbf{k}}$ is the \mathbf{k} -gamma function which has the following relation with the classical Euler's gamma function Γ (see, e.g., [14, Section 1.1])

$$(1.2) \quad \Gamma_{\mathbf{k}}(\gamma) = \mathbf{k}^{\frac{\gamma}{\mathbf{k}}-1} \Gamma\left(\frac{\gamma}{\mathbf{k}}\right).$$

Here and in the following, let \mathbb{C} , \mathbb{R} , \mathbb{R}^+ , \mathbb{N} , and \mathbb{Z}_0^- be the sets of complex numbers, real numbers, positive real numbers, positive integers, and non-positive integers, respectively.

Obviously, the special case $\mathbf{k} = 1$ of (1.1) reduces to the familiar Pochhammer symbol (see, e.g., [14, pp. 2 and 5]):

$$(1.3) \quad (\lambda)_{\nu} := \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}). \end{cases}$$

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Also, the following identity holds:

$$(1.4) \quad \Gamma_{\mathbf{s}}(\gamma) = \left(\frac{\mathbf{s}}{\mathbf{k}}\right)^{\frac{\gamma}{\mathbf{s}}-1} \Gamma_{\mathbf{k}}\left(\frac{\mathbf{k}\gamma}{\mathbf{s}}\right) \quad (\gamma \in \mathbb{C}; \mathbf{k}, \mathbf{s} \in \mathbb{R}^+),$$

which, upon setting $\mathbf{k} = 1$ and replacing \mathbf{s} by \mathbf{k} , yields (1.2). Further, the following relation holds:

$$(1.5) \quad (\gamma)_{nq, \frac{\mathbf{s}}{\mathbf{k}}} = \left(\frac{\mathbf{s}}{\mathbf{k}}\right)^{nq} \left(\frac{\mathbf{k}\gamma}{\mathbf{s}}\right)_{nq} \quad (n, q \in \mathbb{N}; \mathbf{s}, \mathbf{k} \in \mathbb{R}^+; \gamma \in \mathbb{C}),$$

which, upon setting $\mathbf{k} = 1$ and replacing \mathbf{s} by \mathbf{k} , yields

$$(1.6) \quad (\gamma)_{nq, \mathbf{k}} = (\mathbf{k})^{nq} \left(\frac{\gamma}{\mathbf{k}}\right)_{nq} \quad (n, q \in \mathbb{N}; \mathbf{k} \in \mathbb{R}^+; \gamma \in \mathbb{C}).$$

For more details of \mathbf{k} -Pochhammer symbol with \mathbf{k} -special function and fractional Fourier transform, we refer the reader, for example, to [9, 10].

Recently, Romero et al. [11] (see also [2]) introduced the \mathbf{k} -Bessel function of the first kind

$$(1.7) \quad J_{\mathbf{k}, \mu}^{\gamma, \lambda}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n, \mathbf{k}}}{\Gamma_{\mathbf{k}}(\lambda n + \mu + 1)} \frac{(-1)^n}{(n!)^2} \left(\frac{z}{2}\right)^n$$

$$(\gamma, \lambda, \mu \in \mathbb{C} \text{ with } \min\{\Re(\lambda), \Re(\mu)\} > 0; \mathbf{k} \in \mathbb{R}^+).$$

A more generalized form of \mathbf{k} -Bessel function $\omega_{\mathbf{k}, \nu, b, c}^{\gamma, \lambda}(z)$ is given as follows:

$$(1.8) \quad \omega_{\mathbf{k}, \mu, b, c}^{\gamma, \lambda}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n (\gamma)_{n, \mathbf{k}}}{\Gamma_{\mathbf{k}}(\lambda n + \mu + \frac{b+1}{2})} \frac{\left(\frac{z}{2}\right)^{\mu+2n}}{(n!)^2}$$

$$(b, c, \gamma, \lambda, \mu \in \mathbb{C} \text{ with } \min\{\Re(\lambda), \Re(\mu)\} > 0; \mathbf{k} \in \mathbb{R}^+).$$

In (1.7) and (1.8), $(\gamma)_{n, \mathbf{k}}$ and $\Gamma_{\mathbf{k}}$ are the \mathbf{k} -Pochhammer symbol and the \mathbf{k} -gamma function given in (1.1) and (1.2), respectively.

Consider the following product of (1.8)

$$(1.9) \quad \prod_{i=1}^r \omega_{\mathbf{k}_i, \mu_i, b_i, c_i}^{\gamma_i, \lambda_i}(z) = \prod_{i=1}^r \sum_{n=0}^{\infty} \frac{(-1)^n c_i^n (\gamma_i)_{n, \mathbf{k}_i}}{\Gamma_{\mathbf{k}_i}(\lambda_i n + \mu_i + \frac{b_i+1}{2})} \frac{\left(\frac{z}{2}\right)^{\mu_i+2n}}{(n!)^2}$$

$(b_i, c_i, \gamma_i, \lambda_i, \mu_i \in \mathbb{C} \text{ with } \min\{\Re(\lambda_i), \Re(\mu_i)\} > 0; \mathbf{k}_i \in \mathbb{R}^+; i = 1, \dots, r)$, which, upon setting $b_i = c_i = 1$ ($i = 1, \dots, r$), yields

$$(1.10) \quad \prod_{i=1}^r \omega_{\mathbf{k}_i, \mu_i, 1, 1}^{\gamma_i, \lambda_i}(z) = \prod_{i=1}^r \left(\frac{z}{2}\right)^{\mu_i} \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma_i)_{n, \mathbf{k}_i}}{\Gamma_{\mathbf{k}_i}(\lambda_i n + \mu_i + 1)} \frac{\left(\frac{z^2}{4}\right)^n}{(n!)^2}$$

$$= \prod_{i=1}^r \left(\frac{z}{2}\right)^{\mu_i} J_{\mathbf{k}_i, \mu_i}^{\gamma_i, \lambda_i}\left(\frac{z^2}{2}\right)$$

$(\gamma_i, \lambda_i, \mu_i \in \mathbb{C} \text{ with } \min\{\Re(\lambda_i), \Re(\mu_i)\} > 0; \mathbf{k}_i \in \mathbb{R}^+; i = 1, \dots, r)$.

Setting $b_i = -1$ and $c_i = 1$ ($i = 1, \dots, r$), (1.9) reduces to product of the \mathbf{k} -Wright functions

$$(1.11) \quad \prod_{i=1}^r \omega_{\mathbf{k}_i, \mu_i, -1, 1}^{\gamma_i, \lambda_i}(z) = \prod_{i=1}^r \left(\frac{z}{2}\right)^{\mu_i} \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma_i)_{n, \mathbf{k}_i} \left(\frac{z^2}{4}\right)^n}{\Gamma_{\mathbf{k}_i}(\lambda_i n + \mu_i) (n!)^2} \\ = \prod_{i=1}^r \left(\frac{z}{2}\right)^{\mu_i} W_{\mathbf{k}_i, \lambda_i, \mu_i}^{\gamma_i} \left(\frac{-z^2}{2}\right)$$

($\gamma_i, \lambda_i, \mu_i \in \mathbb{C}$ with $\min\{\Re(\lambda_i), \Re(\mu_i)\} > 0; \mathbf{k}_i \in \mathbb{R}^+; i = 1, \dots, r$).

The Fox-Wright function ${}_p\Psi_q$ is defined as follows (see, e.g., [15, p. 21]):

$$(1.12) \quad {}_p\Psi_q \left[\begin{matrix} (\alpha_1, A_1), \dots, (\alpha_p, A_p); \\ (\beta_1, B_1), \dots, (\beta_q, B_q); \end{matrix} z \right] = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p \Gamma(\alpha_j + A_j k) z^k}{\prod_{j=1}^q \Gamma(\beta_j + B_j k) k!},$$

where the coefficients $A_1, \dots, A_p \in \mathbb{R}^+$ and $B_1, \dots, B_q \in \mathbb{R}^+$ satisfying

$$(1.13) \quad 1 + \sum_{j=1}^q B_j - \sum_{j=1}^p A_j \geq 0.$$

A special case of (1.12) is

$$(1.14) \quad {}_p\Psi_q \left[\begin{matrix} (\alpha_1, 1), \dots, (\alpha_p, 1); \\ (\beta_1, 1), \dots, (\beta_q, 1); \end{matrix} z \right] = \frac{\prod_{j=1}^p \Gamma(\alpha_j)}{\prod_{j=1}^q \Gamma(\beta_j)} {}_pF_q \left[\begin{matrix} \alpha_1, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} z \right],$$

where ${}_pF_q$ is the generalized hypergeometric function (see, e.g., [14, Section 1.5]).

Very recently, Chand et al. [3] have used the following generalized \mathbf{k} -Mittag-Leffler function (see also [1])

$$(1.15) \quad E_{\mathbf{k}, \alpha, \beta}^{\gamma, q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq, \mathbf{k}} z^n}{\Gamma_{\mathbf{k}}(n\alpha + \beta) n!}$$

($\mathbf{k}, q \in \mathbb{R}^+, \min\{\Re(\alpha), \Re(\beta), \Re(\gamma)\} > 0$)

to present some Saigo hypergeometric fractional integral formulas for a finite product of the generalized \mathbf{k} -Mittag-Leffler functions (1.15) and gave image formulas of several integral transforms including Beta transform, Laplace transform, and Whittaker transform. In this sequel, we aim to establish certain Saigo hypergeometric fractional integral formulas for a finite product of the generalized \mathbf{k} -Bessel functions (1.8), which are also used to present image formulas of several integral transforms such as Beta transform, Laplace transform, and Whittaker transform.

Here we note the following points: The generalized \mathbf{k} -Mittag-Leffler function (1.15) cannot be reducible to give the generalized \mathbf{k} -Bessel function (1.8) and

vice versa; At best, the generalized \mathbf{k} -Bessel function (1.8) is seen to reduce to

$$\omega_{\mathbf{k},0,-1,-4}^{\gamma,\lambda}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,\mathbf{k}}}{\Gamma_{\mathbf{k}}(\lambda n + \mu)} \frac{z^{2n}}{(n!)^2}.$$

2. Fractional integration

Here, we establish certain fractional integral formulas for the generalized \mathbf{k} -Bessel function. For the basic concepts and detailed results about fractional calculus, among many other things, we refer the reader, for example, to Kiryakova [7], Miller and Ross [8], and Srivastava et al. [16].

For our purpose, we recall the following pair of Saigo hypergeometric fractional integral operators. For $x \in \mathbb{R}^+$, $\sigma, \vartheta, \lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$, we have

$$(2.1) \quad \begin{aligned} & \left(I_{0,x}^{\lambda,\sigma,\vartheta} f(t) \right) (x) \\ &= \frac{x^{-\lambda-\sigma}}{\Gamma(\lambda)} \int_0^x (x-t)^{\lambda-1} {}_2F_1 \left(\lambda + \sigma, -\vartheta; \lambda; 1 - \frac{t}{x} \right) f(t) dt \end{aligned}$$

and

$$(2.2) \quad \begin{aligned} & \left(J_{x,\infty}^{\lambda,\sigma,\vartheta} f(t) \right) (x) \\ &= \frac{1}{\Gamma(\lambda)} \int_x^{\infty} (t-x)^{\lambda-1} t^{-\lambda-\sigma} {}_2F_1 \left(\lambda + \sigma, -\vartheta; \lambda; 1 - \frac{x}{t} \right) f(t) dt, \end{aligned}$$

where ${}_2F_1(\cdot)$ is the Gauss hypergeometric function (see, e.g., [14, Section 1.5]).

The operators $I_{0,x}^{\lambda,\sigma,\vartheta}(\cdot)$ and $J_{x,\infty}^{\lambda,\sigma,\vartheta}(\cdot)$ contain the familiar Riemann-Liouville fractional integral operators $R_{0,x}^{\lambda}(\cdot)$ and $W_{x,\infty}^{\lambda}(\cdot)$, as their respective special cases, by means of the following relationships:

$$(2.3) \quad \left(R_{0,x}^{\lambda} f(t) \right) (x) = \left(I_{0,x}^{\lambda,-\lambda,\vartheta} f(t) \right) (x) = \frac{1}{\Gamma(\lambda)} \int_0^x (x-t)^{\lambda-1} f(t) dt$$

and

$$(2.4) \quad \left(W_{x,\infty}^{\lambda} f(t) \right) (x) = \left(J_{x,\infty}^{\lambda,-\lambda,\vartheta} f(t) \right) (x) = \frac{1}{\Gamma(\lambda)} \int_x^{\infty} (t-x)^{\lambda-1} f(t) dt.$$

It is also noted that the operators (2.1) and (2.2) unify the Erdélyi-Kober fractional integral operators as follows:

$$(2.5) \quad \left(E_{0,x}^{\lambda,\vartheta} f(t) \right) (x) = \left(I_{0,x}^{\lambda,0,\vartheta} f(t) \right) (x) = \frac{x^{-\lambda-\vartheta}}{\Gamma(\lambda)} \int_0^x (x-t)^{\lambda-1} t^{\vartheta} f(t) dt$$

and

$$(2.6) \quad \left(K_{x,\infty}^{\lambda,\vartheta} f(t) \right) (x) = \left(J_{x,\infty}^{\lambda,0,\vartheta} f(t) \right) (x) = \frac{x^{\vartheta}}{\Gamma(\lambda)} \int_x^{\infty} (t-x)^{\lambda-1} t^{-\lambda-\vartheta} f(t) dt.$$

We need to recall the following formulas in Lemmas 1 and 2 (see [6]).

Lemma 1. Let $\lambda, \rho, \sigma, \vartheta \in \mathbb{C}$ such that $\Re(\lambda) > 0$ and $\Re(\rho) > \max[0, \Re(\sigma - \vartheta)]$. Then

$$(2.7) \quad \left(I_{0,x}^{\lambda, \sigma, \vartheta} t^{\rho-1} \right) (x) = \frac{\Gamma(\rho)\Gamma(\rho + \vartheta - \sigma)}{\Gamma(\rho - \sigma)\Gamma(\rho + \lambda + \vartheta)} x^{\rho - \sigma - 1}.$$

Lemma 2. Let $\lambda, \rho, \sigma, \vartheta \in \mathbb{C}$ such that $\Re(\lambda) > 0$ and $\Re(\rho) < 1 + \min[\Re(\sigma), \Re(\vartheta)]$. Then

$$(2.8) \quad \left(J_{x,\infty}^{\lambda, \sigma, \vartheta} t^{\rho-1} \right) (x) = \frac{\Gamma(\sigma - \rho + 1)\Gamma(\vartheta - \rho + 1)}{\Gamma(1 - \rho)\Gamma(\lambda + \sigma + \vartheta - \rho + 1)} x^{\rho - \sigma - 1}.$$

Now we are ready to present our main results asserted by Theorems 3 and 4.

Theorem 3. Let $x, \mathbf{k}_i, q_i \in \mathbb{R}^+$ ($i = 1, \dots, r$). Also let $\lambda, \sigma, \vartheta, \rho, \gamma_i, \mu_i, b_i, c_i \in \mathbb{C}$ such that $\Re(\lambda) > 0, \Re(\rho) > \max\{0, \Re(\sigma - \vartheta)\}$, and $\Re(\mu_i) > 0$ ($i = 1, \dots, r$). Then

$$(2.9) \quad \left(I_{0,x}^{\lambda, \sigma, \vartheta} t^{\rho-1} \prod_{i=1}^r \omega_{\mathbf{k}_i, \mu_i, b_i, c_i}^{\gamma_i, q_i}(t) \right) (x) \\ = x^{\rho - \sigma - 1} \prod_{i=1}^r \frac{\mathbf{k}_i^{1 - \frac{\mu_i}{\mathbf{k}_i} - \frac{b_i + 1}{2\mathbf{k}_i}}}{\Gamma\left(\frac{\gamma_i}{\mathbf{k}_i}\right)} \left(\frac{x}{2}\right)^{\mu_i} \\ \times {}_{r+2}\Psi_{r+3} \left[\begin{matrix} (\gamma_1/\mathbf{k}_1, 1), \dots, (\gamma_r/\mathbf{k}_r, 1), \\ (\mu_1/\mathbf{k}_1 + (b_1 + 1)/\mathbf{k}_1, q_1/\mathbf{k}_1), \dots, (\mu_r/\mathbf{k}_r + (b_r + 1)/\mathbf{k}_r, q_r/\mathbf{k}_r), \\ (\mathbf{u} + \rho, 2r), (\mathbf{u} + \rho + \vartheta - \sigma, 2r); \\ (\mathbf{u} + \rho - \sigma, 2r), (\mathbf{u} + \rho + \lambda + \vartheta, 2r), (1, 1); \\ \frac{1}{4}(-c_1)\mathbf{k}_1^{(1 - q_1/\mathbf{k}_1)} \dots (-c_r)\mathbf{k}_r^{(1 - q_r/\mathbf{k}_r)} x^{2r} \end{matrix} \right],$$

where

$$(2.10) \quad \mathbf{u} := \sum_{i=1}^r \mu_i \quad (r \in \mathbb{N}).$$

Proof. Let \mathcal{I} be the left-hand side of (2.9). Using (1.9) and changing the order of integration and summation, which is verified under the conditions given in the theorem, we have

$$(2.11) \quad \mathcal{I} = \prod_{i=1}^r \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n c_i^n (\gamma_i)_{n, \mathbf{k}_i}}{\Gamma_{\mathbf{k}_i}(q_i n + \mu_i + \frac{b_i + 1}{2})} \frac{\left(\frac{1}{2}\right)^{\mu_i + 2n}}{(n!)^2} \right\} \\ \times \left(I_{0,x}^{\lambda, \sigma, \vartheta} t^{\mu_1 + \dots + \mu_r + 2nr + \rho - 1} \right) (x),$$

which, upon applying (2.7), yields

$$\mathcal{I} = \prod_{i=1}^r \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n c_i^n (\gamma_i)_{n, \mathbf{k}_i}}{\Gamma_{\mathbf{k}_i}(q_i n + \mu_i + \frac{b_i + 1}{2})} \frac{\left(\frac{1}{2}\right)^{\mu_i + 2n}}{(n!)^2} \right\}$$

$$(2.12) \quad \times \frac{\Gamma(\mu_1 + \cdots + \mu_r + 2nr + \rho)\Gamma(\mu_1 + \cdots + \mu_r + 2nr + \rho + \vartheta - \sigma)}{\Gamma(\mu_1 + \cdots + \mu_r + 2nr + \rho - \sigma)\Gamma(\mu_1 + \cdots + \mu_r + 2nr + \rho + \lambda + \vartheta)} \\ \times x^{\mu_1 + \cdots + \mu_r + 2nr + \rho - \sigma - 1}.$$

We simplify (2.12) to give

$$(2.13) \quad \mathcal{J} = x^{\rho - \sigma - 1} \prod_{i=1}^r \frac{k_i^{1 - \frac{\mu_i}{k_i} - \frac{b_i + 1}{2k_i}}}{\Gamma\left(\frac{\gamma_i}{k_i}\right)} \left(\frac{x}{2}\right)^{\mu_i} \left\{ \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{\gamma_i}{k_i} + n\right)}{\Gamma\left(\frac{\mu_i}{k_i} + \frac{b_i + 1}{k_i} + \frac{q_i n}{k_i}\right)} \right. \\ \times \frac{\Gamma(\mu_1 + \cdots + \mu_r + \rho + 2nr)\Gamma(\mu_1 + \cdots + \mu_r + \rho + \vartheta - \sigma + 2nr)}{\Gamma(\mu_1 + \cdots + \mu_r + \rho - \sigma + 2nr)\Gamma(\mu_1 + \cdots + \mu_r + \rho + \lambda + \vartheta + 2nr)\Gamma(n+1)} \\ \left. \times \frac{1}{n!} \left(\frac{x^{2r}(-c_i)k_i^{(1-q_i/k_i)}}{4}\right)^n \right\},$$

which, with the help of (1.12), is expressed in terms of ${}_{r+2}\Psi_{r+3}$ function to yield the right-hand side of (2.9). \square

Theorem 4. Let $x, \mathbf{k}_i, q_i \in \mathbb{R}^+$ ($i = 1, \dots, r$). Also let $\lambda, \sigma, \vartheta, \rho, \gamma_i, \mu_i, b_i, c_i \in \mathbb{C}$ such that $\Re(\lambda) > 0$, $\Re(\rho) < 1 + \min\{\Re(\sigma), \Re(\vartheta)\}$, and $\Re(\mu_i) > 0$ ($i = 1, \dots, r$). Then

$$(2.14) \quad \left(J_{x, \infty}^{\lambda, \sigma, \vartheta} t^{\rho-1} \prod_{i=1}^r \omega_{\mathbf{k}_i, \mu_i, b_i, c_i}^{\gamma_i, q_i}(1/t) \right) (x) \\ = x^{\rho - \sigma - 1} \prod_{i=1}^r \frac{k_i^{1 - \frac{\mu_i}{k_i} - \frac{b_i + 1}{2k_i}}}{\Gamma\left(\frac{\gamma_i}{k_i}\right)} \left(\frac{1}{2x}\right)^{\mu_i} \\ \times {}_{r+2}\Psi_{r+3} \left[\begin{matrix} (\gamma_1/k_1, 1), \dots, (\gamma_r/k_r, 1), \\ (\mu_1/k_1 + (b_1 + 1)/k_1, q_1/k_1), \dots, (\mu_r/k_r + (b_r + 1)/k_r, q_r/k_r), \\ (1 + \sigma + \mathbf{u} - \rho, 2r), (1 + \vartheta + \mathbf{u} - \rho, 2r); \\ (1 + \mathbf{u} - \rho, 2r), (1 + \lambda + \sigma + \vartheta + \mathbf{u} - \rho, 2r), (1, 1); \\ \frac{(-c_1)k_1^{(1-q_1/k_1)} \dots (-c_r)k_r^{(1-q_r/k_r)}}{4x^{2r}} \end{matrix} \right],$$

where \mathbf{u} is given as in (2.10).

Proof. The proof is parallel to that of Theorem 3. We omit the details. \square

The results given in (2.9) and (2.14), being very general, can yield a large number of special cases by assigning some suitable values to the involved parameters. We demonstrate only two formulas as in Corollaries 1 and 2.

Setting $r = 1$ and $\mathbf{k}_i = \mathbf{k}$, $\mu_i = \mu$, $\gamma_i = \gamma$, $q_i = q$, $b_i = b$, and $c_i = c$ in (2.9) and (2.14), we obtain the following two formulas in Corollaries 1 and 2.

Corollary 1. Let $x, \mathbf{k}, q \in \mathbb{R}^+$. Also let $\lambda, \sigma, \vartheta, \rho, \gamma, \mu, b, c \in \mathbb{C}$ such that $\min\{\Re(\lambda), \Re(\mu)\} > 0$ and $\Re(\rho) > \max\{0, \Re(\sigma - \vartheta)\}$. Then

$$(2.15) \quad \left(I_{0,x}^{\lambda, \sigma, \vartheta} t^{\rho-1} \omega_{\mathbf{k}, \mu, b, c}^{\gamma, q}(t) \right) (x)$$

$$\begin{aligned}
 &= x^{\rho-\sigma-1} \frac{\mathbf{k}^{1-\frac{\mu}{\mathbf{k}}-\frac{b+1}{2\mathbf{k}}}}{\Gamma\left(\frac{\gamma}{\mathbf{k}}\right)} \left(\frac{x}{2}\right)^\mu \\
 &\quad \times {}_3\Psi_4 \left[\begin{matrix} (\gamma/\mathbf{k}, 1), (\mu + \rho, 2), (\mu + \rho + \vartheta - \sigma, 2); \\ (\mu/\mathbf{k} + (b+1)/\mathbf{k}, q/\mathbf{k}), (\mu + \rho - \sigma, 2), (\mu + \rho + \lambda + \vartheta, 2), (1, 1); \\ \frac{(-c)\mathbf{k}^{(1-q/\mathbf{k})}x^2}{4} \end{matrix} \right].
 \end{aligned}$$

Corollary 2. Let $x, q, \mathbf{k} \in \mathbb{R}^+$. Also let $\lambda, \sigma, \vartheta, \rho, \gamma, \mu, b, c \in \mathbb{C}$ such that $\min\{\Re(\lambda), \Re(\mu)\} > 0$ and $\Re(\rho) < 1 + \min\{\Re(\sigma), \Re(\vartheta)\}$. Then

$$\begin{aligned}
 (2.16) \quad &\left(J_{x,\infty}^{\lambda,\sigma,\vartheta} t^{\rho-1} \omega_{\mathbf{k},\mu,b,c}^{\gamma,q}(1/t) \right) (x) \\
 &= x^{\rho-\sigma-1} \frac{\mathbf{k}^{1-\frac{\mu}{\mathbf{k}}-\frac{b+1}{2\mathbf{k}}}}{\Gamma\left(\frac{\gamma}{\mathbf{k}}\right)} \left(\frac{1}{2x}\right)^\mu \\
 &\quad \times {}_3\Psi_4 \left[\begin{matrix} (\gamma/\mathbf{k}, 1), (1 + \sigma + \mu - \rho, 2), (1 + \vartheta + \mu - \rho, 2); \\ (\mu/\mathbf{k} + (b+1)/\mathbf{k}, q/\mathbf{k}), (1 + \mu - \rho, 2), (1 + \lambda + \sigma + \vartheta + \mu - \rho, 2), (1, 1); \\ \frac{(-c)\mathbf{k}^{(1-q/\mathbf{k})}}{4x^2} \end{matrix} \right].
 \end{aligned}$$

3. Image formulas associated with integral transforms

Here, by using the results in previous section, we establish certain interesting image formulas associated with such integral transforms as Beta transform, Laplace transform, and Whittaker transform.

We recall the Beta transform of $f(z)$ defined by (see [12])

$$(3.1) \quad B\{f(z) : a, b\} = \int_0^1 z^{a-1} (1-z)^{b-1} f(z) dz.$$

Theorem 5. Let $x, \mathbf{k}_i, q_i \in \mathbb{R}^+$ ($i = 1, \dots, r$). Also let $\lambda, \sigma, \vartheta, \rho, \mu_i, \gamma_i, b_i, c_i \in \mathbb{C}$ such that $\Re(\lambda) > 0$, $\Re(\rho) > \max\{0, \Re(\sigma - \vartheta)\}$ and $\Re(\mu_i) > 0$ ($i = 1, \dots, r$). Further let $l, m \in \mathbb{C}$ with $\Re(m) > 0$ and $\Re\left(l + \sum_{i=1}^r \mu_i\right) > 0$. Then

$$\begin{aligned}
 (3.2) \quad &B \left\{ \left(I_{0,x}^{\lambda,\sigma,\vartheta} t^{\rho-1} \prod_{i=1}^r \omega_{\mathbf{k}_i,\mu_i,b_i,c_i}^{\gamma_i,q_i}(t) \right) (x) : l, m \right\} \\
 &= \Gamma(m) x^{\rho-\sigma-1} \prod_{i=1}^r \frac{\mathbf{k}_i^{1-\frac{\mu_i}{\mathbf{k}_i}-\frac{b_i+1}{2\mathbf{k}_i}}}{\Gamma\left(\frac{\gamma_i}{\mathbf{k}_i}\right)} \left(\frac{x}{2}\right)^{\mu_i} \\
 &\quad \times {}_{r+3}\Psi_{r+4} \left[\begin{matrix} (\gamma_1/\mathbf{k}_1, 1), \dots, (\gamma_r/\mathbf{k}_r, 1), \\ (\mu_1/\mathbf{k}_1 + (b_1+1)/\mathbf{k}_1, q_1/\mathbf{k}_1), \dots, (\mu_r/\mathbf{k}_r + (b_r+1)/\mathbf{k}_r, q_r/\mathbf{k}_r), \\ (\mathbf{u} + \rho, 2r), (\mathbf{u} + \rho + \vartheta - \sigma, 2r), (l + \mathbf{u}, 2r); \\ (\mathbf{u} + \rho - \sigma, 2r), (\mathbf{u} + \rho + \lambda + \vartheta, 2r), (l + m + \mathbf{u}, 2r), (1, 1); \\ \frac{(-c_1)\mathbf{k}_1^{(1-q_1/\mathbf{k}_1)} \dots (-c_r)\mathbf{k}_r^{(1-q_r/\mathbf{k}_r)} x^{2r}}{4} \end{matrix} \right],
 \end{aligned}$$

where \mathbf{u} is given as in (2.10).

Proof. Let \mathcal{B} be the left-hand side of (3.2). Using the definition of Beta transform, we have

$$(3.3) \quad \mathcal{B} = \int_0^1 z^{l-1}(1-z)^{m-1} \left(I_{0,x}^{\lambda,\sigma,\vartheta} t^{\rho-1} \prod_{i=1}^r \omega_{\mathbf{k}_i, \mu_i, b_i, c_i}^{\gamma_i, q_i}(tz) \right) (x) dz,$$

which, using (1.9) and changing the order of integration and summation, which is valid under the conditions of Theorem 3, yields

$$(3.4) \quad \begin{aligned} \mathcal{B} &= \prod_{i=1}^r \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n c_i^n (\gamma_i)_{n, \mathbf{k}_i}}{\Gamma_{\mathbf{k}_i}(q_i n + \mu_i + \frac{b_i+1}{2})} \frac{(\frac{1}{2})^{\mu_i+2n}}{(n!)^2} \right\} \\ &\times \left(I_{0,x}^{\lambda,\sigma,\vartheta} t^{\mu_1+\dots+\mu_r+2nr+\rho-1} \right) (x) \int_0^1 z^{l+\mu_1+\dots+\mu_r+2nr-1} (1-z)^{m-1} dz. \end{aligned}$$

Applying (2.7) to (3.4), after a little simplification, we obtain

$$(3.5) \quad \begin{aligned} \mathcal{B} &= x^{\rho-\sigma-1} \prod_{i=1}^r \frac{\mathbf{k}_i^{1-\frac{\mu_i}{\mathbf{k}_i}-\frac{b_i+1}{2\mathbf{k}_i}}}{\Gamma\left(\frac{\gamma_i}{\mathbf{k}_i}\right)} \left(\frac{x}{2}\right)^{\mu_i} \left\{ \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{\gamma_i}{\mathbf{k}_i} + n\right)}{\Gamma\left(\frac{\mu_i}{\mathbf{k}_i} + \frac{b_i+1}{\mathbf{k}_i} + \frac{q_i n}{\mathbf{k}_i}\right)} \right. \\ &\times \frac{\Gamma(\mu_1+\dots+\mu_r+\rho+2nr)\Gamma(\mu_1+\dots+\mu_r+\rho+\vartheta-\sigma+2nr)}{\Gamma(\mu_1+\dots+\mu_r+\rho-\sigma+2nr)\Gamma(\mu_1+\dots+\mu_r+\rho+\lambda+\vartheta+2nr)\Gamma(n+1)} \\ &\left. \times \frac{1}{n!} \left(\frac{x^{2r}(-c_i)\mathbf{k}_i^{(1-q_i/\mathbf{k}_i)}}{4} \right)^n \right\} \int_0^1 z^{l+\mu_1+\dots+\mu_r+2nr-1} (1-z)^{m-1} dz. \end{aligned}$$

Applying the beta function $B(\alpha, \beta)$ defined by (see, e.g., [14, Section 1.1])

$$(3.6) \quad B(\alpha, \beta) = \begin{cases} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt & (\Re(\alpha) > 0; \Re(\beta) > 0) \\ \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-), \end{cases}$$

to (3.5), we get

$$(3.7) \quad \begin{aligned} \mathcal{B} &= x^{\rho-\sigma-1} \prod_{i=1}^r \frac{\mathbf{k}_i^{1-\frac{\mu_i}{\mathbf{k}_i}-\frac{b_i+1}{2\mathbf{k}_i}}}{\Gamma\left(\frac{\gamma_i}{\mathbf{k}_i}\right)} \left(\frac{x}{2}\right)^{\mu_i} \left\{ \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{\gamma_i}{\mathbf{k}_i} + n\right)}{\Gamma\left(\frac{\mu_i}{\mathbf{k}_i} + \frac{b_i+1}{\mathbf{k}_i} + \frac{q_i n}{\mathbf{k}_i}\right)} \right. \\ &\times \frac{\Gamma(\mu_1+\dots+\mu_r+\rho+2nr)\Gamma(\mu_1+\dots+\mu_r+\rho+\vartheta-\sigma+2nr)}{\Gamma(\mu_1+\dots+\mu_r+\rho-\sigma+2nr)\Gamma(\mu_1+\dots+\mu_r+\rho+\lambda+\vartheta+2nr)\Gamma(n+1)} \\ &\left. \times \frac{1}{n!} \left(\frac{x^{2r}(-c_i)\mathbf{k}_i^{(1-q_i/\mathbf{k}_i)}}{4} \right)^n \right\} \frac{\Gamma(l+\mu_1+\dots+\mu_r+2nr)\Gamma(m)}{\Gamma(l+m+\mu_1+\dots+\mu_r+2nr)}, \end{aligned}$$

which, upon expressing in terms of the Fox-Wright function (1.12), yields the right-hand side of (3.2). \square

Theorem 6. Let $x, \mathbf{k}_i, q_i \in \mathbb{R}^+$ ($i = 1, \dots, r$). Also let $\lambda, \sigma, \vartheta, \rho, \mu_i, \gamma_i, b_i, c_i \in \mathbb{C}$ such that $\Re(\lambda) > 0$, $\Re(\rho) < 1 + \min\{\Re(\sigma), \Re(\vartheta)\}$ and $\Re(\mu_i) > 0$ ($i = 1, \dots, r$). Further let $l, m \in \mathbb{C}$ with $\Re(m) > 0$ and $\Re(l + \sum_{i=1}^r \mu_i) > 0$. Then

$$\begin{aligned}
 (3.8) \quad & B \left\{ \left(J_{x, \infty}^{\lambda, \sigma, \vartheta} t^{\rho-1} \prod_{i=1}^r \omega_{\mathbf{k}_i, \mu_i, b_i, c_i}^{\gamma_i, q_i}(z/t) \right) (x) : l, m \right\} \\
 &= \Gamma(m) x^{\rho-\sigma-1} \prod_{i=1}^r \frac{\mathbf{k}_i^{1-\frac{\mu_i}{\mathbf{k}_i} - \frac{b_i+1}{2\mathbf{k}_i}}}{\Gamma\left(\frac{\gamma_i}{\mathbf{k}_i}\right)} \left(\frac{1}{2x}\right)^{\mu_i} \\
 &\quad \times {}_{r+3}\Psi_{r+4} \left[\begin{matrix} (\gamma_1/\mathbf{k}_1, 1), \dots, (\gamma_r/\mathbf{k}_r, 1), \\ (\mu_1/\mathbf{k}_1 + (b_1 + 1)/\mathbf{k}_1, q_1/\mathbf{k}_1), \dots, (\mu_r/\mathbf{k}_r + (b_r + 1)/\mathbf{k}_r, q_r/\mathbf{k}_r), \\ (1 + \sigma + \mathbf{u} - \rho, 2r), (1 + \vartheta + \mathbf{u} - \rho, 2r), (l + \mathbf{u}, 2r); \\ (1 + \mathbf{u} - \rho, 2r), (1 + \lambda + \sigma + \vartheta + \mathbf{u} - \rho, 2r), (l + m + \mathbf{u}, 2r), (1, 1); \\ \frac{(-c_1)_{\mathbf{k}_1}^{(1-q_1/\mathbf{k}_1)} \dots (-c_r)_{\mathbf{k}_r}^{(1-q_r/\mathbf{k}_r)}}{4x^{2r}} \end{matrix} \right],
 \end{aligned}$$

where \mathbf{u} is given as in (2.10).

Proof. A similar argument as in the proof of Theorem 5 will establish this theorem. We omit the details. \square

We recall the Laplace transform of $f(z)$ (see, e.g., [12, 13]).

$$(3.9) \quad L\{f(z)\} = \int_0^\infty e^{-sz} f(z) dz.$$

Theorem 7. Let $x, \mathbf{k}_i, q_i \in \mathbb{R}^+$ ($i = 1, \dots, r$). Also let $\lambda, \sigma, \vartheta, \rho, \mu_i, \gamma_i, b_i, c_i \in \mathbb{C}$ such that $\Re(\lambda) > 0$, $\Re(\rho) > \max\{0, \Re(\sigma - \vartheta)\}$, and $\Re(\mu_i) > 0$ ($i = 1, \dots, r$). Further let $s, l \in \mathbb{C}$ with $\Re(s) > 0$ and $\Re(l + \sum_{i=1}^r \mu_i) > 0$. Then

$$\begin{aligned}
 (3.10) \quad & L \left\{ z^{l-1} \left(I_{0,x}^{\lambda, \sigma, \vartheta} t^{\rho-1} \prod_{i=1}^r \omega_{\mathbf{k}_i, \mu_i, b_i, c_i}^{\gamma_i, q_i}(tz) \right) (x) \right\} \\
 &= \frac{x^{\rho-\sigma-1}}{s^l} \prod_{i=1}^r \frac{\mathbf{k}_i^{1-\frac{\mu_i}{\mathbf{k}_i} - \frac{b_i+1}{2\mathbf{k}_i}}}{\Gamma\left(\frac{\gamma_i}{\mathbf{k}_i}\right)} \left(\frac{x}{2s}\right)^{\mu_i} \\
 &\quad \times {}_{r+3}\Psi_{r+3} \left[\begin{matrix} (\gamma_1/\mathbf{k}_1, 1), \dots, (\gamma_r/\mathbf{k}_r, 1), \\ (\mu_1/\mathbf{k}_1 + (b_1 + 1)/\mathbf{k}_1, q_1/\mathbf{k}_1), \dots, (\mu_r/\mathbf{k}_r + (b_r + 1)/\mathbf{k}_r, q_r/\mathbf{k}_r), \\ (\mathbf{u} + \rho, 2r), (\mathbf{u} + \rho + \vartheta - \sigma, 2r), (l + \mathbf{u}, 2r); \\ (\mathbf{u} + \rho - \sigma, 2r), (\mathbf{u} + \rho + \lambda + \vartheta, 2r), (1, 1); \\ \frac{(-c_1)_{\mathbf{k}_1}^{(1-q_1/\mathbf{k}_1)} \dots (-c_r)_{\mathbf{k}_r}^{(1-q_r/\mathbf{k}_r)}}{4} \left(\frac{x}{s}\right)^{2r} \end{matrix} \right],
 \end{aligned}$$

where \mathbf{u} is given as in (2.10).

Proof. Let \mathcal{L} be the left-hand side of (3.10). Applying the Laplace transform (3.9), using (1.9), and changing the order of integration and summation, which is valid under the conditions of Theorem 7, we have

$$\begin{aligned}
 (3.11) \quad \mathcal{L} &= x^{\rho-\sigma-1} \prod_{i=1}^r \frac{k_i^{1-\frac{\mu_i}{k_i}-\frac{b_i+1}{2k_i}}}{\Gamma\left(\frac{\gamma_i}{k_i}\right)} \left(\frac{x}{2}\right)^{\mu_i} \left\{ \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{\gamma_i}{k_i} + n\right)}{\Gamma\left(\frac{\mu_i}{k_i} + \frac{b_i+1}{k_i} + \frac{q_i n}{k_i}\right)} \right. \\
 &\quad \times \frac{\Gamma(\mu_1+\dots+\mu_r+\rho+2nr)\Gamma(\mu_1+\dots+\mu_r+\rho+\vartheta-\sigma+2nr)}{\Gamma(\mu_1+\dots+\mu_r+\rho-\sigma+2nr)\Gamma(\mu_1+\dots+\mu_r+\rho+\lambda+\vartheta+2nr)\Gamma(n+1)} \\
 &\quad \left. \times \frac{1}{n!} \left(\frac{x^{2r}(-c_i)k_i^{(1-q_i/k_i)}}{4}\right)^n \right\} \int_0^\infty e^{-sz} z^{l+\mu_1+\dots+\mu_r+2nr-1} dz \\
 &= x^{\rho-\sigma-1} \prod_{i=1}^r \frac{k_i^{1-\frac{\mu_i}{k_i}-\frac{b_i+1}{2k_i}}}{\Gamma\left(\frac{\gamma_i}{k_i}\right)} \left(\frac{x}{2}\right)^{\mu_i} \left\{ \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{\gamma_i}{k_i} + n\right)}{\Gamma\left(\frac{\mu_i}{k_i} + \frac{b_i+1}{k_i} + \frac{q_i n}{k_i}\right)} \right. \\
 &\quad \times \frac{\Gamma(\mu_1+\dots+\mu_r+\rho+2nr)\Gamma(\mu_1+\dots+\mu_r+\rho+\vartheta-\sigma+2nr)}{\Gamma(\mu_1+\dots+\mu_r+\rho-\sigma+2nr)\Gamma(\mu_1+\dots+\mu_r+\rho+\lambda+\vartheta+2nr)\Gamma(n+1)} \\
 &\quad \left. \times \frac{1}{n!} \left(\frac{x^{2r}(-c_i)k_i^{(1-q_i/k_i)}}{4}\right)^n \right\} \frac{\Gamma(l+\mu_1+\dots+\mu_r+2nr)}{s^{l+\mu_1+\dots+\mu_r+2nr}},
 \end{aligned}$$

which, upon using (1.12), leads to the right-hand side of (3.10). □

Theorem 8. Let $x, k_i, q_i \in \mathbb{R}^+$ ($i = 1, \dots, r$). Also let $\lambda, \sigma, \vartheta, \rho, \mu_i, \gamma_i, b_i, c_i \in \mathbb{C}$ such that $\Re(\lambda) > 0$, $\Re(\rho) < 1 + \min[\Re(\sigma), \Re(\vartheta)]$, and $\Re(\mu_i) > 0$ ($i = 1, \dots, r$). Further let $s, l \in \mathbb{C}$ with $\Re(s) > 0$ and $\Re(l + \sum_{i=1}^r \mu_i) > 0$. Then

$$\begin{aligned}
 (3.12) \quad &L \left\{ z^{l-1} \left(J_{x,\infty}^{\lambda,\sigma,\vartheta} t^{\rho-1} \prod_{i=1}^r \omega_{k_i,\mu_i,b_i,c_i}^{\gamma_i,q_i}(z/t) \right) (x) \right\} \\
 &= \frac{x^{\rho-\sigma-1}}{s^l} \prod_{i=1}^r \frac{k_i^{1-\frac{\mu_i}{k_i}-\frac{b_i+1}{2k_i}}}{\Gamma\left(\frac{\gamma_i}{k_i}\right)} \left(\frac{1}{2xs}\right)^{\mu_i} \\
 &\quad \times {}_{r+3}\Psi_{r+3} \left[\begin{matrix} (\gamma_1/k_1, 1), \dots, (\gamma_r/k_r, 1), \\ (\mu_1/k_1 + (b_1+1)/k_1, q_1/k_1), \dots, (\mu_r/k_r + (b_r+1)/k_r, q_r/k_r), \\ (1+\sigma+\mathbf{u}-\rho, 2r), (1+\vartheta+\mathbf{u}-\rho, 2r), (l+\mathbf{u}, 2r); \\ (1+\mathbf{u}-\rho, 2r), (1+\lambda+\sigma+\vartheta+\mathbf{u}-\rho, 2r), (1, 1); \\ (-c_1)k_1^{(1-q_1/k_1)} \dots (-c_r)k_r^{(1-q_r/k_r)} \\ 4(xs)^{2r} \end{matrix} \right],
 \end{aligned}$$

where \mathbf{u} is given as in (2.10).

Proof. The proof would run parallel to that of Theorem 7. We omit the details. □

We recall the following formula (see, e.g., [5, p. 816, Entry 11])

$$(3.13) \quad \int_0^\infty z^{\nu-1} e^{-\frac{1}{2}z} \mathcal{W}_{\eta,\theta}(z) dz = \frac{\Gamma(\frac{1}{2}+\theta+\nu)\Gamma(\frac{1}{2}-\theta+\nu)}{\Gamma(1-\eta+\nu)} \quad (\Re(\nu \pm \theta) > -\frac{1}{2}),$$

where $\mathcal{W}_{\eta,\theta}(\cdot)$ denote the Whittaker functions (see, e.g., [5, Sections 9.22-9.23]).

Theorem 9. *Let $x, \mathbf{k}_i, q_i \in \mathbb{R}^+$ ($i = 1, \dots, r$). Also, let $\lambda, \sigma, \vartheta, \rho, \mu_i, \gamma_i, b_i, c_i \in \mathbb{C}$ such that $\Re(\lambda) > 0$, $\Re(\rho) > \max\{0, \Re(\sigma - \vartheta)\}$, and $\Re(\mu_i) > 0$ ($i = 1, \dots, r$). Further, let $\xi, \eta, \tau, \omega \in \mathbb{C}$ with $\Re(\eta) > 0$ and $\Re(\xi + \omega) > -\frac{1}{2}$. Then*

$$(3.14) \quad \int_0^\infty z^{\xi-1} e^{-\eta z/2} W_{\tau,\omega}(\eta z) \left\{ \left(I_{0,x}^{\lambda,\sigma,\vartheta} t^{\rho-1} \prod_{i=1}^r \omega_{\mathbf{k}_i, \mu_i, b_i, c_i}^{\gamma_i, q_i}(tz) \right) (x) \right\} dz$$

$$= \frac{x^{\rho-\sigma-1}}{\eta^\xi} \prod_{i=1}^r \frac{\mathbf{k}_i^{1-\frac{\mu_i}{\mathbf{k}_i}-\frac{b_i+1}{2\mathbf{k}_i}}}{\Gamma\left(\frac{\gamma_i}{\mathbf{k}_i}\right)} \left(\frac{x}{2\eta}\right)^{\mu_i}$$

$$\times {}_{r+4}\Psi_{r+4} \left[\begin{matrix} (\gamma_1/\mathbf{k}_1, 1), \dots, (\gamma_r/\mathbf{k}_r, 1), \\ (\mu_1/\mathbf{k}_1 + (b_1+1)/\mathbf{k}_1, q_1/\mathbf{k}_1), \dots, (\mu_r/\mathbf{k}_r + (b_r+1)/\mathbf{k}_r, q_r/\mathbf{k}_r), \\ (\mathbf{u} + \rho, 2r), (\mathbf{u} + \rho + \vartheta - \sigma, 2r), (1/2 + \omega + \xi + \mathbf{u}, 2r), (1/2 - \omega + \xi + \mathbf{u}, 2r); \\ (\mathbf{u} + \rho - \sigma, 2r), (\mathbf{u} + \rho + \lambda + \vartheta, 2r), (1 - \tau + \xi + \mathbf{u}, 2r), (1, 1); \\ \frac{(-c_1)_{\mathbf{k}_1}^{(1-q_1/\mathbf{k}_1)} \dots (-c_r)_{\mathbf{k}_r}^{(1-q_r/\mathbf{k}_r)}}{4} \left(\frac{x}{\eta}\right)^{2r} \end{matrix} \right],$$

where \mathbf{u} is given as in (2.10).

Proof. Let \mathcal{W} be the left-hand side of the result (3.14). By using (1.9) and changing the order of integration and summation, which is valid under the conditions of Theorem 9, we have

$$\mathcal{W} = x^{\rho-\sigma-1} \prod_{i=1}^r \frac{\mathbf{k}_i^{1-\frac{\mu_i}{\mathbf{k}_i}-\frac{b_i+1}{2\mathbf{k}_i}}}{\Gamma\left(\frac{\gamma_i}{\mathbf{k}_i}\right)} \left(\frac{x}{2}\right)^{\mu_i} \left\{ \sum_{n=0}^\infty \frac{\Gamma\left(\frac{\gamma_i}{\mathbf{k}_i} + n\right)}{\Gamma\left(\frac{\mu_i}{\mathbf{k}_i} + \frac{b_i+1}{\mathbf{k}_i} + \frac{q_i n}{\mathbf{k}_i}\right)} \right.$$

$$\times \frac{\Gamma(\mu_1 + \dots + \mu_r + \rho + 2nr) \Gamma(\mu_1 + \dots + \mu_r + \rho + \vartheta - \sigma + 2nr)}{\Gamma(\mu_1 + \dots + \mu_r + \rho - \sigma + 2nr) \Gamma(\mu_1 + \dots + \mu_r + \rho + \lambda + \vartheta + 2nr) \Gamma(n+1)}$$

$$\left. \times \frac{1}{n!} \left(\frac{x^{2r} (-c_i)_{\mathbf{k}_i}^{(1-q_i/\mathbf{k}_i)}}{4} \right)^n \right\} \int_0^\infty z^{\xi+\mu_1+\dots+\mu_r+2nr-1} e^{-\eta z/2} W_{\tau,\omega}(\eta z) dz,$$

which, upon replacing ηz by ς , yields

$$(3.15) \quad \mathcal{W} = x^{\rho-\sigma-1} \prod_{i=1}^r \frac{\mathbf{k}_i^{1-\frac{\mu_i}{\mathbf{k}_i}-\frac{b_i+1}{2\mathbf{k}_i}}}{\Gamma\left(\frac{\gamma_i}{\mathbf{k}_i}\right)} \left(\frac{x}{2}\right)^{\mu_i} \left\{ \sum_{n=0}^\infty \frac{\Gamma\left(\frac{\gamma_i}{\mathbf{k}_i} + n\right)}{\Gamma\left(\frac{\mu_i}{\mathbf{k}_i} + \frac{b_i+1}{\mathbf{k}_i} + \frac{q_i n}{\mathbf{k}_i}\right)} \frac{\Gamma(\mu_1 + \dots + \mu_r + \rho + 2nr)}{\Gamma(\mu_1 + \dots + \mu_r + \rho - \sigma + 2nr)} \right.$$

$$\left. \times \frac{\Gamma(\mu_1 + \dots + \mu_r + \rho + \vartheta - \sigma + 2nr)}{\Gamma(\mu_1 + \dots + \mu_r + \rho + \lambda + \vartheta + 2nr) \Gamma(n+1)} \frac{1}{n!} \left(\frac{x^{2r} (-c_i)_{\mathbf{k}_i}^{(1-q_i/\mathbf{k}_i)}}{4} \right)^n \right\}$$

$$\times \frac{1}{\eta^{\xi+\mu_1+\dots+\mu_r+2nr}} \int_0^\infty \zeta^{\xi+\mu_1+\dots+\mu_r+2nr-1} e^{-\zeta/2} W_{\tau,\omega}(\zeta) d\zeta.$$

Upon applying (3.13) to the involved integral in (3.15), we obtain

$$\begin{aligned} \mathscr{W} &= \frac{x^{\rho-\sigma-1}}{\eta^\xi} \prod_{i=1}^r \frac{\mathbf{k}_i^{1-\frac{\mu_i}{\mathbf{k}_i}-\frac{b_i+1}{2\mathbf{k}_i}}}{\Gamma\left(\frac{\gamma_i}{\mathbf{k}_i}\right)} \left(\frac{x}{2\eta}\right)^{\mu_i} \left\{ \sum_{n=0}^\infty \frac{\Gamma\left(\frac{\gamma_i}{\mathbf{k}_i} + n\right)}{\Gamma\left(\frac{\mu_i}{\mathbf{k}_i} + \frac{b_i+1}{\mathbf{k}_i} + \frac{q_i n}{\mathbf{k}_i}\right)} \right. \\ &\times \frac{\Gamma(\mu_1+\dots+\mu_r+\rho+2nr)\Gamma(\mu_1+\dots+\mu_r+\rho+\vartheta-\sigma+2nr)}{\Gamma(\mu_1+\dots+\mu_r+\rho-\sigma+2nr)\Gamma(\mu_1+\dots+\mu_r+\rho+\lambda+\vartheta+2nr)\Gamma(n+1)} \\ &\times \frac{\Gamma(1/2+\omega+\xi+\mu_1+\dots+\mu_r+2nr)\Gamma(1/2-\omega+\xi+\mu_1+\dots+\mu_r+2nr)}{\Gamma(1-\tau+\xi+\mu_1+\dots+\mu_r+2nr)} \\ &\left. \times \frac{1}{n!} \left(\frac{x^{2r}(-c_i)\mathbf{k}_i^{(1-q_i/\mathbf{k}_i)}\right)^n \right\}, \end{aligned}$$

which, upon expressing in terms of (1.12), leads to the right-hand side of (3.14). □

Theorem 10. Let $x, \mathbf{k}_i, q_i \in \mathbb{R}^+$ ($i = 1, \dots, r$). Also, let $\lambda, \sigma, \vartheta, \rho, \mu_i, \gamma_i, b_i, c_i \in \mathbb{C}$ such that $\Re(\lambda) > 0, \Re(\rho) < 1 + \min[\Re(\sigma), \Re(\vartheta)]$, and $\Re(\mu_i) > 0$ ($i = 1, \dots, r$). Further, let $\xi, \eta, \tau, \omega, \in \mathbb{C}$ with $\Re(\eta) > 0$ and $\Re(\xi + \omega) > -\frac{1}{2}$. Then

$$\begin{aligned} (3.16) \quad &\int_0^\infty z^{\xi-1} e^{-\eta z/2} W_{\tau,\omega}(\eta z) \left\{ \left(J_{x,\infty}^{\lambda,\sigma,\vartheta} t^{\rho-1} \prod_{i=1}^r \omega_{\mathbf{k}_i,\mu_i,b_i,c_i}^{\gamma_i,q_i}(z/t) \right) (x) \right\} dz \\ &= \frac{x^{\rho-\sigma-1}}{\eta^\xi} \prod_{i=1}^r \frac{\mathbf{k}_i^{1-\frac{\mu_i}{\mathbf{k}_i}-\frac{b_i+1}{2\mathbf{k}_i}}}{\Gamma\left(\frac{\gamma_i}{\mathbf{k}_i}\right)} \left(\frac{1}{2x\eta}\right)^{\mu_i} \\ &\times {}_{r+4}\Psi_{r+4} \left[\begin{matrix} (\gamma_1/\mathbf{k}_1, 1), \dots, (\gamma_r/\mathbf{k}_r, 1), \\ (\mu_1/\mathbf{k}_1 + (b_1 + 1)/\mathbf{k}_1, q_1/\mathbf{k}_1), \dots, (\mu_r/\mathbf{k}_r + (b_r + 1)/\mathbf{k}_r, q_r/\mathbf{k}_r), \\ (1 + \sigma + \mathbf{u} - \rho, 2r), (1 + \vartheta + \mathbf{u} - \rho, 2r), (1/2 + \omega + \xi + \mathbf{u}, 2r), (1/2 - \omega + \xi + \mathbf{u}, 2r); \\ (1 + \mathbf{u} - \rho, 2r), (1 + \lambda + \sigma + \vartheta + \mathbf{u} - \rho, 2r), (1 - \tau + \xi + \mathbf{u}, 2r), (1, 1); \\ \frac{(-c_1)\mathbf{k}_1^{(1-q_1/\mathbf{k}_1)} \dots (-c_r)\mathbf{k}_r^{(1-q_r/\mathbf{k}_r)}}{4(x\eta)^{2r}} \end{matrix} \right], \end{aligned}$$

where \mathbf{u} is given as in (2.10).

Proof. Here, using (2.2), a similar argument as in the proof of Theorem 9 will establish (3.16). So its detailed account is omitted. □

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