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HARMONIC MAPS AND BIHARMONIC MAPS ON PRINCIPAL BUNDLES AND WARPED PRODUCTS

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Abstract. In this paper, we study harmonic maps and biharmonic maps on the principal G-bundle in Kobayashi and Nomizu [22] and also the warped product $P = M \times_f F$ for a $C^\infty(M)$ function f on M studied by Bishop and O'Neill [4], and Ejiri [11].

1. Introduction

Variational problems play central roles in geometry; Harmonic map is one of important variational problems which is a critical point of the energy functional $E(\varphi) = \frac{1}{2}\int_M |d\varphi|^2\, v_g$ for smooth maps φ of (M, g) into (N, h) (see [7,16,17,20, 32,44,45,47]). The Euler-Lagrange equations are given by the vanishing of the tension filed $\tau(\varphi)$. In 1983, J. Eells and L. Lemaire [10] extended the notion of harmonic map to biharmonic map, which are, by definition, critical points of the bienergy functional

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2\, v_g. \tag{1.1}$$

After G. Y. Jiang [19] studied the first and second variation formulas of E_2, extensive studies in this area have been done (for instance, see [6, 14, 15, 18, 28, 31, 40, 41], etc.). Notice that harmonic maps are always biharmonic by definition. B. Y. Chen raised ([8]) so called B. Y. Chen's conjecture and later, R. Caddeo, S. Montaldo, P. Piu and C. Oniciuc raised ([2,3,6]) the generalized B. Y. Chen's conjecture.

B. Y. Chen's conjecture. *Every biharmonic submanifold of the Euclidean space $\mathbb{R}^n$ must be harmonic* (*minimal*).

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The generalized B. Y. Chen's conjecture. *Every biharmonic submanifold of a Riemannian manifold of non-positive curvature must be harmonic* (*minimal*).

For the generalized Chen's conjecture, Ou and Tang gave ([39,40]) a counter example in a Riemannian manifold of negative curvature. For the Chen's conjecture, affirmative answers were known for the case of surfaces in the three dimensional Euclidean space ([8]), and the case of hypersurfaces of the four dimensional Euclidean space ([9, 13]). K. Akutagawa and S. Maeta gave ([1]) showed a supporting evidence to the Chen's conjecture: *Any complete regular biharmonic submanifold of the Euclidean space* $\mathbb{R}^n$ *is harmonic* (*minimal*). The affirmative answers to the generalized B. Y. Chen's conjecture were shown ([34–36]) under the L^2-condition and completeness of (M, g).

In this paper, we first treat with a principal G-bundle over a Riemannian manifold, and show the following two theorems:

Theorem 3.2. *Let* $\pi : (P, g) \to (M, h)$ *be a principal* G*-bundle over a Riemannian manifold* (M, h) *with non-positive Ricci curvature. Assume* P *is compact so that* M *is also compact. If the projection* π *is biharmonic, then it is harmonic.*

Theorem 4.1. *Let* $\pi : (P, g) \to (M, h)$ *be a principal* G*-bundle over a Riemannian manifold with non-positive Ricci curvature. Assume that* (P, g) *is a non-compact complete Riemannian manifold, and the projection* π *has both finite energy* $E(\pi) < \infty$ *and finite bienergy* $E_2(\pi) < \infty$*. If* π *is biharmonic, then it is harmonic.*

We give two comments on the above theorems: For the generalized B. Y. Chen's conjecture, non-positivity of the sectional curvature of the ambient space of biharmonic submanifolds is necessary. However, it should be emphasized that for the principal G-bundles, we need not the assumption of non-positivity of the sectional curvature. We only assume *non-positivity of the Ricci curvature* of the domain manifolds in the proofs of Theorems 3.2 and 4.1. Second, finiteness of the energy and bienergy is necessary in Theorem 4.1. Otherwise, one have the counter examples due to Loubeau and Ou (cf. Sect. Four, Examples 1 and 2 [29]).

Next, we consider the warped products. For two Riemannian manifolds (M, h), (F, k) and a C^∞ function f on M, $f \in C^\infty(M)$, the warping function on M, let us consider the warped product (P, g) where $\pi : P = M \times F \ni (x, y) \mapsto x \in M$ and $g = \pi^* h + f^2\, k$. Let us consider the following two problems:

Problem 1. When $\pi : (P, g) \to (M, h)$ is harmonic?

Problem 2. In the case $(M, h) = (\mathbb{R}, dt^2)$, a line, can one choose $f \in C^\infty(\mathbb{R})$ such that $\pi : (P, g) \to (M, h)$ is biharmonic but not harmonic?

In this paper, we answer these two problems as follows.

Theorem 5.2. *Let $\pi : (P, g) \to (M, h)$ be the warped product with a warping function $f \in C^\infty(M)$. Then, the tension field $\tau(\pi)$ is given by*

$$\tau(\pi) = \ell \, \frac{\operatorname{grad} f}{f} = \ell \, \frac{\nabla f}{f}, \tag{1.2}$$

where $\ell = \dim F$. Therefore, π is harmonic if and only if f is constant.

Theorem 6.2. *For the warped product $\pi : (P, g) \to (M, h)$, the bitension field $\tau_2(\pi)$ is given by*

$$\tau_2(\pi) = \overline{\Delta}(\tau(\pi)) - \rho^h(\tau(\pi)) - \ell \, \overline{\nabla}_{\frac{\nabla f}{f}} \tau(\pi), \tag{1.3}$$

where $\overline{\Delta}$ is the rough Laplacian and $\overline{\nabla}$ is the induced connection from the Levi-Civita connection ∇^h of (M, h). Therefore, π is biharmonic if and only if

$$\overline{\Delta}(\tau(\pi)) - \rho^h(\tau(\pi)) - \ell \, \overline{\nabla}_{\frac{\nabla f}{f}} \tau(\pi) = 0. \tag{1.4}$$

Here, ρ^h is the Ricci transform $\rho^h(u) := \sum_{i=1}^m R^h(u, e_i')e_i'$, $u \in T_xM$ for an locally defined orthonormal field $\{e_i'\}_{i=1}^m$ on (M, h).

Theorem 7.1. (1) *In the case $(M, h) = (\mathbb{R}, dt^2)$, a line, the warped product $\pi : (P, g) \to (\mathbb{R}, dt^2)$ is biharmonic if and only if $f \in C^\infty(\mathbb{R})$ satisfies the following ordinary equation*:

$$f''' \, f^2 + (\ell - 3) \, f'' \, f' \, f + (-\ell + 2) \, f'^3 = 0. \tag{1.5}$$

(2) *All the solutions f of* (1.5) *are given by*

$$f(t) = c \exp \left(\int_{t_0}^t a \, \tanh \left[\frac{\ell}{2} \, a \, r + b \right] \, dr \right), \tag{1.6}$$

where a, b, $c > 0$ are arbitrary constants.

(3) *In the case $(M, h) = (\mathbb{R}, dt^2)$, a line, let $f(t)$ be C^∞ function defined by* (1.6) *with $a \neq 0$ and $c > 0$. Then, the warped product $\pi : (P, g) \to (M, h)$ is biharmonic but not harmonic.*

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2. Preliminaries

2.1. Harmonic maps and biharmonic maps

We first prepare the materials for the first and second variational formulas for the bienergy functional and biharmonic maps. Let us recall the definition of a harmonic map $\varphi : (M, g) \to (N, h)$, of a compact Riemannian manifold (M, g) into another Riemannian manifold (N, h), which is an extremal of the *energy functional* defined by

$$E(\varphi) = \int_M e(\varphi) \, v_g,$$

where $e(\varphi) := \frac{1}{2}|d\varphi|^2$ is called the energy density of φ. That is, for any variation $\{\varphi_t\}$ of φ with $\varphi_0 = \varphi$,

$$\frac{d}{dt}\bigg|_{t=0} E(\varphi_t) = -\int_M h(\tau(\varphi), V) v_g = 0, \tag{2.1}$$

where $V \in \Gamma(\varphi^{-1}TN)$ is a variation vector field along φ which is given by $V(x) = \frac{d}{dt}\big|_{t=0}\varphi_t(x) \in T_{\varphi(x)}N$, $(x \in M)$, and the *tension field* is given by $\tau(\varphi) = \sum_{i=1}^m B(\varphi)(e_i, e_i) \in \Gamma(\varphi^{-1}TN)$, where $\{e_i\}_{i=1}^m$ is a locally defined orthonormal frame field on (M, g), and $B(\varphi)$ is the second fundamental form of φ defined by

$$\begin{aligned} B(\varphi)(X, Y) &= (\widetilde{\nabla} d\varphi)(X, Y) \\ &= (\widetilde{\nabla}_X d\varphi)(Y) \\ &= \overline{\nabla}_X(d\varphi(Y)) - d\varphi(\nabla_X Y) \end{aligned} \tag{2.2}$$

for all vector fields $X, Y \in \mathfrak{X}(M)$. Here, ∇, and ∇^h, are Levi-Civita connections on TM, TN of (M, g), (N, h), respectively, and $\overline{\nabla}$, and $\widetilde{\nabla}$ are the induced ones on $\varphi^{-1}TN$, and $T^*M \otimes \varphi^{-1}TN$, respectively. By (2.1), φ is *harmonic* if and only if $\tau(\varphi) = 0$.

The second variation formula is given as follows. Assume that φ is harmonic. Then,

$$\frac{d^2}{dt^2}\bigg|_{t=0} E(\varphi_t) = \int_M h(J(V), V) v_g, \tag{2.3}$$

where J is an elliptic differential operator, called the *Jacobi operator* acting on $\Gamma(\varphi^{-1}TN)$ given by

$$J(V) = \overline{\Delta} V - \mathcal{R}(V), \tag{2.4}$$

where $\overline{\Delta} V = \overline{\nabla}^* \overline{\nabla} V = -\sum_{i=1}^m \{\overline{\nabla}_{e_i}\overline{\nabla}_{e_i} V - \overline{\nabla}_{\nabla_{e_i} e_i} V\}$ is the *rough Laplacian* and $\mathcal{R}$ is a linear operator on $\Gamma(\varphi^{-1}TN)$ given by

$$\mathcal{R}(V) = \sum_{i=1}^m R^N(V, d\varphi(e_i)) d\varphi(e_i),$$

and R^N is the curvature tensor of (N, h) given by $R^h(U, V) = \nabla^h{}_U \nabla^h{}_V - \nabla^h{}_V \nabla^h{}_U - \nabla^h{}_{[U,V]}$ for U, $V \in \mathfrak{X}(N)$ (see [22, 27, 28]).

J. Eells and L. Lemaire [10] proposed polyharmonic (k-harmonic) maps and Jiang [19] studied the first and second variation formulas of biharmonic maps. Let us consider the *bienergy functional* defined by

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 v_g, \tag{2.5}$$

where $|V|^2 = h(V, V)$, $V \in \Gamma(\varphi^{-1}TN)$.

The first variation formula of the bienergy functional is given by

$$\left.\frac{d}{dt}\right|_{t=0} E_2(\varphi_t) = -\int_M h(\tau_2(\varphi), V) v_g. \tag{2.6}$$

Here,

$$\tau_2(\varphi) := J(\tau(\varphi)) = \overline{\Delta}(\tau(\varphi)) - \mathcal{R}(\tau(\varphi)), \tag{2.7}$$

which is called the *bitension field* of φ, and J is given in (2.4).

A smooth map φ of (M, g) into (N, h) is said to be *biharmonic* if $\tau_2(\varphi) = 0$. By definition, every harmonic map is biharmonic. We say, for an immersion $\varphi : (M, g) \to (N, h)$ to be *proper biharmonic* if it is biharmonic but not harmonic (minimal) (see [12, 23–27, 30, 33, 37, 38, 42, 43, 48, 49]).

2.2. The principal G-bundle

Recall several notions on principal G-bundles ([5, 21, 22]). A manifold $P = P(M, G)$ is a principal fiber bundle over M with a compact Lie group G, where $p = \dim P$, $m = \dim M$, and $k = \dim G$. By definition, a Lie group G acts on P by right hand side denoted by $(G, P) \ni (a, u) \mapsto u \cdot a \in P$, and, for each point $u \in P$, the tangent space T_uP admits a subspace $G_u := \{A^*{}_u \mid A \in \mathfrak{g}\}$, the vertical subspace at u, and each $A \in \mathfrak{g}$ defines the fundamental vector field $A^* \in \mathfrak{X}(P)$ by

$$A^*{}_u := \left.\frac{d}{dt}\right|_{t=0} u \exp(t\, A) \in T_uP.$$

A Riemannian metric g on P is called *adapted* if it is invariant under all the right action of G, i.e., $R_a{}^*g = g$ for all $a \in G$. An adapted Riemannian metric on P always exists because for every Riemannian metric g' on P, define a new metric g on P by

$$g_u(X_u, Y_u) = \int_G g'(R_{a*}X_u, R_{a*}Y_u)\, d\mu(a),$$

where $d\mu(a)$ is a bi-invariant Haar measure on G. Then, $R_a{}^*g = g$ for all $a \in G$. Each tangent space T_uP has the orthogonal direct decomposition of the tangent space T_uP,

$$T_uP = G_u \oplus H_u, \tag{a}$$

where the subspace G_u of P_u satisfies

$$G_u = \{A^*{}_u \mid A \in \mathfrak{g}\}, \tag{b}$$

and the subspace H_u of P_u satisfies that

$$H_{u\cdot a} = R_{a*}H_u, \quad a \in G,\ u \in P, \tag{C}$$

where the subspace H_u of P_u is called *horizontal subspace* at $u \in P$ with respect to g.

In the following, we fix a locally defined orthonormal frame field $\{e_i\}_{i=1}^p$ corresponding (a), (b) in such a way that $\{e_i\}_{i=1}^m$ is a locally defined orthonormal basis of the horizontal subspace H_u $(u \in P)$, and $\{e_i = A^*{}_{m+i}\}_{i=1}^k$ is a locally defined orthonormal basis of the vertical subspace G_u $(u \in P)$ for an orthonormal basis $\{A_{m+i}\}_{i=1}^k$ of the Lie algebra $\mathfrak{g}$ of a Lie group G with respect to the $\mathrm{Ad}(G)$ invariant inner product $\langle \cdot, \cdot \rangle$.

For each decomposition (a), one can define a $\mathfrak{g}$-valued 1-form ω on P by

$$\omega(X_u) = A, \qquad X_u = {X_u}^{\mathrm{V}} + {X_u}^{\mathrm{H}},$$

where

$${X_u}^{\mathrm{V}} \in G_u, \qquad {X_u}^{\mathrm{H}} \in H_u, \qquad {X_u}^{\mathrm{V}} = {A_u}^*$$

for $u \in P$ and a unique $A \in \mathfrak{g}$. This 1-form ω on P is called a *connection form* of P.

Then, there exist a unique Riemannian metric h on M and an $\mathrm{Ad}(G)$-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ such that

$$g(X_u, Y_u) = h(\pi_* X_u, \pi_* Y_u) + \ \langle \omega(X_u), \omega(Y_u) \rangle, \quad X_u,\ Y_u \in T_u P,\ u \in P,$$

namely,

$$g = \pi^* h + \langle\, \omega(\cdot), \omega(\cdot) \rangle.$$

We call this Riemannian metric g on P, an *adapted* Riemannian metric on P.

Then, let us recall the following definitions for our question:

Definition 2.1. (1) The projection $\pi : (P, g) \to (M, h)$ is to be *harmonic* if the tension field vanishes, $\tau(\pi) = 0$, and

(2) the projection $\pi : (P, g) \to (M, h)$ is to be *biharmonic* if, the bitension field vanishes, $\tau_2(\pi) = J(\tau(\pi)) = 0$.

Here, J is the Jacobi operator for the projection π given by

$$J(V) := \overline{\Delta} V - \mathcal{R}(V), \qquad V \in \Gamma(\pi^{-1} TM),$$

where

$$\begin{aligned}
\overline{\Delta} V &:= -\sum_{i=1}^{p} \left\{ \overline{\nabla}_{e_i} (\overline{\nabla}_{e_i} V) - \overline{\nabla}_{\nabla_{e_i} e_i} V \right\} \\
&= -\sum_{i=1}^{m} \left\{ \overline{\nabla}_{e_i} (\overline{\nabla}_{e_i} V) - \overline{\nabla}_{\nabla_{e_i} e_i} V \right\} \\
&\qquad - \sum_{i=1}^{k} \left\{ \overline{\nabla}_{A^*_{m+i}} (\overline{\nabla}_{A^*_{m+i}} V) - \overline{\nabla}_{\nabla_{A^*_{m+i}} A^*_{m+i}} V \right\}
\end{aligned}$$

for $V \in \Gamma(\pi^{-1} TM)$, i.e., $V(x) \in T_{\pi(x)} M$ $(x \in P)$. Here, $\{e_i\}_{i=1}^p$ is a local orthonormal frame field on (P, g) which is given by that: $\{e_i\}_{i=1}^m$ is an orthonormal horizontal field on the principal G-bundle $\pi : (P, g) \to (M, h)$ and $\{e_{m+i,\,u} = A^*_{m+i,\,u}\}_{i=1}^k$ $(u \in P)$ is an orthonormal frame field on the vertical space $G_u = \{A^*_u \,|\, A \in \mathfrak{g}\}$ $(u \in P)$ corresponding to an orthonormal basis $\{A_{m+i}\}_{i=1}^k$ of $(\mathfrak{g}, \langle , \rangle)$.

2.3. The warped products

On the product manifold $P = M \times F$ for two Riemannian manifolds (M, h) and (F, k), and a C^∞ function, $f \in C^\infty(M)$ on M, let us consider the Riemannian metric

$$g = \pi^* h + f^2\, k, \tag{2.8}$$

where the projection $\pi : P = M \times F \ni (x, y) \mapsto x \in M$. The Riemannian submersion $\pi : (P, g) \to (M, h)$ is called the *warped product* of (M, h) and (F, k) with a *warping function* $f \in C^\infty(M)$ ([4, 11, 46]). In this section, we prepare several notions in order to calculate the tension field and bitension field.

We first construct a locally defined orthonormal frame field $\{e_i\}_{i=1}^{m+\ell}$ on (P, g) where $m = \dim M$ and $\ell = \dim F$ as follows: For $i = 1, \dots, m$,

$$e_{i\,(x,y)} := (e'_{i\,x}, 0_y) \in T_{(x,y)}P = T_xM \times T_yF,$$

and for $i = m+1, \dots, p$,

$$e_{i\,(x,y)} := \frac{1}{f(x)}\,(0_x, e''_{i,\,y}) \in T_{(x,y)}P = T_xM \times T_yF,$$

where $p = m + \ell$.

Recall the O'Neill's formulas on the warped product (cf. [4, 11]). For a C^∞ vector field $X \in \mathfrak{X}(M)$ on M, $X^* \in \mathfrak{X}(P)$, the *horizontal lift* of X which satisfies for $z \in P$,

$$X^*{}_z \in \mathcal{H}_z, \quad \text{and} \qquad \pi_*\,(X^*{}_z) = X_{\pi(z)}, \tag{2.9}$$

where recall the *vertical subspace* $\mathcal{V}_z$ and *horizontal subspace* $\mathcal{H}_z$ of the tangent space T_zP:

$$\mathcal{V}_z = \mathrm{Ker}(\pi_{*\,(x,y)}), \tag{2.10}$$

$$T_zP = \mathcal{V}_z \oplus \mathcal{H}_z, \qquad g(\mathcal{V}_z, \mathcal{H}_z) = 0, \tag{2.11}$$

where $\pi_{*\,(x,y)} : T_{(x,y)}P \to T_xM$ is the differential of the projection $\pi : P \to M$ at $(x, y) \in P$.

Let $q : P = M \times F \ni (x, y) \mapsto y \in F$ be the projection of P onto F. For a vector field V on F, there exists a unique vector field $\widetilde{V}$ on P satisfying that $\widetilde{V} \in \mathcal{V}$ and $q_*(\widetilde{V}) = V$. We identify $V \in \mathfrak{X}(F)$ with $\widetilde{V} \in \mathcal{V}$ denoting by the same letter V in the following.

Lemma 2.1. *Let $X, Y \in \mathfrak{X}(M)$ be vector fields on M, and $V, W \in \mathfrak{X}(F)$, vector fields on F, and ∇^g, ∇^h, ∇^k, the Levi-Civita connections of (P, g), (M, h), and (F, k), respectively. Then,*

1. $\mathrm{grad}\,(f \circ \pi) = \mathrm{grad}\, f$.
2. $\pi_*\,(\nabla^g{}_{X^*} Y^*) = \nabla^h{}_X Y$, *where X^* and Y^* are the horizontal lifts of X and Y, respectively.*
3. $\nabla^g{}_{X^*} V = \nabla^g{}_V X^* = \frac{Xf}{f}\, V$.

(4) $\mathcal{H}\,(\nabla^g{}_V W) = -f\,k(V,W)\,G = -\frac{1}{f}\,g(V,W)\,G$, *where G is the gradient of f and $f \circ \pi$.*
(5) $\mathcal{V}\,(\nabla^g{}_V W) = \nabla^k{}_V W$, *where $\mathcal{H}A$, and $\mathcal{V}A$ are the horizontal part, and the vertical part of A, respectively.*

Lemma 2.2 (O'Neill's formulas)**.**
(1) $g_{(x,y)}(X^*{}_{(x,y)}, Y^*{}_{(x,y)}) = h_x(X_x, Y_x),\ x \in M.$
(2) $\pi_*\,([X^*, Y^*]) = [X, Y].$
(3) $\pi_*\,(\nabla^g{}_{X^*} Y^*) = \nabla^h{}_X Y.$

Lemma 2.3. *For a vector field $X \in \mathfrak{X}(M)$ whose $h(X,X)$ is constant, $\nabla^g{}_{X^*} X^*$ is the horizontal lift of $\nabla^h{}_X X$.*

Proof. By (3) of Lemma 2.2, we only have to see $\nabla^g{}_{X^*} X^*$ is a horizontal vector field. Due to Lemma 2.3(1), for every vertical vector field $X \in \mathfrak{X}(M)$, we have

$$\begin{aligned} 2g(\nabla^g{}_{X^*} X^*, V) &= X^*(g(X^*, V)) + X^*(g(V, X^*)) - V(g(X^*, X^*) \\ &\quad + g(V, [X^*, X^*]) + g(X^*, [V, X^*]) - g(X^*, [X^*, V]) \\ &= 2g(X^*, [X^*, V]) \\ &= 0. \end{aligned} \tag{2.12}$$

Here, the last equality of (2.12) follows as:

$$\begin{aligned} [X^*, V] &= \nabla^g{}_{X^*} V - \nabla^g{}_V X^* \\ &= \frac{Xf}{f}\,V - \frac{Xf}{f}\,V \\ &= 0 \end{aligned} \tag{2.13}$$

by using Lemma 2.1(3). □

Then, we can choose a locally defined orthonormal vector field

$$\{e_1, \dots, e_m, e_{m+1}, \dots, e_{m+\ell}\}$$

on (P, g) in such a way that $\{e_1, \dots, e_m\}$ are orthonormal vector fields which are horizontal lifts of the orthonormal vector fields $e'_1, \dots, e'_m$ on (M, h) and $e_{m+1} = \frac{1}{f}\,e''_{m+1}, \dots, e_{m+\ell} = \frac{1}{f} e''_{m+\ell}$. Then, by Lemma 2.3, $\nabla^g{}_{e_i} e_i$, $i = 1, \dots, m$, are the horizontal lifts of $\nabla^h{}_{e'_i} e'_i$.

For $i = m+1, \dots, m+\ell$, we have the following decomposition:

$$\nabla^g{}_{e_i} e_i = \frac{1}{f^2}\left\{-(e''_i f)\,e_i + \nabla^k_{e''_i} e''_i - f\,\nabla(f \circ \pi)\right\}. \tag{2.14}$$

We first note that $\nabla(f \circ \pi)$ is a horizontal vector field on P. Because,

$$g(\nabla(f \circ \pi), V) = Vf = 0$$

for every $V \in \mathfrak{X}(F)$. And the first two terms of (2.14) are vertical since $\nabla^k_{e''_i} e''_i$, $i = m+1, \dots, m+\ell$, are vertical.

To prove (2.14), for $i = m+1, \dots, m+\ell$, we have

$$\begin{aligned}
\nabla^g{}_{e_i} e_i &= \nabla^g{}_{\frac{1}{f}\, e_i''} \frac{1}{f}\, e_i'' \\
&= \frac{1}{f} \left\{ e_i'' \left(\frac{1}{f} \right) e_i'' + \frac{1}{f}\, \nabla^g{}_{e_i''} e_i'' \right\} \\
&= \frac{1}{f^2} \left\{ -\frac{e_i'' f}{f}\, e_i'' + \nabla^g{}_{e_i''} e_i'' \right\}. \qquad (2.15)
\end{aligned}$$

We decompose $\nabla^g{}_{e_i''} e_i''$ into the vertical and horizontal components:

$$\nabla^g{}_{e_i''} e_i'' = \mathcal{V}\left(\nabla^g{}_{e_i''} e_i'' \right) + \mathcal{H}\left(\nabla^g{}_{e_i''} e_i'' \right). \qquad (2.16)$$

Here, by Lemma 2.1(5), we have

$$\mathcal{V}\left(\nabla^g{}_{e_i''} e_i'' \right) = \nabla^k{}_{e_i''} e_i''. \qquad (2.17)$$

By Lemma 2.1(4) and $k(e_i'', e_j'') = \delta_{ij}$, we have

$$\begin{aligned}
\mathcal{H}\left(\nabla^g{}_{e_i''} e_i'' \right) &= -f\, k(e_i'', e_i'')\, G \\
&= -f\, G \\
&= -f\, \nabla f \\
&= -f\, \nabla (f \circ \pi) \qquad (2.18)
\end{aligned}$$

by Lemma 2.1(1). We obtain (2.14).

3. Proof of Theorem 3.2

If the principal G-bundle $\pi : (P, g) \to (M, h)$ is harmonic, then it is clearly biharmonic. Our main interest is to ask the reverse holds under what conditions:

Problem 3.1. If the projection π of a principal G-bundle $\pi : (P, g) \to (M, h)$ is biharmonic, is π harmonic or not.

In this paper, we show that this problem is affirmative when the Ricci curvature of the base manifold (M, h) is negative definite. Indeed, we show that:

Theorem 3.2. *Let $\pi : (P, g) \to (M, h)$ be a principal G-bundle over a Riemannian manifold (M, h) with non-positive Ricci curvature. Assume P is compact so that M is also compact. If the projection π is biharmonic, then it is harmonic.*

In this section, we give a proof of Theorem 3.2 in case of a compact Riemannian manifold (M, h) and the Ricci tensor of (M, h) is negative definite. We will give the proof of Theorem 4.1 in case of a non-compact complete Riemannian manifold (M, h) in Section 4.

Let us first consider a principal G-bundle $\pi : (P, g) \to (M, h)$ whose the total space P is compact. Assume that the projection $\pi : (P, g) \to (M, h)$ is

biharmonic, which is by definition, $J(\tau(\pi)) \equiv 0$, where $\tau(\pi)$ is the tension field of π which is defined by

$$\tau(\pi) := \sum_{i=1}^{p} \{\nabla^h_{e_i} \pi_* e_i - \pi_*(\nabla_{e_i} e_i)\}, \tag{3.1}$$

the Jacobi operator J is defined by

$$JV := \overline{\Delta} V - \mathcal{R}(V) \qquad (V \in \Gamma(\pi^{-1}TM)), \tag{3.2}$$

$\overline{\Delta}$ is the rough Laplacian defined by

$$\overline{\Delta} V := -\sum_{i=1}^{p} \{\overline{\nabla}_{e_i}(\overline{\nabla}_{e_i} V) - \overline{\nabla}_{\nabla_{e_i} e_i} V\}, \tag{3.3}$$

and

$$\mathcal{R}(V) := R^h(V, \pi_* e_i)\pi_* e_i, \tag{3.4}$$

where $\{e_i\}_{i=1}^p$ is a locally defined orthonormal frame field on (P, g).

The tangent space P_u $(u \in P)$ is canonically decomposed into the orthogonal direct sum of the vertical subspace $G_u = \{{A_u}^* \mid A \in \mathfrak{g}\}$ and the horizontal subspace H_u: $P_u = G_u \oplus H_u$. Then, we have

$$\begin{aligned}
\tau_2(\pi) &= \overline{\Delta}\tau(\pi) - \sum_{i=1}^{p} R^h(\tau(\pi), \pi_* e_i)\pi_* e_i \\
&= \overline{\Delta}\tau(\pi) - \sum_{i=1}^{m} R^h(\tau(\pi), \pi_* e_i)\pi_* e_i \\
&\qquad - \sum_{i=1}^{k} R^h(\tau(\pi), \pi_* A^*_{m+i})\pi_* A^*_{m+i} \\
&= \overline{\Delta}\tau(\pi) - \sum_{i=1}^{m} R^h(\tau(\pi), \pi_* e_i)\pi_* e_i,
\end{aligned}$$

where $p = \dim P$, $m = \dim M$, $k = \dim G$, respectively. Then, we obtain

$$\begin{aligned}
0 &= \int_M \langle J(\tau(\pi)), \tau(\pi)\rangle \, v_g \\
&= \int_M \langle \overline{\nabla}^* \, \overline{\nabla} \, \tau(\pi), \tau(\pi)\rangle \, v_g - \int_M \sum_{i=1}^{m} \langle R^h(\tau(\pi), \pi_* e_i)\pi_* e_i, \tau(\pi)\rangle \, v_g \\
&= \int_M \langle \overline{\nabla} \, \tau(\pi), \overline{\nabla} \, \tau(\pi)\rangle \, v_g - \int_M \sum_{i=1}^{m} \langle R^h(\tau(\pi), \pi_* e_i)\pi_* e_i, \tau(\pi)\rangle \, v_g.
\end{aligned}$$

Therefore, we obtain

$$\int_M \langle \overline{\nabla} \, \tau(\pi), \overline{\nabla} \, \tau(\pi)\rangle \, v_g = \int_M \sum_{i=1}^{m} \langle R^h(\tau(\pi), \pi_* e_i)\pi_* e_i, \tau(\pi)\rangle \, v_g$$

$$= \int_M \sum_{i=1}^m \langle R^h(\tau(\pi), e_i')e_i', \tau(\pi)\rangle \, v_g$$

$$= \int_M \mathrm{Ric}^h(\tau(\pi)) \, v_g, \tag{3.5}$$

where $\{e_i'\}_{i=1}^m$ is a locally defined orthonormal frame field on (M, h) satisfying $\pi_* e_i = e_i'$, and $\mathrm{Ric}(X)$ is the Ricci curvature of (M, h) along $X \in T_x M$. The left hand side of (3.5) is non-negative, and then, the both hand sides of (3.5) must vanish if the Ricci curvature of (M, h) is non-positive. Therefore, we obtain

$$\begin{cases} \overline{\nabla}_X \tau(\pi) = 0 \quad (\forall X \in \mathfrak{X}(P)), \text{ i.e., } \tau(\pi) \text{ is parallel, and} \\ \mathrm{Ric}^h(\tau(\pi)) = 0. \end{cases} \tag{3.6}$$

Let us define a 1-form $\alpha \in A^1(P)$ on P by $\alpha(X) = \langle d\pi(X), \tau(\pi)\rangle, \ X \in \mathfrak{X}(P)$. Then, we have

$$-\delta\alpha = \sum_{i=1}^p (\nabla_{e_i}\alpha)(e_i) = \langle \tau(\pi), \tau(\pi)\rangle + \langle d\pi, \overline{\nabla}\tau(\pi)\rangle. \tag{3.7}$$

Integrate the above (3.7) over P since P is compact without boundary. By (3.6), $\overline{\nabla}_X \tau(\pi) = 0, \ X \in \mathfrak{X}(P)$, we have

$$0 = -\int_P \delta\alpha \, v_g = \int_P \langle \tau(\pi), \tau(\pi)\rangle \, v_g \tag{3.8}$$

which implies that $\tau(\pi) = 0$, i.e., $\pi : (P, g) \to (M, h)$ is harmonic.

4. Proof of Theorem 4.1

In this section, we will show:

Theorem 4.1. *Let $\pi : (P, g) \to (M, h)$ be a principal G-bundle over a Riemannian manifold with non-positive Ricci curvature. Assume that (P, g) is a non-compact complete Riemannian manifold, and the projection π has both finite energy $E(\pi) < \infty$ and finite bienergy $E_2(\pi) < \infty$. If π is biharmonic, then it is harmonic.*

Here, we first recall the following examples:

Example 1 (cf. [29], p. 62)**.** The inversion in the unit sphere $\phi : \mathbb{R}^n \backslash \{o\} \ni \mathbf{x} \mapsto \frac{\mathbf{x}}{|\mathbf{x}|^2} \in \mathbb{R}^n$ is a biharmonic morphism if $n = 4$. It is not harmonic since $\tau(\phi) = -\frac{4\mathbf{x}}{|\mathbf{x}|^4}$.

Here, a C^∞ map $\phi : (M, g) \to (N, h)$ is called to be a *biharmonic morphism* if, for every biharmonic function $f : U \subset N \to \mathbb{R}$ with $\phi^{-1}(U) \neq \emptyset$, the composition $f \circ \phi : \phi^{-1}(U) \subset M \to \mathbb{R}$ is biharmonic.

Example 2 (cf. [29], p. 70)**.** Let (M^2, h) be a Riemannian surface, and let $\beta : M^2 \times \mathbb{R} \to \mathbb{R}^*$ and $\lambda : \mathbb{R} \to \mathbb{R}^*$ be two positive C^∞ functions. Consider the projection $\pi : (M^2 \times \mathbb{R}^*, g = \lambda^{-2}\, h + \beta^2\, dt^2) \ni (p, t) \mapsto p \in (M^2, h)$. Here, we take $\beta = c_2\, e^{\int f(x)\, dx}$, $f(x) = \frac{-c_1\,(1+e^{c_1 x})}{1-e^{c_1 x}}$ with c_1, $c_2 \in \mathbb{R}^*$, and $(M^2, h) = (\mathbb{R}^2, dx^2 + dy^2)$. Then,

$$\pi : (\mathbb{R}^2 \times \mathbb{R}^*, dx^2 + dy^2 + \beta^2(x)\, dt^2) \ni (x, y, t) \mapsto (x, y) \in (\mathbb{R}^2, dx^2 + dy^2)$$

gives a family of *proper biharmonic* (i.e., biharmonic but not harmonic) Riemannian submersions.

For a non-compact and complete Riemannian manifold (N, h) with non-positive Ricci curvature, we will give a proof of Theorem 4.1.

(*The first step*) We first take a cut off function η on (P, g) for a fixed point $p_0 \in P$ as follows:

$$\begin{cases} 0 \le \eta \le 1 & (\text{on } P), \\ \eta = 1 & (\text{on } B_r(p_0)), \\ \eta = 0 & (\text{outside } B_{2r}(p_0)), \\ |\nabla \eta| \le \dfrac{2}{r} & (\text{on } P), \end{cases} \tag{4.1}$$

where $B_r(p_0)$ is the ball in (P, g) of radius r around p_0.

Now assume that the projection $\pi : (P, g) \to (N, h)$ is biharmonic. Namely, we have, by definition,

$$\begin{aligned} 0 = J_2(\pi) &= J_\pi(\tau(\pi)) \\ &= \overline{\Delta}\, \tau(\pi) - \sum_{i=1}^{p} R^h(\tau(\pi), \pi_* e_i)\pi_* e_i, \end{aligned} \tag{4.2}$$

where $\{e_i\}_{i=1}^p$ is a local orthonormal frame field on (P, g) and $\overline{\Delta}$ is the rough Laplacian which is defined by

$$\overline{\Delta} V := \overline{\nabla}^*\, \overline{\nabla} V = -\sum_{i=1}^{p} \left\{ \overline{\nabla}_{e_i}(\overline{\nabla}_{e_i} V) - \overline{\nabla}_{\nabla_{e_i} e_i} V \right\} \tag{4.3}$$

for $V \in \Gamma(\pi^{-1} TM)$.

(*The second step*) By (4.2), we have

$$\begin{aligned} \int_P \langle \overline{\nabla}^*\, \overline{\nabla}\, \tau(\pi), \eta^2\, \tau(\pi) \rangle\, v_g &= \int_P \eta^2 \left\langle \sum_{i=1}^{p} R^h(\tau(\pi), \pi_* e_i)\pi_* e_i, \tau(\pi) \right\rangle v_g \\ &= \int_P \eta^2 \sum_{i=1}^{p} \left\langle R^h(\tau(\pi), \pi_* e_i)\pi_* e_i, \tau(\pi) \right\rangle v_g \\ &= \int_P \eta^2 \sum_{i=1}^{m} \left\langle R^h(\tau(\pi), e'_i)e'_i, \tau(\pi) \right\rangle v_g \end{aligned}$$

$$= \int_P \eta^2 \operatorname{Ric}^h(\tau(\pi))\, v_g, \tag{4.4}$$

where $\{e'_i\}_{i=1}^m$ is a local orthonormal frame field on (M,h), and $\operatorname{Ric}^h(u)$ $u \in T_yM$, $(y \in M)$ is the Ricci curvature of (M,h) which is non-positive by our assumption.

(*The third step*) Therefore, we obtain

$$\begin{aligned}
0 &\geq \int_P \langle \overline{\nabla}^* \overline{\nabla}\, \tau(\pi), \eta^2\, \tau(\pi)\rangle\, v_g \\
&= \int_P \langle \overline{\nabla}\, \tau(\pi), \overline{\nabla}(\eta^2\, \tau(\pi))\rangle\, v_g \\
&= \int_P \sum_{i=1}^p \langle \overline{\nabla}_{e_i}\, \tau(\pi), \overline{\nabla}_{e_i}(\eta^2\, \tau(\pi))\rangle\, v_g \\
&= \int_P \sum_{i=1}^p \left\{ \eta^2\, \langle \overline{\nabla}_{e_i} \tau(\pi), \overline{\nabla}_{e_i} \tau(\pi)\rangle + e_i(\eta^2)\, \langle \overline{\nabla}_{e_i} \tau(\pi), \tau(\pi)\rangle \right\}\, v_g \\
&= \int_P \eta^2 \sum_{i=1}^p |\overline{\nabla}_{e_i} \tau(\pi)|^2\, v_g + 2 \int_P \sum_{i=1}^p \langle \eta\, \overline{\nabla}_{e_i} \tau(\pi), e_i(\eta)\, \tau(\pi)\rangle\, v_g. \qquad (4.5)
\end{aligned}$$

Therefore, we obtain by (4.5).

(*The fourth step*) Then, we have

$$\begin{aligned}
\int_P \eta^2 \sum_{i=1}^p \left|\overline{\nabla}_{e_i} \tau(\pi)\right|^2\, v_g &\leq -2 \int_P \sum_{i=1}^p \langle \eta\, \overline{\nabla}_{e_i} \tau(\pi), e_i(\eta)\, \tau(\pi)\rangle\, v_g \\
&= -2 \int_P \sum_{i=1}^p \langle V_i, W_i\rangle\, v_g, \qquad (4.6)
\end{aligned}$$

where $V_i := \eta\, \overline{\nabla}_{e_i} \tau(\pi)$, and $W_i := e_i(\eta)\, \tau(\pi)$ $(i = 1, \ldots, p)$. Then, the right hand side of (4.6) is estimated by the Cauchy-Schwarz inequality,

$$\pm 2\, \langle V_i, W_i\rangle \leq \epsilon\, |V_i|^2 + \frac{1}{\epsilon}\, |W_i|^2 \tag{4.7}$$

since

$$0 \leq |\sqrt{\epsilon}\, V_i \pm \frac{1}{\sqrt{\epsilon}}\, W_i|^2 = \epsilon\, |V_i|^2 \pm\, 2\, \langle V_i, W_i\rangle + \frac{1}{\epsilon}\, |W_i|^2,$$

so that

$$\mp 2\, \langle V_i, W_i\rangle \leq \epsilon\, |V_i|^2 + \frac{1}{\epsilon}\, |W_i|^2.$$

Therefore, the right hand side of (4.6) is estimated as follows:

$$\begin{aligned}
\text{RHS of (4.6)} &:= -\int_P \sum_{i=1}^p \langle V_i, W_i\rangle\, v_g \\
&\leq \epsilon \int_P \sum_{i=1}^p |V_i|^2\, v_g + \frac{1}{\epsilon} \int_P \sum_{i=1}^p |W_i|^2\, v_g. \qquad (4.8)
\end{aligned}$$

(*The fifth step*) By putting $\epsilon = \frac{1}{2}$, we have

$$\int_P \eta^2 \sum_{i=1}^p |\overline{\nabla}_{e_i}\tau(\pi)|^2 v_g \leq \frac{1}{2}\int_P \sum_{i=1}^p \eta^2 |\overline{\nabla}_{e_i}\tau(\pi)|^2 v_g + 2\int_P \sum_{i=1}^p e_i(\eta)^2 |\tau(\pi)|^2 v_g. \tag{4.9}$$

Therefore, we obtain

$$\frac{1}{2}\int_P \eta^2 \sum_{i=1}^p |\overline{\nabla}_{e_i}\tau(\pi)|^2 v_g \leq 2\int_P |\nabla\eta|^2 |\tau(\pi)|^2 v_g. \tag{4.10}$$

Substituting (4.1) into (4.10), we obtain

$$\int_P \eta^2 \sum_{i=1}^p |\overline{\nabla}_{e_i}\tau(\pi)|^2 v_g \leq 4\int_P |\nabla\eta|^2 |\tau(\pi)|^2 v_g \leq \frac{16}{r^2}\int_P |\tau(\pi)|^2 v_g. \tag{4.11}$$

(*The sixth step*) Tending $r \to \infty$ by the completeness of (P, g) and $E_2(\pi) = \frac{1}{2}\int_P |\tau(\pi)|^2 v_g < \infty$, we obtain that

$$\int_P \sum_{i=1}^p |\overline{\nabla}_{e_i}\tau(\pi)|^2 v_g = 0, \tag{4.12}$$

which implies that

$$\overline{\nabla}_X \tau(\pi) = 0 \qquad (\forall\, X \in \mathfrak{X}(P)). \tag{4.13}$$

(*The seventh step*) Therefore, we obtain

$$|\tau(\pi)| \quad \text{is constant, say } c \tag{4.14}$$

because

$$X\,|\tau(\pi)|^2 = 2\,\langle \overline{\nabla}_X \tau(\pi), \tau(\pi)\rangle = 0 \qquad (\forall\ X \in \mathfrak{X}(M))$$

by (4.13).

(*The eighth step*) In the case that $\mathrm{Vol}(P, g) = \infty$ and $E_2(\pi) < \infty$, c must be zero. Because, if $c \neq 0$,

$$E_2(\pi) = \frac{1}{2}\int_P |\tau(\pi)|^2 v_g = \frac{c}{2}\,\mathrm{Vol}(P, g) = \infty$$

which is a contradiction.

Thus, if $\mathrm{Vol}(P, g) = \infty$, then $c = 0$, i.e., $\pi : (P, g) \to (M, h)$ is harmonic.

(*The ninth step*) In the case $E(\pi) < \infty$ and $E_2(\pi) < \infty$, let us define a 1-form $\alpha \in A^1(P)$ on P by

$$\alpha(X) := \langle d\pi(X), \tau(\pi)\rangle, \qquad (X \in \mathfrak{X}(P)). \tag{4.15}$$

Then, we obtain

$$\begin{aligned}\int_P |\alpha|\, v_g &= \int_P \left(\sum_{i=1}^p |\alpha(e_i)|^2\right)^{1/2} \leq \int_P |d\pi|\ |\tau(\pi)|\, v_g \\ &\leq \left(\int_P |d\pi|^2 v_g\right)^{1/2} \left(\int_P |\tau(\pi)|^2 v_g\right)^{1/2}\end{aligned}$$

$$= 2\sqrt{E(\pi)\,E_2(\pi)} < \infty. \tag{4.16}$$

For the function $\delta\alpha := -\sum_{i=1}^{p}(\nabla_{e_i}\alpha)(e_i) \in C^\infty(P)$, we have

$$\begin{aligned}
-\delta\alpha &= \sum_{i=1}^{p}(\nabla_{e_i}\alpha)(e_i) = \sum_{i=1}^{p}\left\{e_i(\alpha(e_i)) - \alpha(\nabla_{e_i}e_i)\right\}\\
&= \sum_{i=1}^{p}\left\{e_i\left\langle d\pi(e_i),\tau(\pi)\right\rangle - \left\langle d\pi(\nabla_{e_i}e_i),\tau(\pi)\right\rangle\right\}\\
&= \sum_{i=1}^{p}\left\{\left\langle\overline{\nabla}_{e_i}d\pi(e_i),\tau(\pi)\right\rangle + \left\langle d\pi(e_i),\overline{\nabla}_{e_i}\tau(\pi)\right\rangle - \left\langle d\pi(\nabla_{e_i}e_i,\tau(\pi)\right\rangle\right\}\\
&= \left\langle\sum_{i=1}^{p}\left\{\overline{\nabla}_{e_i}d\pi(e_i) - d\pi(\nabla_{e_i}e_i)\right\},\tau(\pi)\right\rangle + \sum_{i=1}^{p}\left\langle d\pi(e_i),\overline{\nabla}_{e_i}\tau(\pi)\right\rangle\\
&= \left\langle\tau(\pi),\tau(\pi)\right\rangle + \left\langle d\pi,\overline{\nabla}\tau(\pi)\right\rangle\\
&= |\tau(\pi)|^2 \qquad (4.17)
\end{aligned}$$

since $\overline{\nabla}\tau(\pi) = 0$. By (4.17), we obtain

$$\int_P |\delta\alpha|\,v_g = \int_P |\tau(\pi)|^2\,v_g = 2\,E_2(\pi) < \infty. \tag{4.18}$$

By (4.16), (4.18) and the completeness of (P,g), we can apply Gaffney's theorem which implies that

$$0 = \int_P (-\delta\,\alpha)\,v_g = \int_P |\tau(\pi)|^2\,v_g. \tag{4.19}$$

Thus, we obtain

$$\tau(\pi) = 0, \tag{4.20}$$

that is, $\pi : (P,g) \to (M,h)$ is harmonic. We obtain Theorem 4.1.

5. The tension fields of the warped products

In this section, we calculate the tension field $\tau(\pi)$. Let us recall the definition of the tension field:

Definition 5.1.

$$\begin{aligned}
\tau(\pi) &= \sum_{i=1}^{m+\ell}\left\{\overline{\nabla}_{e_i}\pi_*e_i - \pi_*\left(\nabla^g{}_{e_i}e_i\right)\right\}\\
&= \sum_{i=1}^{m+\ell}\left\{\nabla^h{}_{\pi_*e_i}\pi_*e_i - \pi_*\left(\nabla^g{}_{e_i}e_i\right)\right\}. \qquad (5.1)
\end{aligned}$$

Since $\nabla^g{}_{e_i}e_i$ are the horizontal lifts of $\nabla^h{}_{e'_i}e'_i$ for $i = 1, \dots, m$, and (2.14), we have

$$\begin{aligned}
\tau(\pi) &= \sum_{i=1}^{m} \left\{ \nabla^h{}_{\pi_* e_i} \pi_* e_i - \pi_* \left(\nabla^g{}_{e_i} e_i \right) \right\} \\
&\quad + \sum_{i=m+1}^{m+\ell} \left\{ \nabla^h{}_{\pi_* e_i} \pi_* e_i - \pi_* \left(\nabla^g{}_{e_i} e_i \right) \right\} \\
&= \sum_{i=1}^{m} \left\{ \nabla^h{}_{e'_i} e'_i - \nabla^h{}_{e'_i} e'_i \right\} + \sum_{i=m+1}^{m+\ell} \left\{ 0 - \left(-\frac{1}{f} \nabla (f \circ \pi) \right) \right\} \\
&= \frac{\ell}{f} \nabla (f \circ \pi). \qquad (5.2)
\end{aligned}$$

Indeed, we obtain the second equality of (5.2) as follows: The first sum vanishes since $\pi_* e_i = e'_i$ and $\pi_* \nabla^g{}_{e_i} e_i = \nabla^h_{e'_i} e'_i$, $(i = 1, \dots, m)$. The second sum coincides with $\frac{\ell}{f} \nabla (f \circ \pi)$ since $\pi_* e_i = 0$ and also $\pi_* \nabla^g{}_{e_i} e_i = -\frac{1}{f} \nabla (f \circ \pi)$ $(i = m+1, \dots, m+\ell)$. Therefore, we obtain:

Theorem 5.1. *Let $\pi : (P, g) \to (M, h)$ be the warped product. Then, we have*

$$\tau(\pi) = \frac{\ell}{f} \nabla (f \circ \pi). \qquad (5.3)$$

Then, π is harmonic if and only if f is constant.

6. The bitension fields of the warped products

Let us recall the definition of the bitension field for a C^∞ mapping $\varphi : (P, g) \to (M, h)$ which is given by

$$\tau_2(\varphi) := \overline{\Delta}\, \tau(\varphi) - \mathcal{R}^h(\tau(\varphi)). \qquad (6.1)$$

Here, recall, for $V \in \Gamma(\varphi^{-1}TM)$,

$$\overline{\Delta} V := -\sum_{i=1}^{p} \left\{ \overline{\nabla}_{e_i} (\overline{\nabla}_{e_i} V) - \overline{\nabla}_{\nabla^g{}_{e_i} e_i} V \right\}, \qquad (6.2)$$

$$\mathcal{R}^h V := \sum_{i=1}^{p} R^h(V, \varphi_* e_i) \varphi_* e_i, \qquad (6.3)$$

where $\{e_i\}_{i=1}^p$ is a locally defined orthonormal frame field on (P, g), $p = \dim P$, $\overline{\nabla}$ is the induced connection on the induced bundle $\varphi^{-1}TM$, and the curvature tensor of (N, h) is given by $R^h(U, V)W := \nabla^h{}_U(\nabla^h{}_V W) - \nabla^h{}_V(\nabla^h{}_U W) - \nabla^h{}_{[U,V]} W$ for $U, V, W \in \mathfrak{X}(M)$ (cf. [5, 21, 22]).

Definition 6.1. $\pi : (P, g) \to (M, h)$ is *biharmonic* if $\tau_2(\pi) = 0$.

Let us $\pi : (P, g) \to (M, h)$ be the warped product whose Riemannian metric g is given by (2.8). For $V = \tau(\pi)$, then,

$$\begin{aligned} \mathcal{R}^h V &= \sum_{i=1}^{p} R^h(\tau(\pi), \pi_* e_i)\pi_* e_i \\ &= \sum_{i=1}^{m} R^h(\tau(\pi), e_i')e_i' \\ &= \rho^h(\tau(\pi)), \end{aligned} \tag{6.4}$$

where $m = \dim M$ and ρ^h is Ricci transform of (M, h) given by $\rho^h(u) := \sum_{i=1}^{m} R^h(u, e_i')e_i'$, $u \in T_x M$, and $\{e_i'\}_{i=1}^{m}$ is a locally defined orthonormal field on (M, h).

In the following, we calculate the rough Laplacian $\overline{\Delta}$ for $V = \tau(\pi)$.

(*The first step*) We calculate $\overline{\nabla}_{e_i}\tau(\pi)$ and $\overline{\nabla}_{e_i}(\overline{\nabla}_{e_i}\tau(\pi))$ as follows:

$$\overline{\nabla}_{e_i}\tau(\pi) = \nabla^h_{\pi_* e_i}\tau(\pi) = \begin{cases} \nabla^h_{e_i'}\tau(\pi) & (i = 1, \dots, m = \dim M), \\ 0 & (i = m+1, \dots, m+\ell), \end{cases} \tag{6.5}$$

where $p := \dim P = m + \ell$, $m = \dim M$, and $\ell = \dim F$. Furthermore,

$$\overline{\nabla}_{e_i}(\overline{\nabla}_{e_i}\tau(\pi)) = \begin{cases} \nabla^h_{e_i'}(\nabla^h_{e_i'}\tau(\pi)) & (i = 1, \dots, m), \\ 0 & (i = m+1, \dots, m+\ell = p). \end{cases} \tag{6.6}$$

(*The second step*) We calculate $\overline{\nabla}_{\nabla^g_{e_i} e_i}\tau(\pi)$ by the similar way as the first step:

For $i = 1, \dots, m$,

$$\overline{\nabla}_{\nabla^g_{e_i} e_i}\tau(\pi) = \nabla^h_{\pi_*(\nabla^g_{e_i} e_i)}\tau(\pi) = \nabla^h_{\nabla^h_{e_i'} e_i'}\tau(\pi), \tag{6.7}$$

and for $i = m+1, \dots, m+\ell$, by (5.1),

$$\overline{\nabla}_{\nabla^g_{e_i} e_i}\tau(\pi) = \nabla^h_{\pi_*(\nabla^g_{e_i} e_i)}\tau(\pi) = \nabla^h_{-\frac{1}{f}\nabla(f\circ\pi)}\tau(\pi). \tag{6.8}$$

(*The third step*) Therefore, we calculate (6.2) for $V = \tau(\pi)$ as follows.

$$\begin{aligned} \overline{\Delta}\tau(\pi) :=& -\sum_{i=1}^{p}\left\{\overline{\nabla}_{e_i}(\overline{\nabla}_{e_i}\tau(\pi)) - \overline{\nabla}_{\nabla^g_{e_i} e_i}\tau(\pi)\right\} \\ =& -\sum_{i=1}^{m}\left\{\overline{\nabla}_{e_i}(\overline{\nabla}_{e_i}\tau(\pi)) - \overline{\nabla}_{\nabla^g_{e_i} e_i}\tau(\pi)\right\} \\ &- \sum_{i=m+1}^{m+\ell}\left\{\overline{\nabla}_{e_i}(\overline{\nabla}_{e_i}\tau(\pi)) - \overline{\nabla}_{\nabla^g_{e_i} e_i}\tau(\pi)\right\} \\ =& -\sum_{i=1}^{m}\left\{\nabla^h{}_{e_i'}(\nabla^h{}_{e_i'}\tau(\pi)) - \nabla^h{}_{\nabla^h{}_{e_i'} e_i'}\tau(\pi)\right\} \end{aligned}$$

$$
\begin{aligned}
&\quad - \sum_{i=m+1}^{m+\ell} \left\{ 0 - \nabla^h{}_{-\frac{1}{f}\nabla(f\circ\pi)}\tau(\pi) \right\} \\
&= -\sum_{i=1}^{m} \left\{ \nabla^h{}_{e_i'} \left(\nabla^h{}_{e_i'}\tau(\pi) \right) - \nabla^h{}_{\nabla^h_{e_i'}e_i'}\tau(\pi) \right\} - \ell\, \nabla^h{}_{\frac{1}{f}\nabla(f\circ\pi)}\tau(\pi). \qquad (6.9)
\end{aligned}
$$

(*The fourth step*) Therefore, by (6.1), (6.4) and (6.9), we obtain

$$
\begin{aligned}
\tau_2(\pi) &= -\sum_{i=1}^{m} \left\{ \overline{\nabla}_{e_i'} \left(\overline{\nabla}_{e_i'}\tau(\pi) \right) - \overline{\nabla}_{\nabla^h{}_{e_i'}e_i'}\tau(\pi) \right\} - \ell\, \nabla^h{}_{\frac{1}{f}\nabla f}\tau(\pi) - \rho^h(\tau(\pi)) \\
&= \ell\, J_{\mathrm{id}} \left(\frac{1}{f}\,\nabla f \right) - \ell^2\, \nabla^h{}_{\frac{1}{f}\nabla f} \left\{ \frac{\nabla f}{f} \right\} \qquad (6.10) \\
&= \ell\, J_{\mathrm{id}} \left(\frac{1}{f}\,\nabla f \right) + \frac{\ell^2}{f^3}\, h(\nabla f, \nabla f)\, \nabla f - \frac{\ell^2}{f^2}\, \nabla^h{}_{\nabla f}\nabla f. \qquad (6.11)
\end{aligned}
$$

Therefore, we can summarize the above by recalling the following definitions:

$$
J_{\mathrm{id}} := \overline{\Delta}^h - \rho^h, \qquad (6.12)
$$

is the Jacobi operator of the identity of (M,h), $\mathrm{id}:\ (M,h) \to (M,h)$ acting on the space $\mathfrak{X}(M)$ of C^∞ vector fields on M, and the operator $\overline{\Delta}^h$ is defined by

$$
\overline{\Delta}^h(X) := -\sum_{i=1}^{m} \left(\nabla^h{}_{e_i'}\, \nabla^h{}_{e_i'} - \nabla^h{}_{\nabla^h{}_{e_i'}e_i'} \right) X \qquad (X \in \mathfrak{X}(M)), \qquad (6.13)
$$

and ρ^h is the Ricci operator of (M,h) given by

$$
\rho^h(X) = R^h(X, e_i')e_i' \qquad (X \in \mathfrak{X}(M)). \qquad (6.14)
$$

Therefore, due to (6.1) and (6.2), we have:

Theorem 6.1. *For the warped product $\pi:\ (P,g) \to (M,h)$, the bitension field $\tau_2(\pi)$ is given by*

$$
\tau_2(\pi) = \overline{\Delta}(\tau(\pi)) - \rho^h(\tau(\pi)) - \ell\, \overline{\nabla}_{\frac{\nabla f}{f}}\tau(\pi), \qquad (6.15)
$$

where $\overline{\Delta}$ is the rough Laplacian and $\overline{\nabla}$ is the induced connection from the Levi-Civita connection ∇^h of (M,h). Therefore, the warped product $\pi:\ (P,g) \to (M,h)$ is biharmonic, i.e., $\tau_2(\pi) = 0$, if and only if the following hold:

$$
J_{\mathrm{id}} \left(\frac{\nabla f}{f} \right) = \ell\, \nabla^h_{\frac{1}{f}\nabla f} \left\{ \frac{\nabla f}{f} \right\} = -\frac{\ell}{f^3}\, h(\nabla f, \nabla f)\, \nabla f + \frac{\ell}{f^2}\, \nabla^h{}_{\nabla f}\nabla f. \qquad (6.16)
$$

Corollary 6.2. *For a positive C^∞ function f on M, let $\pi:\ (P,g) = (M \times_f F, g) \to (M,h)$ be the warped product with $g = \pi^* h + f^2\, k$ over a Riemannian manifold (M,h) whose Ricci curvature is non-positive. If π is biharmonic, then*

$$
\int_M (\nabla f) \left(h(\frac{\nabla f}{f}, \frac{\nabla f}{f}) \right) v_h = 2 \int_M h \left(\nabla^h_{\frac{\nabla f}{f}} \frac{\nabla f}{f}, \frac{\nabla f}{f} \right) v_h \geq 0. \qquad (6.17)
$$

Proof of Corollary 6.2. If $\pi : (P, g) \to (M, h)$ is biharmonic, by (6.15), it holds that

$$0 = \tau_2(\pi) = J_{\mathrm{id}}\left(\ell\,\frac{\nabla f}{f}\right) - \ell^2\,\nabla^h{}_{\frac{\nabla f}{f}}\,\frac{\nabla f}{f}, \tag{6.18}$$

which implies that

$$0 \le \int_M h\left(J_{\mathrm{id}}\left(\frac{\nabla f}{f}\right), \frac{\nabla f}{f}\right) v_h = \ell \int_M h\left(\nabla^h{}_{\frac{\nabla f}{f}}\,\frac{\nabla f}{f}, \frac{\nabla f}{f}\right) v_h. \tag{6.19}$$

Because all the eigenvalues of J_{id} are non-negative since $J_{\mathrm{id}} = \overline{\Delta}^h - \rho^h$ and the Ricci transform ρ^h are non-positive (cf. [4], [6, p. 161]). □

7. The solutions of the ordinary differential equation

Assume that $(M, h) = (\mathbb{R}, dt^2)$, a line, and $(P, g) = F \times_f \mathbb{R}$, the warped product of a Riemannian manifold (F, k) and the line $(\mathbb{R}, dt^2)$, that is,

$$g = \pi^*(dt^2) + f^2\,k \tag{7.1}$$

for a C^∞ function $f \in C^\infty(\mathbb{R})$.

In this case, it holds that

$$\begin{cases} J_{\mathrm{id}}(\pi) = J_{\mathrm{id}}\left(\ell\,\dfrac{\nabla f}{f}\right) = -\ell\left(\dfrac{f'}{f}\right)''\dfrac{\partial}{\partial t}, \\ \ell^2\,\nabla^h{}_{\frac{\nabla f}{f}}\,\dfrac{\nabla f}{f} = \ell^2\,\dfrac{f'}{f}\,\nabla^h{}_{\frac{\partial}{\partial t}}\left(\dfrac{f'}{f}\,\dfrac{\partial}{\partial t}\right) = \ell^2\,\dfrac{f'}{f}\,\dfrac{\partial}{\partial t}\left(\dfrac{f'}{f}\right)\dfrac{\partial}{\partial t} \\ \qquad = \ell^2\left(\dfrac{f'\,f''}{f^2} - \dfrac{f'^3}{f^3}\right)\dfrac{\partial}{\partial t}. \end{cases} \tag{7.2}$$

Therefore, $\pi : (F \times_f \mathbb{R}, g) \to (\mathbb{R}, dt^2)$ is biharmonic, i.e.,

$$\tau_2(\pi) = J_{\mathrm{id}}\left(\ell\,\frac{\nabla f}{f}\right) - \ell^2\,\nabla^h{}_{\frac{\nabla f}{f}}\,\frac{\nabla f}{f} = 0 \tag{7.3}$$

if and only if

$$\begin{aligned} 0 &= -\ell\left(\frac{f'}{f}\right)'' - \ell^2\left(\frac{f'\,f''}{f^2} - \frac{f'^3}{f^3}\right) \\ &= -\ell\left(\frac{f''f - f'^2}{f^2}\right)' - \ell^2\left(\frac{f'\,f''}{f^2} - \frac{f'^3}{f^3}\right) \\ &= -\ell\,\frac{f'''\,f^2 - 3\,f''\,f'\,f + 2\,f'^3}{f^3} - \ell^2\left(\frac{f'\,f''}{f^2} - \frac{f'^3}{f^3}\right) \\ &= -\frac{\ell}{f^3}\left\{f'''\,f^2 + (\ell - 3)\,f''\,f'\,f + (-\ell + 2)\,f'^3\right\} \end{aligned} \tag{7.4}$$

if and only if

$$f'''\,f^2 + (\ell - 3)\,f''\,f'\,f + (-\ell + 2)\,f'^3 = 0. \tag{7.5}$$

Therefore, we have:

Theorem 7.1. *Let (F, k) be a Riemannian manifold.*

(1) *the warped product $\pi : (F \times_f \mathbb{R}, g) \to (\mathbb{R}, dt^2)$ is biharmonic if and only if* (7.5) *holds.*

(2) *All the positive C^∞ solution f of* (7.5) *on $\mathbb{R}$ are given by*

$$f(t) = c \exp\left(\int_{t_0}^{t} a \tanh\left[a \frac{\ell}{2} r + b\right] dr\right), \tag{7.6}$$

where $a \neq 0$, b, $c > 0$ are arbitrary constants.

(3) *In the case $(M, h) = (\mathbb{R}, dt^2)$, let $f(t)$ be a C^∞ function defined by* (7.6) *with $a \neq 0$, b any real number and $c > 0$. Then, the warped product $\pi : (\mathbb{R} \times_f F, g) \to (\mathbb{R}, dt^2)$ with the Riemannian metric*

$$g = \pi^* dt^2 + f^2 k \tag{7.7}$$

is biharmonic but not harmonic.

In order to solve (7.5), we put $u := (\log f)' = \frac{f'}{f}$. Then (7.5) turns into the ordinary differential equation on u:

$$u'' + \frac{\ell}{2} (u^2)' = 0. \tag{7.8}$$

A general solution u of (5.6) is given by

$$u(t) = a \tanh\left[a \frac{\ell}{2} t + b\right], \tag{7.9}$$

where a and b are arbitrary constants. Thus, every positive solution $f(t)$ is given by

$$f(t) = c \exp\left(\int_{t_0}^{t} a \tanh\left[a \frac{\ell}{2} r + b\right] dr\right), \tag{7.10}$$

where a, b, $c > 0$ are arbitrary constants.

Therefore, we obtain Theorem 7.1 together with Theorem 6.1, □

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