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Summability Results for Mapping Matrices

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Abstract

For topological vector spaces X and Y, let $\mathbb{F}_0(X,Y)=\{f\in Y^X : f(0)=0\}$. Then it is an extremely large family and the family of linear operators is a very small subfamily of $\mathbb{F}_0(X,Y)$. In this paper, we establish the characterizations of $\mathbb{F}_0(X,Y)$ - matrix families $(l^\infty(X), l^\infty(Y))$, $(c_0(X), c_0(Y))$ and $(c_0(X), l^\infty(Y))$.

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1. Introduction

For topological vector spaces X and Y, $\mathbb{F}_0(X,Y)=\{f\in Y^X : f(0)=0\}$ is an extremely large family and the family of linear operators is a very small subfamily of $\mathbb{F}_0(X,Y)$.

For sequence families $\lambda(X)\subset X^{\mathbb{N}}$, $\mu(Y)\subset Y^{\mathbb{N}}$ and mappings $f_{ij}\in\mathbb{F}_0(X,Y)(i,j\in\mathbb{N})$, the matrix $(f_{ij})_{i,j\in\mathbb{N}}\in(\lambda(X),\mu(Y))$ means that $\sum_{j=1}^{\infty}f_{ij}(x_j)$ converges whenever $i\in\mathbb{N}$, $(x_j)\in\lambda(X)$ and $\left\{\sum_{j=1}^{\infty}f_{ij}(x_j)\right\}_{i=1}^{\infty}\in\mu(Y)$, $\forall (x_j)\in\lambda(X)$.

Let $l^\infty(X)=\{(x_j)\in X^{\mathbb{N}} : \{x_j\}\text{ is bounded}\}$ and $c_0(X)=\{(x_j)\in X^{\mathbb{N}} : x_j\to 0\}$.

In 2001 and 2002, a mapping family $QH_f(X,Y)$ including all linear operators and many non-linear mappings were defined in the literature[1,2] which gave the characterization of $QH_f(X,Y)$ - matrix families $(c_0(X), c_0(Y))$, $(c_0(X), l^\infty(Y))$ and $(l^\infty(X), l^\infty(Y))$. However, $QH_f(X,Y)$ is also a small subfamily of $\mathbb{F}_0(X,Y)$ and so the quality lower of the results in the literature[1,2] is lover than Theorem B[3].

If $Y=(Y,\|\cdot\|)$ is a seminormed space and $\{f_j\}\subset\mathbb{F}_0(X,Y)$, then for $n\in\mathbb{N}$ and $B\subset X$ let $R_n=(f_n, f_{n+1}, f_{n+2},\cdots)$ and

$$\|R_n\|_B=\sup_{m\in\mathbb{N},\{x_j\}\subset B}\left\|\sum_{k=0}^{m-1}f_{n+k}(x_j)\right\|=\sup_{m\ge n,\{x_j\}\subset B}\left\|\sum_{j=n}^{m}f_j(x_j)\right\|.$$

Obviously, $\|R_n\|_B=\sup\left\{\left\|\sum_{k\in\triangle}f_{n+k}(x_j)\right\| : \triangle\subset\mathbb{N}\text{ finite}, \{x_j\}\subset B\right\}$ whenever $0\in B$. If both X and Y are seminormed, then $\|R_n\|=\|R_n\|_{\{x\in X:\|x\|\le 1\}}$ is just the group norm of $(f_j)_{j\ge n}$[4].

For $f_{ij}\in\mathbb{F}_0(X,Y)$, $i,j\in\mathbb{N}$, R_{in} denotes the sequence $(f_{in}, f_{i\,n+1}, f_{i\,n+2},\cdots)$.

In the paper[3], we have established the characterizations of $\mathbb{F}_0(X,Y)$ - matrix families $(l^\infty(X), c_0(Y))$ and $(l^\infty(X), l^\infty(Y))$ as follows.

Proposition 1. $f_{ij}\in\mathbb{F}_0(X,Y)$ for $i,j\in\mathbb{N}$. Then $(f_{ij})_{i,j\in\mathbb{N}}\in(l^\infty(X), c_0(Y))$ if and only if

(1) $\lim_i f_{ij}(x)=0$, $\forall j\in\mathbb{N}$, $x\in X$ and

(2) for every bounded $B\subset X$, $\sum_{j=1}^{\infty}f_{ij}(x_j)$ converges uniformly with respect to both $i\in\mathbb{N}$ and $\{x_j\}\subset B$.

Proposition 2. Let Y be a Banach space and $f_{ij}\in\mathbb{F}_0(X,Y)$, $\forall i,j\in\mathbb{N}$. Then $(f_{i,j})_{i,j\in\mathbb{N}}\in(l^\infty(X), c_0(Y))$ if and only if

(1) $\lim_i f_{ij}(x)=0$, $\forall j\in\mathbb{N}$, $x\in X$ and

(2) for every bounded $B\subset X$, $\lim_m\sup_{i\in\mathbb{N}}\|R_{im}\|_B=0$.

If, in addition, X is seminormed and each $f_{ij}: X\to Y$ is linear, then $(f_{ij})_{i,j\in\mathbb{N}}\in(l^\infty(X), c_0(Y))$ if and only if (1) and

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(3) $\lim_m \sup_{i\in\mathbb{N}} \| R_{im} \| = 0.$

Proposition 3. $f_{ij} \in \mathbb{F}_0(X, Y),\ \forall i, j \in \mathbb{N}$. Then the following (a), (b) and (c) are equivalent.

(a) $(f_{i,j})_{i,j\in\mathbb{N}} \in (l^\infty(X), l^\infty(Y))$.

(b) $(s_i f_{ij})_{i,j\in\mathbb{N}} \in (l^\infty(X), c_0(Y)),\ \forall (s_i) \in c_0 = \{(s_i) \in \mathbb{C}^{\mathbb{N}} : s_i \to 0\}$.

(c) (4) $\{f_{ij}(x)\}_{i=1}^{\infty}$ is bounded, $\forall j \in \mathbb{N},\ x \in X$ and

(5) for every bounded $B \subset X$ and $(s_i) \in c_0$, $\sum_{j=1}^{\infty} s_i f_{ij}(x_j)$ converges uniformly with respect to both $i \in \mathbb{N}$ and $\{x_j\} \subset B$.

Proposition 4. If Y is seminormed and $f_{ij} \in \mathbb{F}_0(X, Y)$, $\forall i, j \in \mathbb{N}$, then $(f_{ij})_{i,j\in\mathbb{N}} \in (l^\infty(X), l^\infty(Y))$ if and only if

(4) $\{f_{ij}(x)\}_{i=1}^{\infty}$ is bounded, $\forall j \in \mathbb{N},\ x \in X$,

(6) $\sum_{j=1}^{\infty} f_{ij}(x_j)$ converges, $\forall i \in \mathbb{N},\ (x_j) \in l^\infty(X)$ and

(7) for every $(x_j) \in l^\infty(X)$ and integer sequences $i_1 < i_2 < \cdots,\ m_1 < n_1 < m_2 < n_2 < \cdots$,

$$\sup_{k\in\mathbb{N}} \left\| \sum_{j=m_k}^{n_k} f_{i_k j}(x_j) \right\| < +\infty.$$

In this paper we would like to give the additional characterizations of $(l^\infty(X), l^\infty(Y))$, $(c_0(X), c_0(Y))$ and $(c_0(X), l^\infty(Y))$ for matrices in $\mathbb{F}_0(X, Y)$.

The characterizations we ascertain in this paper is different from the summability results of matrices of linear operators during 1950-1992 that C. Swartz[5] gave an epoch-making result of Theorem A[3].

2. Main Results

Throughout this paper, X and Y are topological vector spaces, and let $L(X, Y) = \{T \in Y^X : T$ is linear and continuous$\}$.

We begin with the corollary of Proposition 3 which is a more clear-cut characterization of the family $(l^\infty(X), l^\infty(Y))$ for matrices of linear operators on Banach spaces as follows.

Corollary 1. Let X, Y be Banach spaces and $T_{ij} \in L(X, Y),\ \forall i, j \in \mathbb{N}$. Then $(T_{ij})_{i,j\in\mathbb{N}} \in (l^\infty(X), l^\infty(Y))$ if and only if

(8) $\sup_{i\in\mathbb{N}} \| T_{ij} \| < +\infty,\ \forall j \in \mathbb{N}$ and

(9) $\lim_m \| R_{im} \| = \lim_m \| (T_{im}, T_{im+1}, \cdots) \| = 0,\ \forall i \in \mathbb{N}$ and

$$\sup_{i\in\mathbb{N}} \| R_{i1} \| = \sup_{i,n\in\mathbb{N},\ \|x_j\|\le 1} \left\| \sum_{j=1}^{n} T_{ij}(x_j) \right\| < +\infty.$$

Proof. Suppose $(T_{ij})_{ij\in\mathbb{N}} \in (l^\infty(X), l^\infty(Y))$. Since $\{T_{ij(x)}\}_{i=1}^{\infty}$ is bounded for every $x \in X$ and $j \in \mathbb{N}$, (8) follows from the resonance theorem. Since $\sum_{j=1}^{\infty} T_{ij}(x_j)$ converges whenever $(x_j) \in l^\infty(X)$ and $i \in \mathbb{N}$, $\lim_m \| R_{im} \| = 0,\ \forall \in \mathbb{N}$ (see[4], p. 21, Proposition 3.3).

Suppose $\sup_i \| R_{i1} \| = +\infty$. Then $\| R_{i_1 1} \| = \sup_{n\in\mathbb{N},\ \|x_j\|\le 1} \left\| \sum_{j=1}^{n} T_{i_1 j}(x_j) \right\| > 1 + \sup_i \| T_{i1} \|$ for some $i_1 \in \mathbb{N}$ and, hence, $\left\| \sum_{j=1}^{n_1} T_{i_1 j}(x_{1j}) \right\| > 1 + \sup_i \| T_{i1} \|$ for some $n_1 > 1$ and $\{x_{1j} : 1 \le j \le n_1\} \subset B = \{x \in X : \| x \| \le 1\}$. Since $\sum_{j=1}^{\infty} T_{ij}(x_j)$ converges for each $\{x_j\} \subset B$ and $i \in \mathbb{N}$, it follows from Lemma 1[3] that there exists an $m_0 > n_1 + 1$ such that $\left\| \sum_{j=m_0+1}^{\infty} T_{ij}(x_j) \right\| < \frac{1}{2},\ \forall 1 \le i \le i_1,\ \{x_j\} \subset B$. By $\sup_i \| R_{i1} \| = +\infty$ again, there is an $i_2 \in \mathbb{N}$ for which $\| R_{i_2 1} \| = \sup_{n\in\mathbb{N},\ \|x_j\|\le 1} \left\| \sum_{j=1}^{n} T_{i_2 j}(x_j) \right\| > 2 + \frac{1}{2} + \sum_{j=1}^{m_0} \sup_i \| T_{ij} \|$ and, hence, $\left\| \sum_{j=1}^{n_2} T_{i_2 j}(x_{2j}) \right\| > 2 + \frac{1}{2} + \sum_{j=1}^{m_0} \sup_i \| T_{ij} \|$ for some $n_2 \in \mathbb{N}$ and $\{x_{2j} : 1 \le j \le n_2\} \subset B$. Obviously, $n_2 > m_0$. If $i_2 \le i_1$, then

$$\begin{aligned} \left\| \sum_{j=1}^{m_0} T_{i_2 j}(x_{2j}) \right\| &\ge \left\| \sum_{j=1}^{n_2} T_{i_2 j}(x_{2j}) \right\| - \left\| \sum_{j=m_0+1}^{n_2} T_{i_2 j}(x_{2j}) \right\| \\ &> 2 + \frac{1}{2} + \sum_{j=1}^{m_0} \sup_i \| T_{ij} \| - \frac{1}{2} \\ &\ge 2 + \left\| \sum_{j=1}^{m_0} T_{i_2 j}(x_{2j}) \right\|. \end{aligned}$$

This is a contradiction and we have that $i_2 > i_1$. Since $n_2 > m_0 > n_1 + 1$,

$$\begin{aligned} \left\| \sum_{j=n_1+1}^{n_2} T_{i_2 j}(x_{2j}) \right\| &\ge \left\| \sum_{j=1}^{n_2} T_{i_2 j}(x_{2j}) \right\| - \left\| \sum_{j=1}^{n_1} T_{i_2 j}(x_{2j}) \right\| \\ &> 2 + \frac{1}{2} + \sum_{j=1}^{m_0} \sup_i \| T_{ij} \| - \left\| \sum_{j=1}^{n_1} T_{i_2 j}(x_{2j}) \right\| \\ &\ge 2 + \frac{1}{2}, \end{aligned}$$

i.e., letting $m_2 = n_1 + 1$, $\left\| \sum_{j=m_2}^{n_2} T_{i_2 j}(x_{2j}) \right\| > 2$.

Continuing by induction, we have integer sequences $i_1 < i_2 < i_3 < \cdots$, $1 = m_1 < n_1 < m_2 < n_2 < m_3 < n_3 < \cdots$ and $\{x_{kj} : m_k \le j \le n_k, k \in \mathbb{N}\} \subset B$ such that

$$\left\| \sum_{j=m_k}^{n_k} T_{i_k j}(x_{kj}) \right\| > k, \quad k = 1,2,3,\cdots.$$

Letting

$$x_j = \begin{cases} x_{kj}, & m_k \le j \le n_k, \ k = 1,2,3,\cdots, \\ 0, & otherwise, \end{cases}$$

$(x_j) \in l^\infty(X)$ and $\sup_k \left\| \sum_{j=m_k}^{n_k} T_{i_k j}(x_j) \right\| = +\infty$. This contradicts Proposition 4. Hence, $\sup_{i\in\mathbb{N}} \| R_{i1} \| < +\infty$.

Conversely, suppose that (8) and (9) hold. Then $\sum_{j=1}^{\infty} T_{ij}(x_j)$ converges whenever $(x_j) \in l^\infty(X)$ and $i \in \mathbb{N}$ (see [4], p. 21). Since

$$\sup_{\|x_j\| \le 1} \left\| \sum_{j=m}^{n} T_{ij}(x_j) \right\| \le \sup_{\|x_j\| \le 1} \left\| \sum_{j=1}^{n} T_{ij}(x_j) \right\|$$
$$\le \sup_{k\in\mathbb{N}, \|x_j\|\le 1} \left\| \sum_{j=1}^{k} T_{ij}(x_j) \right\| \le \sup_{i\in\mathbb{N}} \| R_{i1} \|,$$

$\forall\, m \le n,\ i \in \mathbb{N}$, $(8)+(9) \Rightarrow (4)+(6)+(7)$ and $(T_{ij})_{i,j\in\mathbb{N}} \in (l^\infty(X), l^\infty(Y))$ by Proposition 4. □

X is said to be braked if for every $(x_j) \in c_0(X)$ there exist $(t_j) \in c_0$ and $(z_j) \in c_0(X)$ such that $x_j = t_j z_j$ for all j (see [6], p. 43). Metrizable spaces are braked and the non-metrizable (l^1, weak) is also braked. Especially, each (LF) space is not metrizable but braked.

Each $t \in \mathbb{C}$ gives a continuous linear operator $t : X \to X$ by letting $t(x) = tx$, $x \in X$. In fact, if $t = 0$, then $t : X \to X$ is the zero operator, and for $t \ne 0$, $t : X \to X$ is a linear homeomorphism from X onto X. Then for $f \in Y^X$ and $t \in \mathbb{C}$, $f \circ t \in Y^X$ and $(f \circ t)(x) = f(tx)$, $x \in X$.

Theorem 1. If X is braked and $f_{ij} \in \mathbb{F}_0(X, Y)$, $\forall i, j \in \mathbb{N}$, then the following (d), (e) and (f) are equivalent.

(d) $(f_{ij})_{i,j\in\mathbb{N}} \in (c_0(X), c_0(Y))$.

(e) $(f_{ij} \circ t_j)_{i,j\in\mathbb{N}} \in (l^\infty(X), c_0(Y))$, $\forall\, (t_j) \in c_0$.

(f) (1) $\lim_i f_{ij}(x) = 0$, $\forall\, j \in \mathbb{N}$, $x \in X$ and

(10) for every $(t_j) \in c_0$ and bounded $B \subset X$, $\sum_{j=1}^{\infty} f_{ij}(t_j x_j)$ converges uniformly with respect to both $i \in \mathbb{N}$ and $\{x_j\} \subset B$.

Proof. Obviously, (d)⇒(e). (e)⇒(f) by Proposition 1.

(e)⇒(d): Let $(x_j) \in c_0(X)$. Since X is braked, $(x_j) = (t_j z_j)$ where $(t_j) \in c_0$, $(z_j) \in c_0(X) \subset l^\infty(X)$. By (e), $\left\{ \sum_{j=1}^{\infty} f_{ij}(x_j) \right\}_{i=1}^{\infty} = \left\{ \sum_{j=1}^{\infty} (f_{ij} \circ t_j)(z_j) \right\}_{i=1}^{\infty} \in c_0(Y)$. □

The braked condition of X can not be omitted in Theorem 1.

Example 1. $X = (l^\infty, \sigma(l^\infty, l^1))$ is not braked : $e_j = (0, \cdots, 0, 1^{(j)}, 0, 0, \cdots) \to 0$ in X but $\lambda_j e_j \not\to 0$ in X for every $\lambda_j \to \infty$. For $i, j \in \mathbb{N}$ define $f_{ij} : X \to \mathbb{C}$ by

$$f_{ij}((\alpha_k)_{k=1}^{\infty}) = \begin{cases} \alpha_i, & i = j, \\ 0, & i \ne j, \end{cases} \quad (\alpha_k) \in l^\infty.$$

If $(t_j) \in c_0$, then $(f_{ij} \circ t_j)_{i,j\in\mathbb{N}} \in (l^\infty(X), c_0)$. In fact, if $\left\{ (\alpha_{jk})_{k=1}^{\infty} \right\}_{j=1}^{\infty} \in l^\infty(X)$, then $\sup_{j,k\in\mathbb{N}} |\alpha_{jk}| = M < +\infty$ by the resonance theorem and, hence, $\sum_{j=1}^{\infty} f_{ij}(t_j(\alpha_{jk})_{k=1}^{\infty}) = f_{ii}(t_i(\alpha_{ik})_{k=1}^{\infty}) = t_i \alpha_{ii} \to 0$.

However, $(f_{ij})_{i,j\in\mathbb{N}} \notin (c_0(X), c_0) : (e_j) \in c_0(X)$ but $\sum_{j=1}^{\infty} f_{ij}(e_j) = f_{ii}(e_i) = 1$, $\forall i \in \mathbb{N}$.

Combining Theorem 1 and Proposition 2, we have

Corollary 2. If X is braked and Y is a Banach space and $f_{ij} \in \mathbb{F}_0(X, Y)$, $\forall i, j \in \mathbb{N}$, then $(f_{ij})_{i,j\in\mathbb{N}} \in (c_0(X), c_0(Y))$ if and only if

(1) $\lim_i f_{ij}(x) = 0$, $\forall j \in \mathbb{N}$, $x \in X$ and

(2') for every $(t_j) \in c_0$ and bounded $B \subset X$,
$\lim_m \sup_{i\in\mathbb{N}} \| (f_{im} \circ t_m, f_{i\,m+1} \circ t_{m+1}, \cdots) \|_B$
$= \lim_m \sup_i \sup_{n \ge m, \{x_j\} \subset B} \left\| \sum_{j=m}^{n} f_{ij}(t_j x_j) \right\| = 0.$

For the case of linear operators, we have a more clear-cut result as follows.

Corollary 3. If X is braked and Y is a Banach space and $T_{ij} : X \to Y$ is linear, $\forall i, j \in \mathbb{N}$, then $(T_{ij})_{i,j\in\mathbb{N}} \in (c_0(X), c_0(Y))$ if and only if

(1) $\lim_i T_{ij}(x)=0,\ \forall j\in\mathbb{N},\ x\in X$ and

(2*) for every bounded $B\subset X$ there is an $m_0\in\mathbb{N}$ such that

$$\sup_{i\in\mathbb{N}}\ \|(T_{i\,m_0}, T_{i\,m_0+1}, \cdots)\|_B < +\infty.$$

If, in addition, X is seminormed, then (2*) can be replaced by the following (2**) $\exists m_0\in\mathbb{N}$ such that $\sup_{i\in\mathbb{N}}\|(T_{i\,m_0}, T_{i\,m_0+1}, \cdots)\| < +\infty$.

Proof. Let $c_{00}=\{(t_j)\in c_0 : t_j=0 \text{ eventually}\}$ and $(t_j)\in c_0\setminus c_{00}$. Then $\delta_m = \sup_{j\geq m}|t_j|>0,\ \delta_m\to 0$. Observe that $B\subset X$ is bounded if and only if $B_0=\{tx : |t|\leq 1,\ x\in B\}$ is bounded.

Since $\|(T_{i\,m}, T_{i\,m+1}, \cdots)\|_{B_0} \geq \|(T_{i\,n}, T_{i\,n+1}, \cdots)\|_{B_0}$ for $m<n$, if (2*) holds and $B\subset X$ is bounded, then

$$\begin{aligned}
&\lim_m \sup_{i\in\mathbb{N}} \sup \left\{\left\|\sum_{j=m}^{n} T_{ij}(t_j x_j)\right\| : n\geq m,\ x_j\in B \text{ for } m\leq j\leq n\right\}\\
&=\lim_m \delta_m \sup_{i\in\mathbb{N}} \sup \left\{\left\|\sum_{j=m}^{n} T_{ij}\Big(\frac{t_j}{\delta_m}x_j\Big)\right\| : n\geq m,\ x_j\in B \text{ for } m\leq j\leq n\right\}\\
&\leq \lim_m \delta_m \sup_{i\in\mathbb{N}} \sup \left\{\left\|\sum_{j=m}^{n} T_{ij}(z_j)\right\| : n\geq m,\ z_j\in B_0 \text{ for } m\leq j\leq n\right\}\\
&\leq \lim_m \delta_m \sup_{i\in\mathbb{N}} \sup \left\{\left\|\sum_{j=m_0}^{n} T_{ij}(z_j)\right\| : n\geq m_0,\ z_j\in B_0 \text{ for } m_0\leq j\leq n\right\}=0,
\end{aligned}$$

i.e., (2*)⇒(2').

Conversely, if (2*) fails, then there is a bounded $B\subset X$ such that

$$\sup_{i\in\mathbb{N}}\|R_{im}\|_B = \sup_{i\in\mathbb{N}}\|(T_{i\,m}, T_{i\,m+1}, \cdots)\|_B = +\infty,\ \forall m\in\mathbb{N}.$$

Since $\sup_{i\in\mathbb{N}}\|R_{i1}\|_B=+\infty$, there exist $i_1, n_1\in\mathbb{N}$ and $x_{1j}\in B$ for $1\leq j\leq n_1$ such that $\left\|\sum_{j=1}^{n_1} T_{i_1 j}(x_{1j})\right\|>1$. But $\sup_{i\in\mathbb{N}}\|R_{i\,n+1}\|=+\infty$ so there exist integers $i_2, n_2\geq n_1+1$ and $x_{2j}\in B$ for $n_1+1\leq j\leq n_2$ such that $\left\|\sum_{j=n_1+1}^{n_2} T_{i_2 j}(x_{2j})\right\|>2$. Proceeding inductively, we have integer sequences $\{i_k\}$, $0=n_0<n_1<n_2<n_3<\cdots$ and $\{x_{k+1\,j} : n_k+1\leq j\leq n_{k+1},\ k=0,1,2,\cdots\}\subset B$ such that

$$\left\|\sum_{j=n_k+1}^{n_k+1} T_{i_{k+1}j(x_{k+1j})}\right\|>k+1,\quad k=0,1,2,3,\cdots.$$

Now let $t_j=\dfrac{1}{k+1}$ if $n_k<j\leq n_{k+1}$, $k=0,1,2,\cdots$, then $(t_j)\in c_0$ and

$$\begin{aligned}
&\sup_{i\in\mathbb{N}}\sup_{n\geq n_k+1,\{z_j\}\subset B}\left\|\sum_{j=n_k+1}^{n} T_{ij}(t_j z_j)\right\|\\
&\geq \left\|\sum_{j=n_k+1}^{n_{k+1}} T_{i_{k+1}j}\Big(\frac{1}{k+1}x_{k+1j}\Big)\right\|>1,\ k=0,1,2,\cdots.\\
&\geq \left\|\sum_{j=n_k+1}^{n_{k+1}} T_{i_{k+1}j}\Big(\frac{1}{k+1}x_{k+1\,j}\Big)\right\|>1,\ k=0,1,2,\cdots.
\end{aligned}$$

This contradicts (2′) so (2′)⇒(2*) holds. □

Theorem 2. $f_{ij}\in\mathbb{F}_0(X,Y),\ \forall i,j\in\mathbb{N}$. If X is braked, then the following (g), (h) and (i) are equivalent.

(g) $(f_{ij})_{i,j\in\mathbb{N}}\in(c_0(X), l^\infty(Y))$.

(h) $(f_{ij}\circ t_j)_{i,j\in\mathbb{N}}\in(l^\infty(X), l^\infty(Y),\ \forall (t_j)\in c_0$.

(i) (4) $\{f_{ij}(x)\}_{i=1}^{\infty}$ is bounded, $\forall j\in\mathbb{N},\ x\in X$ and

(11) for every bounded $B\subset X$ and (s_i), $(t_j)\in c_0$, $\sum_{j=1}^{\infty} s_i f_{ij}(t_j x_j)$ converges uniformly with respect to both $i\in\mathbb{N}$ and $\{x_j\}\subset B$.

Proof. Since X is braked, (g)⇔(h) is obvious. By Proposition 3, (h)⇔(i). □

Corollary 4. Let X, Y be Banach spaces and $T_{ij}\in L(X,Y),\ \forall i,j\in\mathbb{N}$. Then $(T_{ij})_{i,j\in\mathbb{N}}\in(c_0(X), l^\infty(Y))$ if and only if

(8) $\sup_{i\in\mathbb{N}}\|T_{ij}\|<+\infty,\ \forall j\in\mathbb{N}$ and

(12) $\sup_{i\in\mathbb{N}}\|R_{i1}\| =$

$$\sup_{i,n\in\mathbb{N},\ \|x_j\|\leq 1}\left\|\sum_{j=1}^{n} T_{ij}(x_j)\right\|<+\infty.$$

Proof. Suppose that (8) and (12) hold. Then $\sum_{j=1}^{\infty} T_{ij}(x_j)$ converges for every $(x_j)\in c_0(X)$ and $i\in\mathbb{N}$ (see [4], p. 19). If $0\neq(x_j)\in c_0(X)$, then

$$\begin{aligned}
&\left\|\sum_{j=1}^{\infty} T_{kj}(x_j)\right\| = \lim_n\left\|\sum_{j=1}^{n} T_{kj}(x_j)\right\|\\
&= \|(x_j)\|_\infty \lim_n\left\|\sum_{j=1}^{n} T_{kj}\Big(\frac{x_j}{\|x_j\|_\infty}\Big)\right\|\\
&\leq \|(x_j)\|_\infty \sup_{i\in\mathbb{N}}\|R_{i1}\|,\ \forall k\in\mathbb{N},
\end{aligned}$$

i.e., $\left\{\sum_{j=1}^{\infty} T_{ij}(x_j)\right\}_{i=1}^{\infty} \in l^{\infty}(Y)$.

Conversely, suppose that $(T_{ij})_{i,j\in \mathbb{N}} \in (c_0(X), l^{\infty}(Y))$ but $\sup_{i\in \mathbb{N}} \| R_{i1} \| = +\infty$. Observe that (8) holds bt the resonance theorem. As in the proof of Corollary 1, there exist integer sequences $i_1 < i_2 < \cdots$, $m_1 < n_1 < m_2 < n_2 < \cdots$ and $\{x_{kj} : m_k \le j \le n_k, k\in \mathbb{N}\} \subset B = \{x\in X : \|x\| \le 1\}$ such that

$$\left\| \sum_{j=m_k}^{n_k} T_{i_k j}(x_{kj}) \right\| > k,\ k=1,2,3,\cdots.$$

Let

$$s_i = \begin{cases} 1/\sqrt{k}, & i=i_k,\ k=1,2,3,\cdots, \\ 0, & otherwise, \end{cases}$$

$$t_j = \begin{cases} 1/\sqrt{k}, & m_k \le j \le n_k,\ k=1,2,3,\cdots, \\ 0, & oterwise. \end{cases}$$

Then $s_i \to 0$, $t_j \to 0$ but

$$\left\| \sum_{j=m_k}^{n_k} s_{i_k} T_{i_k j}(t_j x_{kj}) \right\| = \left\| \frac{1}{k} \sum_{j=m_k}^{n_k} T_{i_k j}(x_{kj}) \right\| > 1,\ k=1,2,3,\cdots.$$

This contradicts Theorem 2 so $\sup_{i\in \mathbb{N}} \| R_{i1} \| < +\infty$. □

$(l^{\infty}(X), l^{\infty}(Y)) \subset (c_0(X), l^{\infty}(Y))$ and, in general, the containment is strict. We now characterize $(l^{\infty}(X), l^{\infty}(Y)) = (c_0(X), l^{\infty}(Y))$ as follows.

For a matrix family $(\lambda(X), \mu(Y))$, let

$(\lambda(X), \mu(Y))|_{L(X,Y)}$
$= \{(T_{ij}) \in (\lambda(X), \mu(Y)) :$ each $T_{ij} : X \to Y$ is linear and continuous$\}$.

Theorem 3. A Banach space Y contains no copy of c_0 if and only if $(l^{\infty}(X), l^{\infty}(Y))|_{L(X,Y)} = (c_0(X), l^{\infty}(Y))|_{L(X,Y)}$ for every Banach space X.

Proof. $\Rightarrow$: Let X be a Banach space and $(T_{ij})_{i,j\in \mathbb{N}} \in (c_0(X), l^{\infty}(Y))|_{L(X,Y)}$. If $(x_j) \in l^{\infty}(X)$ and $(t_j) \in c_0$, then $(t_j x_j) \in c_0(X)$ so $\sum_{j=1}^{\infty} t_j T_{ij}(x_j) = \sum_{j=1}^{\infty} T_{ij}(t_j x_j)$ converges, $\forall i \in \mathbb{N}$. Since Y contains no copy of c_0, it follows from Theorem 4[7] that $\sum_{j=1}^{\infty} T_{ij}(x_j)$ converges for every $i\in \mathbb{N}$ and $(x_j) \in l^{\infty}(X)$.

By Corollary 4, $\sup_{i\in \mathbb{N}} \| (T_{i1}, T_{i2}, \cdots) \| < +\infty$. For $0 \ne (x_j) \in l^{\infty}(X)$,

$$\left\| \sum_{j=1}^{\infty} T_{kj}(x_j) \right\| = \|(x_j)\|_{\infty} \lim_n \left\| \sum_{j=1}^{n} T_{kj}\left(\frac{x_j}{\|(x_j)\|_{\infty}}\right) \right\|$$
$$\le \|(x_j)\|_{\infty} \sup_{i\in \mathbb{N}} \| (T_{i1}, T_{i2}, \cdots) \|,\ \forall k \in \mathbb{N},$$

i.e., $(T_{ij})_{i,j\in \mathbb{N}} \in (l^{\infty}(X), l^{\infty}(Y))$.

$\Leftarrow$: Suppose that Y contains a copy of c_0. We may assume that $(c_0, \|\cdot\|_{\infty})$ is a subspace of Y.

Define $T_{ij} : \mathbb{C} \to Y$ by $T_{ij}(t) = te_j$, $\forall i,j \in \mathbb{N}$, $t \in \mathbb{C}$. Then $\sum_{j=1}^{\infty} T_{ij}(t_j) = \sum_{j=1}^{\infty} t_j e_j$ converges in $(c_0, \|\cdot\|_{\infty}) \subset Y$ for each $(t_j) \in c_0$ and $\left\| \sum_{j=1}^{\infty} T_{ij}(t_j) \right\|_{\infty} = \left\| \sum_{j=1}^{\infty} t_j e_j \right\|_{\infty} = \|(t_j)\|_{\infty}$, $\forall \in \mathbb{N}$, i.e., $(T_{ij})_{i,j\in \mathbb{N}} \in (c_0, l^{\infty}(Y))$. However, $\sum_{j=1}^{\infty} T_{ij}(1) = \sum_{j=1}^{\infty} e_j$ diverges for $(1,1,1,\cdots) \in l^{\infty}$ so $(l^{\infty}, l^{\infty}(Y)) \ne (c_0, l^{\infty}(Y))$. □

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References

[1] R. Li, L. Li, and M. K. Shin, "Summability results for operator matrices on topological vector spaces", Sci. China Ser. A Math., Vol. 44, pp. 1300-1311, 2001.

[2] R. Li, M. K. Shin, and C. Swartz, "Operator matrices on topological vector spaces", J. Math. Anal. Appl., Vol. 274, pp. 645-658, 2002.

[3] W. S. Yoo, "The characterizations of matrix families $(c_0(X), l^{\infty}(Y))$ and $(l^{\infty}(X), l^{\infty}(Y))$", Applied Mathematical Sciences, Vol. 10, pp. 653-659, 2016.

[4] I. J. Maddox, "Infinite matrices of operators", ed. A. Dold and B. Eckmann, New York: Springer-Verlag Berlin-Heidelberg, 1980.

[5] R. Li and C. Swartz, “A nonlinear Schur theorem”, Acta Scientiarum Mathematicarum, Vol. 58, pp. 497-508, 1993.

[6] S. M. Khaleelulla, “Counterexamples in topological vector spaces”, ed. A. Dold and B. Eckmann, New York: Springer-Verlag Berlin-Heidelberg, 1982.

[7] R. L. Li and Q. Y. Bu, “Locally convex spaces containing no copy of c_0”, J. Math. Anal. Appl., Vol. 172, pp. 205-211, 1993.