

## NONCONSTANT WARPING FUNCTIONS ON EINSTEIN WARPED PRODUCT MANIFOLDS WITH 2-DIMENSIONAL BASE

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ABSTRACT. In this paper, we study nonconstant warping functions on an Einstein warped product manifold  $M = B \times_{f^2} F$  with a warped product metric  $g = g_B + f(t)^2 g_F$ . And we consider a 2-dimensional base manifold  $B$  with a metric  $g_B = dt^2 + (f'(t))^2 du^2$ . As a result, we prove the following: if  $M$  is an Einstein warped product manifold with a 2-dimensional base, then there exist generally nonconstant warping functions  $f(t)$ .

### 1. Introduction

In [1], A.L. Besse, the author studies an Einstein manifold. And he considers an Einstein warped product manifold  $M = B \times_{f^2} F$  with a warped product metric  $g = g_B + f(t)^2 g_F$ . In addition to, he may also consider an Einstein warped product manifold with a 2-dimensional base manifold.

In a recent study, we have various results on an Einstein warped product manifold by several authors( [4, 6–10]). And we get results on an Einstein warped product manifold with a 2-dimensional base by several authors( [8–10]).

In this paper, we study the following question:

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MAIN THEOREM : If  $M = B \times_{f^2} F$  is an Einstein warped product manifold with a 2-dimensional base, then there exist generally nonconstant warping functions  $f(t)$ .

DEFINITION 1.1. Let  $(B, g_B)$  and  $(F, g_F)$  be two manifolds. Let  $g_B$  be metric tensors of  $B$  and  $g_F$  be metric tensors of  $F$ . We denote by  $\pi$  and  $\sigma$  the projections of  $B \times F$  onto  $B$  and  $F$ , respectively.

For a positive smooth function  $f$  on  $B$  the warped product manifold  $M = B \times_f F$  is the product manifold  $M = B \times F$  furnished with the metric tensor  $g$  defined by  $g = \pi^*(g_B) + (f \circ \pi)^2 \sigma^*(g_F)$ . We denote by  $\pi^*$  and  $\sigma^*$  the pullback  $\pi$  and  $\sigma$ , respectively. Here  $B$  is called the base of  $M$  and  $F$  the fiber ([1–3, 11]).

We denote by  $Ric_F$  is the Ricci curvature of  $(F, g_F)$  and  $Ric_B$  is the Ricci curvature of  $(B, g_B)$ . We denote by  $Ric^B$  and  $Ric^F$  the lifts to  $M$  of Ricci curvatures of  $B$  and  $F$ , respectively. Let  $p$  be a dimension of  $F$  ([1–3, 11]).

PROPOSITION 1.2. (See Proposition 9.106 in [1].) *The Ricci curvature Ric of the warped product manifold  $M = B \times_{f^2} F$  satisfies*

$$(i) Ric(V, W) = Ric^F(V, W) + g(V, W) \left[ \left( \frac{\Delta f}{f} - (p-1) \frac{\|df\|^2}{f^2} \right) \pi \right],$$

$$(ii) Ric(X, V) = 0,$$

$$(iii) Ric(X, Y) = Ric^B(X, Y) - \frac{p}{f} H^f(X, Y)$$

for any vertical vectors  $V, W$  and any horizontal vectors  $X, Y$ . We are defined by  $df$  is the gradient of  $f$  for  $g_B$ ,  $H^f$  is the Hessian of  $f$  for  $g_B$ . We denote by  $\Delta f$  is the Laplacian of  $f$  for  $g_B$  and  $p$  is a dimension of  $F$ .

COROLLARY 1.3. (See Corollary 9.107 in [1].) *The warped product manifold  $M = B \times_{f^2} F$  is an Einstein manifold (with  $Ric = \lambda g$ ) if and only if  $g_F, g_B$  and  $f$  satisfy*

$$(i) (F, g_F) \text{ is Einstein (with } Ric_F = \lambda_0 g_F),$$

$$(ii) \frac{\Delta f}{f} - (p-1) \frac{\|df\|^2}{f^2} + \frac{\lambda_0}{f^2} = \lambda,$$

$$(iii) Ric_B - \frac{p}{f} H^f = \lambda g_B.$$

Obviously, (i) gives a condition on  $(F, g_F)$  alone, whereas (ii) and (iii) are two differential equations for  $f$  on  $(B, g_B)$ .

REMARK 1.4. (See 9.108 in [1].) Using Corollary 1.3 (ii) and (iii), we replace the unique equation

$$(1.1) \quad Ric_B - \frac{p}{f}H^f = \frac{1}{2} \left[ s_B + 2p\frac{\Delta f}{f} - p(p-1)\frac{\|df\|^2}{f^2} + p\frac{\lambda_0}{f^2} - (p+q-2)\lambda \right] g_B,$$

where  $q = \dim B$ .

REMARK 1.5. (See 9.116 in [1].) In a special case of a warped product manifold  $M = B \times_{f^2} F$  over a 2-dimensional base. We denote by  $s_B$  is a Gaussian curvature of  $B$ ,  $Ric_B = \frac{1}{2}s_B g_B$ , and  $q = 2$ . Hence equation (1.1) simplifies to

$$(1.2) \quad H^f = -\frac{1}{2} \left[ 2\Delta f - (p-1)\frac{\|df\|^2}{f} + \frac{\lambda_0}{f} - \lambda f \right] g_B.$$

LEMMA 1.6. (See Lemma 9.117 in [1].) *On a 2-dimensional manifold  $(B, g_B)$  the equation  $H^f = f''g_B$  admits a nonconstant solution  $f$  if and only if, locally at points where  $df \neq 0$ , there exist local coordinates  $(t, u)$  such that  $f$  is a function of  $t$  alone and  $g_B = dt^2 + f'(t)^2 du^2$ .*

REMARK 1.7. (See 9.117a in [1].) With the notations of the lemma, equation (1.2) becomes an ordinary differential equation for  $f$  in the variable  $t$  and  $\|df\|^2 = (f')^2$ . Then we have an equation

$$(1.3) \quad 2f'' + (p-1)\frac{f'^2}{f} - \frac{\lambda_0}{f} + \lambda f = 0.$$

If we denote  $f(t) = u(t)^{\frac{2}{p+1}}$ , where  $u(t)$  is a positive function and  $\dim F = p > 1$ , then equation (1.3) can be changed into

$$u''(t) = \frac{(p+1)\lambda_0}{4} u(t)^{1-\frac{4}{p+1}} - \frac{(p+1)\lambda}{4} u(t).$$

By multiplies both side  $u'(t)$  and an integration gives, then we obtain

$$(1.4) \quad (u'(t))^2 = \frac{(p+1)^2 \lambda_0}{4(p-1)} u(t)^{2-\frac{4}{p+1}} - \frac{(p+1)\lambda}{4} (u(t))^2.$$

We study generally nonconstant warping functions of equation (1.4) on an Einstein warped product manifold  $M = B \times_{f_2} F$  with a 2-dimensional base  $B$  and the metric  $g_B = dt^2 + f'(t)^2 du^2$ .

## 2. Fiber manifold with $\lambda_0 = 0$

Let  $\dim F = p > 1$ . In case that  $\lambda_0 = 0$ , we consider to the following theorem according to the signs of  $\lambda$ .

**THEOREM 2.1.** *In case that  $\lambda_0 = 0$ . If  $\lambda$  is a constant, then there exist solutions of equation (1.4).*

(i) For  $\lambda = 0$ ,  $u(t) = c$ , where  $c$  is a positive constant.

(ii) For  $\lambda > 0$ , there does not exist a solution of equation (1.4).

(iii) For  $\lambda < 0$ ,  $u(t) = e^{\pm \sqrt{\frac{-(p+1)\lambda}{4}} t+c}$ , where  $c$  is a constant.

*Proof.* For  $\lambda_0 = 0$ , equation (1.4) implies that

$$(2.1) \quad (u'(t))^2 = -\frac{(p+1)\lambda}{4}(u(t))^2.$$

(i) For  $\lambda = 0$ , equation (2.1) implies that  $(u'(t))^2 = 0$  and  $u'(t) = 0$ . An integration gives  $u(t) = c$ , where  $c$  is a positive constant.

(ii) For  $\lambda > 0$ , equation (2.1) implies that  $(u'(t))^2 < 0$ , which is a contradiction. Hence there does not exist a solution of equation (1.4).

(iii) For  $\lambda < 0$ , then we get  $u'(t) = \pm \sqrt{\frac{-(p+1)\lambda}{4}} u(t)$ , where  $u(t)$  is a positive function. Multiplying both sides of equation by  $\frac{1}{u(t)}$  and an integration gives

$$\ln |u(t)| = \pm \sqrt{\frac{-(p+1)\lambda}{4}} t + c,$$

where  $c$  is a constant. Then we have  $u(t) = e^{\pm \sqrt{\frac{-(p+1)\lambda}{4}} t+c}$ , where  $c$  is a constant and  $u(t)$  is a positive function.

Therefore we have  $u(t) = e^{\pm \sqrt{\frac{-(p+1)\lambda}{4}} t+c}$ , where  $c$  is a constant.  $\square$

From above Theorem 2.1, the following remark considers that equation (1.3) satisfies generally nonconstant warping function  $f(t)$ .

REMARK 2.2. For  $\lambda_0 = 0$ , there exist generally nonconstant warping functions  $f(t)$  of equation (1.3) according to the signs of  $\lambda$  :

- (i) For  $\lambda = 0$ ,  $f(t) = c$ , where  $c$  is a positive constant. Because  $c$  is not nonconstant, thus  $f(t) = c$  is not our nonconstant solution.
- (ii) For  $\lambda > 0$ , there does not exist a solution of equation (1.3).
- (iii) For  $\lambda < 0$ ,  $f(t) = e^{\pm\sqrt{\frac{-\lambda}{p+1}} t + \frac{2c}{p+1}}$ , where  $c$  is a constant.

The following example shows that our results are satisfied with well-known special cases of equation (1.3) besides the constant cases.

EXAMPLE 2.3. Let  $\dim F = p > 1$ . From Remark 2.2, for the well-known special cases  $\lambda_0$  and  $\lambda$ , we have a nonconstant warping function of equation (1.3). For  $\lambda_0 = 0$  and  $\lambda = -(p+1)$ , then  $f(t) = e^{\pm t}$  when  $c = 0$ .

### 3. Fiber manifold with $\lambda_0 > 0$

Let  $\dim F = p > 1$ . In case that  $\lambda_0 > 0$ , we consider to the following theorem according to the signs of  $\lambda$ .

THEOREM 3.1. *In case that  $\lambda_0 > 0$ . If  $\lambda$  is a constant, then there exist solutions  $u(t)$  of equation (1.4):*

- (i) For  $\lambda = 0$ ,  $u(t) = \left( \pm\sqrt{\frac{\lambda_0}{p-1}} t + \frac{2c}{p+1} \right)^{\frac{p+1}{2}}$ ,
  - (ii) for  $\lambda > 0$ ,  $u(t) = \left( \sqrt{\frac{(p-1)\lambda}{(p+1)\lambda_0}} \sec\left(\pm\sqrt{\frac{\lambda}{p+1}} t - \sqrt{\frac{\lambda}{p+1}} c\right) \right)^{-\frac{p+1}{2}}$ ,
  - (iii) for  $\lambda < 0$ ,  $u(t) = \left( \sqrt{\frac{-(p+1)\lambda_0}{(p-1)\lambda}} \sinh\left(\pm\sqrt{\frac{-\lambda}{p+1}} t + \sqrt{\frac{-\lambda}{p+1}} c\right) \right)^{\frac{p+1}{2}}$ ,
- where  $c$  is a constant.

*Proof.* (i) For  $\lambda = 0$ , equation (1.4) implies that we have equation

$$(u'(t))^2 = \frac{(p+1)^2\lambda_0}{4(p-1)} u(t)^{2-\frac{4}{p+1}}.$$

Here we get  $u'(t) = \pm\sqrt{\frac{(p+1)^2\lambda_0}{4(p-1)}} u(t)^{1-\frac{2}{p+1}}$ . Multiplying both sides of equation by  $u(t)^{-1+\frac{2}{p+1}}$  and an integration gives

$$\frac{p+1}{2}u(t)^{\frac{2}{p+1}} = \pm \sqrt{\frac{(p+1)^2\lambda_0}{4(p-1)}} t + c,$$

where  $c$  is a constant. Then we obtain  $u(t)^{\frac{2}{p+1}} = \pm \sqrt{\frac{\lambda_0}{p-1}} t + \frac{2c}{p+1}$ , where  $c$  is a constant. Therefore we have

$$u(t) = \left( \pm \sqrt{\frac{\lambda_0}{p-1}} t + \frac{2c}{p+1} \right)^{\frac{p+1}{2}},$$

where  $c$  is a constant.

(ii) For  $\lambda > 0$ , first of all, equation (1.4) simplifies that we rewritten as

$$(3.1) \quad \int \frac{1}{u(t) \sqrt{\frac{(p+1)^2\lambda_0}{4(p-1)} u(t)^{-\frac{4}{p+1}} - \frac{(p+1)\lambda}{4}}} du = \pm \int dt.$$

Putting  $\frac{(p+1)^2\lambda_0}{4(p-1)} = a > 0$  and  $\frac{(p+1)\lambda}{4} = b > 0$ , then we have

$$\int \frac{1}{u(t) \sqrt{a u(t)^{-\frac{4}{p+1}} - b}} du = \pm \int dt.$$

By using trigonometric substitution,  $u(t)^{\frac{-2}{p+1}} = \frac{\sqrt{b}}{\sqrt{a}} \sec \theta$ , then we get

$$\int -\frac{p+1}{2} \frac{1}{\sqrt{b}} d\theta = \pm \int dt.$$

Upon integration, we obtain  $-\frac{p+1}{2} \frac{1}{\sqrt{b}} \theta = \pm t + c$ , where  $c$  is a constant. Now we have

$$u(t)^{\frac{-2}{p+1}} = \frac{\sqrt{b}}{\sqrt{a}} \sec\left(\pm \frac{2\sqrt{b}}{p+1} t - \frac{2\sqrt{b}}{p+1} c\right),$$

where  $c$  is a constant. Then we become

$$u(t) = \left( \frac{\sqrt{b}}{\sqrt{a}} \sec\left(\pm \frac{2\sqrt{b}}{p+1} t - \frac{2\sqrt{b}}{p+1} c\right) \right)^{-\frac{p+1}{2}},$$

where  $c$  is a constant. Therefore we have

$$u(t) = \left( \sqrt{\frac{(p-1)\lambda}{(p+1)\lambda_0}} \sec\left(\pm\sqrt{\frac{\lambda}{p+1}} t - \sqrt{\frac{\lambda}{p+1}} c\right) \right)^{-\frac{p+1}{2}},$$

where  $c$  is a constant.

(iii) For  $\lambda < 0$ , by a proof similar to Theorem 3.1 (ii), putting  $\frac{(p+1)^2\lambda_0}{4(p-1)} = a > 0$  and  $-\frac{(p+1)\lambda}{4} = b > 0$ , equation (3.1) implies that we have the equation

$$\int \frac{1}{u(t) \sqrt{a u(t)^{\frac{-4}{p+1}} + b}} du = \pm \int dt.$$

By using trigonometric substitution,  $u(t)^{\frac{-2}{p+1}} = \frac{\sqrt{b}}{\sqrt{a}} \tan \theta$ , then we obtain

$$\int -\frac{p+1}{2} \frac{1}{\sqrt{b}} \csc \theta d\theta = \pm \int dt.$$

Upon integration, we become  $\ln |\csc \theta + \cot \theta| = \pm \frac{2\sqrt{b}}{p+1} t + \frac{2\sqrt{b}}{p+1} c$ , where  $c$  is a constant. Here we obtain

$$\ln \left| \frac{\sqrt{a + b u(t)^{\frac{4}{p+1}}}}{\sqrt{a}} + \frac{\sqrt{b} u(t)^{\frac{2}{p+1}}}{\sqrt{a}} \right| = \pm \frac{2\sqrt{b}}{p+1} t + \frac{2\sqrt{b}}{p+1} c,$$

where  $c$  is a constant. Now we get

$$\left| \sqrt{a + b u(t)^{\frac{4}{p+1}}} + \sqrt{b} u(t)^{\frac{2}{p+1}} \right| = e^{\pm \frac{2\sqrt{b}}{p+1} t + \frac{2\sqrt{b}}{p+1} c + \ln \sqrt{a}},$$

where  $c$  is a constant and  $e^{\pm \frac{2\sqrt{b}}{p+1} t + \frac{2\sqrt{b}}{p+1} c + \ln \sqrt{a}}$  are positive functions.

Hence we have

$$u(t)^{\frac{2}{p+1}} = \frac{\sqrt{a}}{\sqrt{b}} \sinh\left(\pm \frac{2\sqrt{b}}{p+1} t + \frac{2\sqrt{b}}{p+1} c\right),$$

where  $c$  is a constant. Therefore we have

$$u(t) = \left( \sqrt{\frac{-(p+1)\lambda_0}{(p-1)\lambda}} \sinh\left(\pm \sqrt{\frac{-\lambda}{p+1}} t + \sqrt{\frac{-\lambda}{p+1}} c\right) \right)^{\frac{p+1}{2}},$$

where  $c$  is a constant.  $\square$

**REMARK 3.2.** For  $\lambda_0 > 0$ , there exist generally nonconstant warping functions  $f(t)$  of the equation (1.3) according to the signs of  $\lambda$ :

- (i) For  $\lambda = 0$ , then  $f(t) = \pm \sqrt{\frac{\lambda_0}{p-1}} t + \frac{2c}{p+1}$ ,
- (ii) for  $\lambda > 0$ , then  $f(t) = \sqrt{\frac{(p+1)\lambda_0}{(p-1)\lambda}} \cos\left(\pm \sqrt{\frac{\lambda}{p+1}} t - \sqrt{\frac{\lambda}{p+1}} c\right)$ ,
- (iii) for  $\lambda < 0$ , then  $f(t) = \sqrt{\frac{-(p+1)\lambda_0}{(p-1)\lambda}} \sinh\left(\pm \sqrt{\frac{-\lambda}{p+1}} t + \sqrt{\frac{-\lambda}{p+1}} c\right)$ ,

where  $c$  is a constant.

**EXAMPLE 3.3.** Let  $\dim F = p > 1$ . From the Remark 3.2, we have nonconstant warping functions  $f(t)$  of equation (1.3) depending on well-known special constants  $\lambda_0$  and  $\lambda$  when  $c = 0$ .

- (i) For  $\lambda_0 = p - 1$  and  $\lambda = 0$ , then  $f(t) = \pm t$ .
- (ii) For  $\lambda_0 = p - 1$  and  $\lambda = p + 1$ , then  $f(t) = \cos t$ .
- (iii) For  $\lambda_0 = p - 1$  and  $\lambda = -(p + 1)$ , then  $f(t) = \pm \sinh t$ .

#### 4. Fiber manifold with $\lambda_0 < 0$

Let  $\dim F = p > 1$ . In case that  $\lambda_0 < 0$ , we consider to the following theorem according to the signs of  $\lambda$ .

**THEOREM 4.1.** *In case that  $\lambda_0 < 0$ . If  $\lambda$  is a constant, then there exists a solution equation (1.4):*

- (i) For  $\lambda \geq 0$ , there does not exist a solution of equation (1.4).

- (ii) For  $\lambda < 0$ ,  $u(t) = \left( \sqrt{\frac{(p+1)\lambda_0}{(p-1)\lambda}} \cosh\left(\pm \sqrt{\frac{-\lambda}{p+1}} t + \sqrt{\frac{-\lambda}{p+1}} c\right) \right)^{\frac{p+1}{2}}$ ,

where  $c$  is a constant.



*Proof.* (i) For  $\lambda \geq 0$ , equation (1.4) implies that  $(u'(t))^2 < 0$ . Therefore there does not exist a solution of equation (1.4).

(ii) For  $\lambda < 0$ , by a proof similar to Theorem 3.1 (ii) and (iii), putting  $-\frac{(p+1)^2\lambda_0}{4(p-1)} = a > 0$  and  $-\frac{(p+1)\lambda}{4} = b > 0$ , equation (3.1) implies that we get the equation

$$\int \frac{1}{u(t) \sqrt{-a u(t)^{\frac{-4}{p+1}} + b}} du = \pm \int dt.$$

By using trigonometric substitution,  $u(t)^{\frac{-2}{p+1}} = \frac{\sqrt{b}}{\sqrt{a}} \sin \theta$ , then we obtain

$$\int -\frac{p+1}{2} \frac{1}{\sqrt{b}} \csc \theta d\theta = \pm \int dt.$$

Upon integration, we have  $\ln |\csc \theta + \cot \theta| = \pm \frac{2\sqrt{b}}{p+1} t + \frac{2\sqrt{b}}{p+1} c$ , where  $c$  is a constant. Here we become

$$\ln \left| \frac{\sqrt{b} u(t)^{\frac{2}{p+1}}}{\sqrt{a}} + \frac{\sqrt{-a + b u(t)^{\frac{4}{p+1}}}}{\sqrt{a}} \right| = \pm \frac{2\sqrt{b}}{p+1} t + \frac{2\sqrt{b}}{p+1} c,$$

where  $c$  is a constant. Here we get

$$\left| \sqrt{b} u(t)^{\frac{2}{p+1}} + \sqrt{-a + b u(t)^{\frac{4}{p+1}}} \right| = e^{\pm \frac{2\sqrt{b}}{p+1} t + \frac{2\sqrt{b}}{p+1} c + \ln \sqrt{a}},$$

where  $c$  is a constant and  $e^{\pm \frac{2\sqrt{b}}{p+1} t + \frac{2\sqrt{b}}{p+1} c + \ln \sqrt{a}}$  are positive functions. Then we have

$$u(t)^{\frac{2}{p+1}} = \frac{\sqrt{a}}{\sqrt{b}} \cosh\left(\pm \frac{2\sqrt{b}}{p+1} t + \frac{2\sqrt{b}}{p+1} c\right),$$

where  $c$  is a constant. Therefore we have

$$u(t) = \left( \sqrt{\frac{(p+1)\lambda_0}{(p-1)\lambda}} \cosh\left(\pm \sqrt{\frac{-\lambda}{p+1}} t + \sqrt{\frac{-\lambda}{p+1}} c\right) \right)^{\frac{p+1}{2}},$$

where  $c$  is a constant. □

REMARK 4.2. For  $\lambda_0 < 0$ , there exists generally nonconstant warping function  $f(t)$  of equation (1.3) for the signs of  $\lambda$ :

(i) For  $\lambda \geq 0$ , there does not exist a positive solution of equation (1.3).

(ii) For  $\lambda < 0$ , then  $f(t) = \sqrt{\frac{(p+1)\lambda_0}{(p-1)\lambda}} \cosh\left(\pm \sqrt{\frac{-\lambda}{p+1}} t + \sqrt{\frac{-\lambda}{p+1}} c\right)$ ,

where  $c$  is a constant.

EXAMPLE 4.3. Let  $\dim F = p > 1$ . From the Remark 4.2, for the well-known special cases  $\lambda_0 = -(p-1)$ ,  $\lambda = -(p+1)$ , and  $c = 0$ , then  $f(t) = \cosh(t)$ .

From above Remark 2.2, Remark 3.2, and Remark 4.2, the following remark shows that equation (1.3) satisfies nonconstant warping functions  $f(t)$ .

REMARK 4.4. If  $M = B \times_{f^2} F$  is an Einstein warped product manifold with a 2-dimensional base  $B$  and a metric  $g_B = dt^2 + f'(t)^2 du^2$ , then there exist generally nonconstant warping functions  $f(t)$  of equation (1.3). The behavior of the solutions depends on the signs of  $\lambda_0$  and  $\lambda$ . We are reduced to the following five cases (besides the constant case), where  $c = 0$ :

- (i) For  $\lambda_0 = 0$  and  $\lambda < 0$ , then  $f(t) = e^{\pm \sqrt{\frac{-\lambda}{p+1}} t}$ .
- (ii) For  $\lambda_0 > 0$  and  $\lambda = 0$ , then  $f(t) = \pm \sqrt{\frac{\lambda_0}{p-1}} t$ .
- (iii) For  $\lambda_0 > 0$  and  $\lambda > 0$ , then  $f(t) = \sqrt{\frac{(p+1)\lambda_0}{(p-1)\lambda}} \cos\left(\pm \sqrt{\frac{\lambda}{p+1}} t\right)$ .
- (iv) For  $\lambda_0 > 0$  and  $\lambda < 0$ , then  $f(t) = \sqrt{\frac{-(p+1)\lambda_0}{(p-1)\lambda}} \sinh\left(\pm \sqrt{\frac{-\lambda}{p+1}} t\right)$ .
- (v) For  $\lambda_0 < 0$  and  $\lambda < 0$ , then  $f(t) = \sqrt{\frac{(p+1)\lambda_0}{(p-1)\lambda}} \cosh\left(\pm \sqrt{\frac{-\lambda}{p+1}} t\right)$ .

Thus main theorem is verified.

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