Journal for History of Mathematics
Vol. 31 No. 1 (Feb. 2018), 1–17
http://dx.doi.org/10.14477/jhm.2018.31.1.001

Eigenspaces of Max-Plus Matrices: An Overview

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In this expository paper, we present an abridged report on the max-plus eigenspaces of max-plus matrices with its brief history. At the end of our work, a number of examples are presented with maple codes, and then we make a claim from the observation of these examples, which is on the euclidean dimension of the max-plus eigenspaces of strongly definite matrices.

Keywords: Max-plus algebra, max-plus geometry, definite matrix, max-plus eigenspace.

MSC: 15A18, 06F15, 52A99 ZDM: 2000

1 Brief history on Max-Plus Algebra

Max-plus algebra has appeared for the first time in A. Shimbel's [19] and S. C. Kleene's works [15] in 1950's. In 1979 R. A. Cunninghame-Green produced a pioneering work on the max-plus algebra [6]. However it is not easy to follow his work because not only his algebraic setting is too general but also teminologies he employed are quite unconventional. Because of this obstacle we reproduced a part of his work over the max-plus algebra [14]. R.A. Cunninghame-Green who observed the development of this early event describes the birth of max-plus algebra as follows [6]:

> In the past 20 years a number of different authors, often apparently unaware of one another's work, have discovered that a very attractive formulation language is provided for a surprisingly wide class of problems by setting up an algebra of real numbers in which, however, the usual operations of multiplication and addition of two numbers are replaced by the operations: (i) arithmetical addition, and (ii) selection of the greater of the two numbers, respectively.

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Received on Jan. 26, 2018, revised on Feb. 19, 2018, accepted on Feb. 25, 2018.

Max-plus algebra is one of many fields in mathematics which have been applied to the industrial area. It has been used in control theory, machine scheduling, discrete event processes, manufacturing systems, telecommunication networks, parallel processing systems and traffic control [1, 2, 9, 10, 11, 13]. Many equations in these applications are nonlinear in conventional algebra but become linear in max-plus algebra. This is the primary reason why max-plus algebra has been studied in such various fields.

Many theorems and techniques in ordinary linear algebra have analogues in max-plus algebra. Cunningham-Green, S. Gaubert, M. Gondran and M. Minoux, and P. Butkovič are among many researchers who have devoted to create much of the current max-plus algebra theory [3, 6, 10, 11, 12]. They have studied concepts such as solving systems of linear equations, the eigenvalue problem, the linear independence and characteristic polynomials over the max-plus algebra.

In 2000's a new mathematics research field, Tropical Geometry has arisen over min-plus algebra, which is isomorphic to max-plus algebra, and it now became one of the major active mathematics fields with AMS Mathematics Subject Classification, 14Txx Tropical Geometry [7, 8, 16, 17].

2 Introduction

The motivation for this work is S. Sergeev's work [18], where it is shown that $n \times n$ strongly definite matrices have strong permanents, which means that they have only one optimal permutation, if and only if their max-plus eigenspaces have an interior, which is equivalent to saying that the maximum euclidean dimension of their eigenspaces is n. So we naively expected that the number of optimal permutations of strongly definite matrices could be related to the maximal dimension of the eigenspaces. S. Sergeev presented two 3×3 matrices A and B as examples, A with only one optimal permutation and B with two optimal permutations [18]. The eigenspace of the matrix A has an interior, but the eigenspace of the matrix B does not have an interior.

After the inspection on a number of 4×4 matrices, we rather found that the euclidean dimension of the eigenspaces are not related with the number of optimal permutations, but rather on the number of special optimal permutations we will describe at the end of this work. We follow closely [3, 5, 18] for notations and general descriptions on the max-plus eigenspaces.

Our algebraic setting is the max-plus algebra whose set is $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$ with binary operations $\oplus$ and $\otimes$ defined as follows

$$\begin{aligned} a \oplus b &= \max(a, b) \\ a \otimes b &= a + b \end{aligned}$$

for $a, b \in \overline{\mathbb{R}}$. For the max-plus summation, we use the notation $\sum_{\oplus}$.

Max-plus algebra $(\overline{\mathbb{R}}, \oplus, \otimes)$ is algebraically isomorphic to min-plus algebra $(\mathbb{R} \cup \{+\infty\}, \min, +)$ and max-times algebra $(\mathbb{R}_+, \max, \times)$ respectively, where $\mathbb{R}_+ = \{r \in \mathbb{R} \,|\, r \geq 0\}$.

The binary operations of max-plus algebra could be extended to vectors and matrices over $\overline{\mathbb{R}}$ just like the ordinary linear algebra over $\mathbb{R}$. If $A = (a_{ij}), B = (b_{ij}) \in \overline{\mathbb{R}}^{n \times n}$ then $C = A \oplus B \in \overline{\mathbb{R}}^{n \times n}$, where $C = (c_{ij}), c_{ij} = a_{ij} \oplus b_{ij}$, and $D = A \otimes B$, where $D = (d_{ij}), d_{ij} = \sum_{\oplus \, k=1}^{n} a_{ik} \otimes b_{kj} = \max_{1 \leq k \leq n} (a_{ik} + b_{kj})$.

The vector in $\overline{\mathbb{R}}^n$ whose entries are all equal to $-\infty$ will be denoted by $-\infty$ again and called a *zero vector*, and similarly a *zero matrix*, denoted by $-\infty$, is the matrix with all of its entries consisted of $-\infty$. From now on all vectors and matrices in this work will be max-plus vectors and matrices defined over $\overline{\mathbb{R}}$.

3 Graph Theory Terminology

A *directed graph* or simply *digraph* is an ordered pair $D = (V, E)$ where V is a nonempty finite set of nodes and $E \subseteq V \times V$, a set of arcs. For a given digraph $D = (V, E)$, a sequence $\pi = (v_1, \ldots, v_p)$ of nodes in D is called a *path* in D if $(v_i, v_{i+1}) \in E$ for all i. A digraph $D = (V,\ E)$ is called *strongly connected* if there is a path from u to v for all pairs of nodes $(u, v) \in V \times V$, which means that there are paths between u and v bidirectionally for all $u \neq v \in V$. *Weighted digraphs* are digraphs with weights assigned to their arcs.

For a given weighted digraph $D = (V,\ E)$, we associate a corresponding $n \times n$ max-plus matrix $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$ whose (i, j) entry a_{ij} is a weight of the arc $(i, j) \in E$ of the digraph D.

Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$ be an $n \times n$ max-plus matrix. The digraph $D_A = (V(A), E(A))$, with a set of nodes $V(A) = \{1, 2, \ldots, n\}$ and a set of arcs $E(A) \subseteq V(A) \times V(A)$ with weights $w(i, j) = a_{ij} > -\infty$, is called the weighted digraph associated with the max-plus matrix A.

Suppose that $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$ and $\pi = (i_1, ..., i_p)$ is a path in D_A, then the *weight* of π is

$$w(\pi, A) = \begin{cases} a_{i_1 i_2} \otimes a_{i_2 i_3} \otimes \ldots \otimes a_{i_{p-1} i_p} & \text{if} \quad p > 1 \\ -\infty & \text{if} \quad p = 1 \end{cases}$$

The length of a path π, denoted by $\ell(\pi)$, is the number of arcs in π. π is called *finite* if $w(\pi, A) \neq -\infty$. A path which begins at i and ends at j is called an *i-j path*. If $\sigma = (i_1, ..., i_p)$, $p > 1$ and $i_1 = i_p$, then the path σ is called a *cycle*, and a cycle σ is called *elementary* if its nodes are all distinct except the initial and the final nodes. Each cycle could be decomposed as a product of elementary cycles. A loop at a node is considered to be a cycle of length 1.

The (i,j) entry of A^k, $(A^k)_{ij}$ represents the maximum weight of i-j paths of length k in D_A, and the entry $(A^k)_{ii}$ represents the maximum weight of cycles of length k which passes through the node i in D_A.

4 Maximum Cycle Mean and Transitive Closure

The *maximum cycle mean* of a matrix $A \in \overline{\mathbb{R}}^{n\times n}$, denoted by $\lambda(A)$, is defined by the formula

$$\lambda(A) = \max_{\sigma} \mu(\sigma, A), \tag{1}$$

where the maximum is taken over all elementary cycles σ in the associated digraph D_A and

$$\mu(\sigma, A) = \frac{1}{\ell(\sigma)}\, w(\sigma, A)$$

denotes the mean of the cycle σ, where $\ell(\sigma)$ is the length of the cycle σ.

The same maximum cycle mean could be obtained not just over elementary cycles but also over arbitrary cycles [3]. From the convention $\max\emptyset = -\infty$, it follows that $\lambda(A) = -\infty$ if and only if the associated digraph D_A is acyclic.

The following inequality is clear from the definition,

$$\lambda(A) \geq \mu(\sigma, A) \ \text{ and } \ \ell(\sigma)\,\lambda(A) \geq w(\sigma, A) \tag{2}$$

for each cycle σ in D_A.

We note that the maximum cycle mean of a matrix A, $\lambda(A)$ could also be described as

$$\sum_{k=1}^{d}{}_{\oplus}\, \frac{1}{k}\Big(\text{trace}(A^k)\Big),$$

where $\text{trace}(A^k) = \sum_{i=1}^{d}{}_{\oplus} (A^k)_{ii}$ represents the maximum weight of cycles of length k.

For a given $n \times n$ matrix $A \in \overline{\mathbb{R}}^{n\times n}$, let

$$\Gamma(A) = A \oplus A^2 \oplus A^3 \oplus \cdots \ \text{ and}$$
$$\Delta(A) = I \oplus A \oplus A^2 \oplus A^3 \oplus \cdots,$$

where I is an identity matrix in which $I_{ij} = -\infty$ if $i \neq j$ and $I_{ii} = 0 \ \forall\, i$. If the sum of these series converges to matrices that do not contain $+\infty$, then $\Gamma(A)$ is called the *weak transitive closure* of A and $\Delta(A)$ is called the *strong transitive closure* [3].

Suppose that $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ and $\pi = (i_1, ..., i_p)$ is a path in D_A. If the length of π is greater than or equal to n, that is, $\ell(\pi) \geq n$, then since the number of nodes of the digraph D_A is n, some of nodes of the path π must be repeated, which means that the path π contains a cycle. From this observation and the above inequality (2), we deduce the following lemma.

Lemma 4.1. [3] *Let* $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ *be a matrix with* $\lambda(A) \le 0$. *If at least one* i-j *path exists, then the heaviest* i-j *path not containing a cycle exists, where* $i, j \in V(A)$. *Furthermore the length of the heaviest path is less than* n *and*

$$A^k \le A \oplus A^2 \oplus \cdots \oplus A^{n-1} \quad \textit{for all} \quad k \ge 1,$$

where for $A = (a_{ij}), B = (b_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ *we denote* $A \le B$ *if* $a_{ij} \le b_{ij}$ *for each* $1 \le i, j \le n$.

Definition 4.1. A matrix $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$ with $\lambda(A) = 0$ is called *definite*. If $\lambda(A) = 0$ and further $a_{ii} = 0$ for all $1 \le i \le n$, then A is called *strongly definite*.

A matrix A is definite if and only if weights of all cycles in D_A are nonpositive and at least one has weight zero. Strongly definite matrices are the type of matrices we will focus on at the end of this work. Here are two results related with weak and strong transitive closures.

Proposition 4.2. [3] *If* $A \in \overline{\mathbb{R}}^{n\times n}$ *is strongly definite then*

$$\Delta(A) = \Gamma(A) = A^{n-1} = A^n = \cdots$$

Proposition 4.3. [3] *If* $A \in \overline{\mathbb{R}}^{n\times n}$ *and* $\lambda(A) \le 0$ *then*

1. $\Delta(A) = I \oplus A \oplus \cdots \oplus A^{n-1}$
2. $(\Delta(A))^k = \Delta(A)$, *for a positive integer* k.

We note that in contrast to $(A^k)_{ij}$, $\Gamma(A)_{ij}$ represents the weights of the heaviest i-j path of any length, if the infinite series converges or $\lambda(A) \le 0$. The next proposition gives a necessary and sufficient condition for a matrix to be a strong transitive closure.

Proposition 4.4. [3, 18] *Let* $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$. *The following are equivalent:*

1. *A is a strong transitive closure of some matrix;*
2. $\Delta(A) = A$;
3. $A^2 = A$ *and* $a_{ii} = 0$ *for all* $1 \le i \le n$.

Proof. $3 \Rightarrow 2$. By mathematical induction, $A^2 = A$ implies $A^k = A$, for all $k \ge 2$. And from the condition $a_{ii} = 0$ for all $i, I \oplus A = A$. Therefore $A = \Delta(A)$

$2 \Rightarrow 1$. Obvious.

$1 \Rightarrow 3$. This follows from Proposition 4.3, but we give its detailed proof.

Let $A = \Delta(B)$ for some matrix $B \in \overline{\mathbb{R}}^{n\times n}$. From the inequality

$$(I \oplus B \oplus \ldots \oplus B^k)^2 = I \oplus B \oplus \ldots \oplus B^{2k} \le \Delta(B) \quad \text{for all} \quad k \ge 1,$$

we conclude that $(\Delta(B))^2 = \Delta(B)$. Then $A^2 = (\Delta(B))^2 = \Delta(B) = A$. And $a_{ii} \ge 0$ for all $1 \le i \le n$ since $A = \Delta(B) = I \oplus B \oplus B^2 \oplus \cdots \ge I$. □

The maximum cycle mean of a matrix is of fundamental importance in max-plus algebra because for any square matrix $A \in \overline{\mathbb{R}}^{n\times n}$ it is the greatest max-plus eigenvalue of A, and every max-plus eigenvalue of A is the maximum cycle mean of some principal submatrix B of A [3]. A principal submatrix of A is a submatrix $A_K = (a_{st})$ of A where $s, t \in K \subseteq N$ for some subset K of $N = \{1, 2, \ldots, n\}$.

5 Max-Plus Permanent

Definition 5.1. Let $A \in \overline{\mathbb{R}}^{n\times n}$. The max-plus permanent of A is

$$\mathrm{mper}(A) = \sum\nolimits_{\oplus}_{\pi \in P_n} a_{1\pi(1)} \otimes a_{2\pi(2)} \otimes a_{3\pi(3)} \otimes \cdots \otimes a_{n\pi(n)},$$

where P_n is the set of all permutations of the set $N = \{\, 1, 2, 3, \ldots, n \,\}$.

Similarly to the weight of a path in a digraph D_A, the weight of the permutation $\pi \in P_n$ with respect to the matrix A is defined by

$$w(\pi, A) = a_{1\pi(1)} \otimes a_{2\pi(2)} \otimes a_{3\pi(3)} \otimes \cdots \otimes a_{n\pi(n)}$$

Then the max-plus permanent of A could also be expressed using the weights of permutations as

$$\mathrm{mper}(A) = \max_{\pi \in P_n} w(\pi, A),$$

the maximum weight of permutations $\pi \in P_n$ with respect to the matrix A. The permanent of a matrix is related to the assignment problem [3]. If a permutation $\pi \in P_n$ has its weight with respect to the matrix A equal to the max-plus permanent of the matrix A, that is $w(\pi, A) = \mathrm{mper}(A)$, then π is called an *optimal permutation*. The set of optimal permutations of a matrix A is denoted by

$$\mathrm{op}(A) = \{\, \pi \in P_n \mid w(\pi, A) = \mathrm{mper}(A) \,\}$$

If an optimal permutation to the matrix A is unique, that is, $|\mathrm{op}(A)| = 1$, then we say that the matrix A has a *strong permanent*. A matrix with nonpositive entries and zero diagonals is called *normal*. It is obvious that every normal matrix is strongly definite but not conversely.

There are many interesting results related to max-plus permanents of max-plus matrices. For those results, we refer to P. Butkovič [3].

6 Max-Plus Eigenspaces

Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$. For some $\lambda \in \overline{\mathbb{R}}$ and $x \in \overline{\mathbb{R}}^{n} \setminus \{-\infty\}$, if

$$A \otimes x = \lambda \otimes x,$$

then λ and x are called *max-plus eigenvalue* and *max-plus eigenvector* of A associated with λ respectively.

The next theorem explains some of the interplay between the maximum cycle mean $\lambda(A)$, the strong transitive closure $\Delta(A)$, and the max-plus eigenvalues. We give its detailed proof, even if it is well known, because it displays some of basic techniques in the study of max-plus eigenspaces.

Proposition 6.1. [3] *Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$. Then*

1. *the series $I \oplus A \oplus A^2 \oplus \ldots$ converges to a finite matrix $\Delta(A)$ if and only if $\lambda(A) \leq 0$, and then $\Delta(A) = I \oplus A \oplus A^2 \oplus \ldots \oplus A^{n-1}$ and $\lambda(\Delta(A)) = 0$;*
2. *$\lambda(A)$ is the greatest max-plus eigenvalue of A.*

Proof. 1. If $\lambda(A) \leq 0$, then from Lemma 4.1, $\Delta(A) \leq I \oplus A \oplus A^2 \oplus \ldots \oplus A^{n-1}$, and hence $\Delta(A) = I \oplus A \oplus A^2 \oplus \ldots \oplus A^{n-1}$. Now suppose that $\lambda(A) > 0$. Then there exists a cycle σ with a positive weight in D_A. If $(i,j) \in E(\sigma)$, the set of arcs of the cycle σ, then $(A^k)_{ij} \longrightarrow \infty$ as $k \longrightarrow \infty$. Hence the series $I \oplus A \oplus A^2 \oplus \ldots$ does not converge to a finite matrix.

Let $\lambda(A) \leq 0$ and $\sigma = (i_1, \ldots, i_p, i_1)$ be an arbitrary cycle in the associated digraph $D_{\Delta(A)}$ of $\Delta(A) = (\delta_{ij})$. Since $w(\sigma, \Delta(A)) = \delta_{i_1 i_2} \otimes \delta_{i_2 i_3} \otimes \ldots \otimes \delta_{i_{p-1} i_p} \otimes \delta_{i_p i_1}$ and each term $\delta_{i_j i_{j+1}}$ represents the weight of the heaviest i_j-i_{j+1} path of any length in the digraph D_A, $w(\sigma, \Delta(A))$ is especially the weight of one of cycles in the digraph D_A. Here we assumed that $i_j \neq i_{j+1}$ since $\delta_{i_j i_j} = 0$ for all j, and then from the fact that $\Delta(A) = I \oplus \Gamma(A)$, $\delta_{i_j i_{j+1}} \in \Gamma(A)$ for each j. From the inequality (2), $w(\sigma, \Delta(A)) \leq \ell(\sigma)\lambda(A) \leq 0$, which implies $\lambda(\Delta(A)) \leq 0$. From the condition $\lambda(A) \leq 0$, $\delta_{ii} = 0$ for all i. Since loops are considered to be cycles of length one, $\lambda(\Delta(A)) = 0$.

2. We show that $\lambda \leq \lambda(A)$ for all max-plus eigenvalue λ of A and $\lambda(A)$ is also a max-plus eigenvalue of A. Assume that $\lambda(A) \neq -\infty$ and $A \neq -\infty$ because if $\lambda(A) = -\infty$, then it is shown that the unique eigenvalue of A is $-\infty$ [3].

Suppose that $\lambda \in \overline{\mathbb{R}}$ is a max-plus eigenvalue of the matrix A, and $\lambda \neq -\infty$ since otherwise it is trivial. Then there exists a non-zero vector $x \in \overline{\mathbb{R}}^n \backslash \{-\infty\}$ such that $A \otimes x = \lambda \otimes x$, which can be written using conventional notations as

$$\max_{1\leq j\leq n} (a_{ij} + x_j) = \lambda + x_i, \ 1 \leq i \leq n, \tag{3}$$

which in turn tells us that for each $i \in N = \{1, \ldots, n\}$, there is a $j \in N$ such that

$$a_{ij} + x_j = \lambda + x_i$$

Since $x = (x_1, \ldots, x_n)^T \neq -\infty$, there exists an index $i_1 \in N$ that $x_{i_1} \neq -\infty$. Take $i = i_1$ in the above equation, then there are $i_2, i_3, \ldots \in N$ such that

$$\begin{aligned} a_{i_1 i_2} + x_{i_2} &= \lambda + x_{i_1} \\ a_{i_2 i_3} + x_{i_3} &= \lambda + x_{i_2} \\ &\vdots \end{aligned} \tag{4}$$

where $x_{i_1}, x_{i_2}, \ldots \neq -\infty$. Then $S = \{i_1, i_2, \ldots\}$ is a set of nodes of the associated digraph D_A, and contains a cycle σ. For the convenience, we may say that $\sigma = (i_1, i_2, \ldots, i_k, i_{k+1} = i_1)$. Then the last equation in (4) will be

$$a_{i_k i_1} + x_{i_1} = \lambda + x_{i_k}$$

Keeping in mind that both sides of (4) are all finite, we add them all side by side as follows

$$\begin{aligned} a_{i_1 i_2} + a_{i_2 i_3} + \cdots + a_{i_{k-1} i_k} + a_{i_k i_1} + x_{i_1 i_2} + x_{i_2 i_3} + \cdots + x_{i_{k-1} i_k} + x_{i_k i_1} \\ = k\lambda + x_{i_1 i_2} + x_{i_2 i_3} + \cdots + x_{i_{k-1} i_k} + x_{i_k i_1} \end{aligned}$$

After eliminating the common term from both sides and simplification, we get

$$\lambda = \frac{a_{i_1 i_2} + a_{i_2 i_3} + \cdots + a_{i_{k-1} i_k} + a_{i_k i_1}}{k} = \mu(\sigma, A) \leq \lambda(A)$$

We note that by taking a principal submatrix A_σ of A corresponding to nodes of the cycle σ, it is easy to confirm that $\lambda = \lambda(A_\sigma)$, the maximum cycle mean of the principal submatrix A_σ.

Next we show that the maximum cycle mean of A, $\lambda(A)$ is a max-plus eigenvalue of A. From the definition, there is a cycle $\tau = (j_1, j_2, \ldots, j_k, j_1)$ such that

$$\lambda(A) = \mu(\tau, A) = \frac{a_{j_1 j_2} + a_{j_2 j_3} + \cdots + a_{j_{k-1} j_k} + a_{j_k j_1}}{k}$$

If we apply a node set of τ, $\{j_1, j_2, \ldots, j_k, j_1\}$ to the equation (3), we get a system of inequalities instead of equalities like (4) as follows

$$\begin{aligned} a_{j_1 j_2} + x_{j_2} &\leq \lambda + x_{j_1} \\ a_{j_2 j_3} + x_{j_3} &\leq \lambda + x_{j_2} \\ &\vdots \\ a_{j_k j_1} + x_{j_1} &\leq \lambda + x_{j_k} \end{aligned}$$

After adding both sides and eliminating the common term as before, we get the inequality,

$$\lambda(A) = \mu(\tau, A) = \frac{a_{j_1 j_2} + a_{j_2 j_3} + \cdots + a_{j_{k-1} j_k} + a_{j_k j_1}}{k} \leq \lambda$$

Thus $\lambda(A) = \lambda$, a max-plus eigenvalue of A. □

The maximum cycle mean of a matrix is of fundamental importance in max-plus algebra because $\lambda(A)$ is the only possible max-plus eigenvalue corresponding to max-plus finite eigenvectors of A whose components are all finite. As pointed out in the above proposition's proof, every eigenvalue of A is the maximum cycle mean of some principal submatrix of A.

Note that $\lambda(A) = -\infty$ implies that A contains a zero vector column whose entries are all $-\infty$, and then eigenvectors are just vectors x with $x_i = -\infty$ whenever the corresponding i-th column $A_i \neq -\infty$. In what follows, we will not treat this trivial case and assume that $\lambda(A) \neq -\infty$ [3].

The spaces that we consider in max-plus algebra are subsets of $\overline{\mathbb{R}}^n$ closed under max-plus addition which is componentwise maximization, and max-plus scalar multiplication which is componentwise scalar addition. They are called *max-plus spaces*. For a given $n \times n$ max-plus matrix A, the set consisting of the zero vector $-\infty$ and all eigenvectors associated with λ, denoted by $V_\lambda(A)$, is a max-plus space, called the *max-plus eigenspace* of A. Further we denote by span(A) the *max-plus column span* of A, which is the set of max combinations

$$\sum_{i=1}^{n}{}_{\oplus}\, \lambda_i \otimes A_i\,,\ \ \lambda_i \in \overline{\mathbb{R}}$$

of the columns A_i of A. It is obvious that $V_\lambda(A) \subseteq$ span(A) for any matrix A.

A set S is called a *generating set* for a max-plus space W, written $W =$ span(S), if every vector $y \in W$ can be expressed as a max combination

$$y = \sum_{x \in S}{}_{\oplus}\, \lambda_x \otimes x, \ \ x \in S,\ \lambda_x \in \overline{\mathbb{R}},$$

where finite number of λ_x are finite. We set span$(\emptyset) = \{-\infty\}$.

S is called *dependent* if v is a max-plus combination of $S \setminus \{v\}$ for some $v \in S$. Otherwise S is *independent*. A set $S \subseteq \overline{\mathbb{R}}^n$ is called a *basis* of $T \subseteq \overline{\mathbb{R}}^n$ if it is an independent generating set for T. A vector $v \in S$ is called an *extremal* in S if $v = u \oplus w,\ u, w \in S$ implies that $v = u$ or $v = w$. For a non-zero vector $v = (v_1, \ldots, v_n)^T \in \overline{\mathbb{R}}^n$, the *norm* of v is $||v|| = \max_i v_i$, and the vector v is called *scaled* if $||v|| = 0$. A set S is called scaled if all of its elements are scaled [4].

It is known that if a subspace has a (scaled) basis then it must be the set of (scaled) extremals, hence the basis is essentially unique. In contrast to the ordinary linear algebra, a maximal independent set in a subspace $T \subseteq \overline{\mathbb{R}}^n$ may not be a basis for T [3].

The number of vectors in any basis of a finitely generated subspace $T \subseteq \overline{\mathbb{R}}^n$ is called the *dimension* of T, denoted by dim(T). However unlike in linear algebra, the dimensions of max-plus subspaces are not related to the numbers of components of the vectors in these subspaces [3].

Next we describe the eigenspace of a matrix $A \in \overline{\mathbb{R}}^{n\times n}$. For this purpose we need the following notions and notations.

Let $A \in \overline{\mathbb{R}}^{n\times n}$ be a given matrix. A cycle σ in the digraph D_A is called *critical* if $\mu(\sigma, A) = \lambda(A)$, and the nodes and the arcs of D_A that belong to critical cycles are called *critical*. The sets of critical nodes and critical arcs are denoted by $N_c(A)$ and $E_c(A)$ respectively. The *critical digraph* of A, denoted by $C(A)$, is the digraph which consists of all critical nodes and critical arcs of D_A, that is, $C(A) = (N_c(A), E_c(A))$.

Two critical nodes $i, j \in N_c(A)$ are called equivalent, denoted $i \sim j$, if i and j belong to the same critical cycle. Critical nodes and critical digraphs play a central role in determining the eigenspace of matrices.

The weak transitive closure $\Gamma(A)$ of $A \in \overline{\mathbb{R}}^{n\times n}$ was defined by the infinite series $A \oplus A^2 \oplus \cdots$, and from Lemma 4.1 and Proposition 6.1,

$$\Gamma(A) = A \oplus A^2 \oplus \cdots \oplus A^n \iff \lambda(A) \leq 0$$

Let $\Gamma(A) = [g_1\, g_2\, \cdots\, g_n]$, where g_i are columns of $\Gamma(A) = (\gamma_{ij})$. We note that if $\lambda(A) \leq 0$, which includes the case when A is definite, then γ_{ij} represents the heaviest weight of i-j paths in D_A. Furthermore if $\lambda(A) \leq 0$, then $\Gamma(A)$ plays an important role in the eigenspace as follows

$$A \otimes \Gamma(A) = A^2 \oplus A^3 \oplus \cdots \oplus A^{n+1} \leq \Gamma(A),$$

from which we get the inequalities

$$A \otimes g_j \leq g_j \quad \text{for all} \quad j \in N$$

The following lemma claims that these inequalities become equalities if A is definite and $j \in N_c(A)$. From the following lemma, we could deduce that the maximum cycle mean is a max-plus eigenvalue, which was shown at Proposition 6.1.

Lemma 6.2. [3] *For a matrix $A \in \overline{\mathbb{R}}^{n\times n}$, if A is definite and $j \in N_c(A)$, then*

$$A \otimes g_j = g_j$$

We note that for a definite matrix A, the (j, j)-diagonal entry of the weak transitive closure $\Gamma(A)$, γ_{jj} represents the heaviest weight of a cycle in D_A passing through the node j, thus critical nodes are nodes corresponding to zero diagonal entries of $\Gamma(A)$, that is,

$$N_c(A) = \{j \in N \mid \gamma_{jj} = 0\}$$

If A is a definite matrix, then since $\lambda(A) = 0$, from the definition of the maximum cycle mean, weights of cycles in D_A are all nonpositive and there is at least one cycle in D_A whose weight is zero. This tells us that $N_c(A) \neq \emptyset$, which implies that at least one of the main diagonals of $\Gamma(A)$ is zero. With the additional facts on the eigenspaces of definite matrices, we get to the following conclusion on the max-plus eigenspaces.

Theorem 6.3. [3] *Let $A \in \overline{\mathbb{R}}^{n\times n}$ be a definite matrix, and S denote a fixed set of indices of N such that for each connected component of the critical digraph $C(A)$, there corresponds a unique representative index of that component in S. Then the max-plus eigenspace of A, denoted by $V(A)$, is described by*

$$V(A) = \left\{ \sum_{i\in S} \lambda_i \otimes g_i \mid \lambda_i \in \mathbb{R} \right\},$$

where generators g_i are columns of $\Gamma(A)$, one from each connected components of $C(A)$.

If we denote the number of connected components of $C(A)$ by #$(C(A))$, then the above Theorem 6.3 yields the following corollary.

Corollary 6.4. *For any definite matrix $A \in \overline{\mathbb{R}}^{n\times n}$ with $\lambda(A) > -\infty$, the max-plus dimension of $V(A)$ is equal to #$(C(A))$.*

Now we focus on our main intersect in this work, strongly definite matrices. For a given max-plus matrix $A \in \overline{\mathbb{R}}^{n\times n}$ and an optimal permutation σ of A, S. Sergeev introduces a *strongly definite form* of a matrix A to be a strongly definite matrix $A' = (a'_{ij})$ where $a'_{ij} = a_{i\sigma(j)} \otimes a^{-1}_{j\sigma(j)}$ [18]. A strongly definite form A' of A is obtained by changing columns of a matrix A with scaling so that an optimal permutation σ becomes an identity permutation. There could be more than one strongly definite form of a matrix with more than one optimal permutation, but he claimed that

Proposition 6.5. [18] *Strong transitive closures of all strongly definite forms of any matrix with nonzero permanent coincide.*

For a vector $x \in \overline{\mathbb{R}}^{n}$, we name the index set $K = \{i \in N \,|\, x_i \neq -\infty\}$ to be the *support* of x, and denote it by supp(x).

Lemma 6.6. [18] *For a matrix $A = (a_{ij}) \in \overline{\mathbb{R}}^{n\times n}$, if A is definite, then*

$$x \in \overline{\mathbb{R}}^{n} \text{ is a max-plus eigenvector of } A \text{ and } supp(x) = K \Longleftrightarrow$$
$$a_{ij} = -\infty \text{ for all } i \in N \setminus K \text{ and all } j \in K.$$

Because of Lemma 6.6, S. Sergeev suggests that we may assume that max-plus eigenvectors to have the full support, which means that max-plus eigenvectors are vectors in $\mathbb{R}^n$, not in $\overline{\mathbb{R}}^{n}$. He then describes eigenspaces of strongly definite matrices by presenting the following propositions.

Proposition 6.7. [18] *Let $A = (a_{ij}) \in \overline{\mathbb{R}}^{n}$ be a strongly definite matrix. Then the max-plus eigenspace of A is*

$$V(A) = \{x \in \mathbb{R}^n \,|\, a_{ij} \leq x_i x_j^{-1},\ i \neq j,\ a_{ij} = a^*_{ij}\},$$

where a^*_{ij} is an (i, j) entry of a strong transitive closure of A, that is, $\Delta(A) = (a^*_{ij})$.

Let A be a strongly definite matrix, then diagonals of A are all 0. We introduce a new matrix $\tilde{A}$, the matrix obtained from A by replacing diagonals of A by $-\infty$.

Proposition 6.8. [18] *Let $A \in \overline{\mathbb{R}}^{n}$ be a strongly definite matrix and $\lambda(\tilde{A}) \neq -\infty$. Then*

$$A \text{ has a strong permanent.} \iff V(A) \text{ has an interior.}$$

We note that A has a strong permanent if the number of optimal permutations of A is only one, and the expression of $V(A)$ having an interior is that $V(A)$ contains an euclidean n dimensional part in it.

We are interested in Proposition 6.8 because it claims that the dimension of the eigenspace of a strongly definite matrix is full dimensional if the matrix has only one optimal permutation. Expecting that the number of optimal permutations might be related to the dimension of the max-plus eigenspace, we examined a number of max-plus matrices, using Maple program, then we found that the euclidean dimension of the max-plus eigenspace is not related with the number of optimal permutations of a max-plus matrix, but rather on the number of special optimal permutations as in the following main claim.

Definition 6.1. Let A be an $n \times n$ max-plus matrix. Then a transposition optimal permutation of A is a transposition $\sigma = (k\,\ell)$ which is an optimal permutation of A, where $1 \leq k \neq \ell \leq n$.

If $\sigma = (k\,\ell)$ is a transposition optimal permutation of a strongly definite matrix $A = (a_{ij})$, then $a_{k\ell} \otimes a_{\ell k} = \mathrm{mper}(A) = 0$. Especially $a_{k\ell} = -a_{\ell k}$.

In a strongly definite matrix, any optimal permutation can be decomposed into critical cycles. Conversely, any critical cycle can be extended to an optimal permutation, using the diagonal entries, P. Butkovič, H. Schneider, S. Sergeev [5].

Main Claim. *Let A be a strongly definite $n \times n$ max-plus matrix. Then the euclidean dimension of $V(A)$ is equal to $n - p$, where p is the number of transposition optimal permutations of A.*

The above claim is very intersecting because, as we pointed out before, the max-plus dimension is not like the dimension in ordinary linear algebra. We cannot expect any geometric information from the max-plus dimension. The main reason why we considered that a transposition optimal permutation $\sigma = (k\,\ell)$ plays a major factor in the dimension of the max-plus eigenspace is that in the equation of the eigenspace described in Proposition 6.7, a transposition optimal permutation $\sigma = (k\,\ell)$ combines two inequalities into an equality as follows

$$a_{k\ell} \leq x_k - x_\ell \quad \text{and} \quad a_{\ell\, k} \leq x_\ell - x_k \quad \Longrightarrow \quad a_{k\ell} = x_k - x_\ell,$$

using the conventional arithmetic operations.

In the following examples, M_1 in Example 1 does not have any transposition optimal permutation, and in Example 2, M_2 has one transposition optimal permutation, and lastly in Example 3, M_3 has two transposition optimal permutations. When we draw the figures of the definite eigensapces of max-plus matrices after scaling each vector so that its last coordinate becomes 0, we are able to find that each transposition optimal permutation indeed reduces the dimension of the definite eigensapce by one.

7 Examples of 4×4 Strongly Definite Matrices

Notation. We use the following notations for the next examples.

1. $\mathrm{ed}(M)$: euclidean dimension of the max-plus eigenspace of M.
2. $\mathrm{epd}(M)$: euclidean projective dimension of the max-plus eigenspace of M.
3. $\mathrm{md}(M)$: max-plus dimension of M.
4. M' : strongly definite form of M.
5. M^* : strong transitive closure of M'.

Example 7.1.

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & -1 & 0 & 0 \\ 0 & 1 & -5 & -1 \\ -1 & 2 & -3 & 1 \end{bmatrix} \quad M_1' = \begin{bmatrix} 0 & 0 & 1 & -2 \\ -3 & 0 & 2 & -3 \\ -4 & -5 & 0 & -1 \\ -2 & -3 & -1 & 0 \end{bmatrix} \quad (M_1')^* = \begin{bmatrix} 0 & \textcircled{0} & 2 & 1 \\ -1 & 0 & \textcircled{2} & 1 \\ -3 & -3 & 0 & \textcircled{-1} \\ \textcircled{-2} & -2 & 0 & 0 \end{bmatrix}$$

1. $\mathrm{mper}(M_1) = 5$.
2. Maximal permutations of M_1: $(4, 3, 1, 2)$.
3. Maximal permutations of M_1': $(1)(2)(3)(4)$.
4. Critical cycles of $(M_1')^*$: $(1), (2), (3), (4)$.
5. $\mathrm{ed}(M_1') = 4$, $\mathrm{epd}(M_1') = 3$, $\mathrm{md}((M_1')^*) = 4$
6. Boxed and circled entries are pairs (i, j) where $(M_1')_{ij} = (M_1')^*_{ij}$, $i \neq j$.
7. Boxed entries are a pair where $(M_1')^*_{ij} = -(M_1')^*_{ji}$.
8. The number of boxed pairs is zero.

Example 7.2.

$$M_2 = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 1 & 1 & 3 & 1 \\ 0 & -3 & -2 & -1 \\ -1 & -2 & 0 & 1 \end{bmatrix} \quad M_2' = \begin{bmatrix} 0 & -1 & 2 & -1 \\ 1 & 0 & 1 & 0 \\ -3 & -5 & 0 & -2 \\ -2 & -3 & -1 & 0 \end{bmatrix} \quad (M_2')^* = \begin{bmatrix} 0 & \boxed{-1} & \textcircled{2} & 0 \\ \boxed{1} & 0 & 3 & 1 \\ \textcircled{-3} & -4 & 0 & \textcircled{-2} \\ \textcircled{-2} & \textcircled{-3} & 0 & 0 \end{bmatrix}$$

1. $\mathrm{mper}(M_2) = 4$.
2. Maximal permutations of M_2: $(1, 2, 3)(4), (1, 3)(2)(4)$.
3. Maximal permutations of M_2': $(1)(2)(3)(4), (1, 2)(3)(4)$.
4. Critical cycles of $(M_2')^*$: $(1), (2), (3), (4), (1, 2)$.
5. $\mathrm{ed}(M_2') = 3$, $\mathrm{epd}(M_2') = 2$, $\mathrm{md}((M_2')^*) = 3$
6. Boxed and circled entries are pairs (i, j) where $(M_2')_{ij} = (M_2')^*_{ij}$, $i \neq j$.
7. Boxed entries are a pair where $(M_2')^*_{ij} = -(M_2')^*_{ji}$.
8. The number of boxed pairs is one.

Example 7.3.

$$M_3 = \begin{bmatrix} 0 & 0 & 1 & -1 \\ -1 & -1 & 0 & -1 \\ -2 & 0 & 1 & -2 \\ -2 & -1 & -3 & 0 \end{bmatrix} \quad M_3' = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & -1 & -1 \\ -2 & 1 & 0 & -2 \\ -2 & -3 & -1 & 0 \end{bmatrix} \quad (M_3')^* = \begin{bmatrix} 0 & \boxed{1} & \textcircled{0} & 0 \\ \boxed{-1} & 0 & \boxed{-1} & \textcircled{-1} \\ 0 & \boxed{1} & 0 & 0 \\ -1 & 0 & \textcircled{-1} & 0 \end{bmatrix}$$

1. $\text{mper}(M_3) = 0$.
2. Maximal permutations of M_3: $(1)(2)(3)(4), (2,3)(1)(4), (1,2)(3)(4), (1,3,2)(4)$.
3. Maximal permutations of M_3': $(1)(2)(3)(4), (2,3)(1)(4), (1,2)(3)(4), (1,3,2)(4)$.
4. Critical cycles of $(M_3')^*$: $(1), (2), (3), (4), (1,2), (2,3), (1,3,2), (1,3)$.
5. $\text{ed}(M_3') = 2$, $\text{epd}(M_3') = 1$, $\text{md}(M_3') = 2$
6. Boxed and circled entries are pairs (i,j) where $M_3' = (M_3')^*_{ij}$, $i \neq j$.
7. Boxed entries are the pairs where $(M_3')^*_{ij} = -(M_3')^*_{ij}$.
8. The number of boxed pairs is two.

8 Maple Codes

```
> restart:
> with(LinearAlgebra):
> with(combinat):
> with(ListTools):
> # mpmul returns a max-plus multiplication of two matrices.
> mpmul := proc(A,B)
   local m, n, l, C;
   m := RowDimension(A);
   n := ColumnDimension(A);
   l := ColumnDimension(B);
   C := Matrix(m,l,(i,j) -> max(seq(A[i,k]+B[k,j],k=1..n)));
   return C;
  end:
> # mpsum returns a max-plus addition of two matrices.
> mpsum := proc(A,B)
   local m, n, l, C;
   m := RowDimension(A);
   n := ColumnDimension(A);
   l := ColumnDimension(B);
   C := Matrix(m,l,(i,j) -> max(A[i,j],B[i,j]));
   return C;
  end:
> # mpper returns a max-plus permanent of a matrix, and produces
> # optimal permutations.
> mpper := proc(A,pos:=false)
```

```
  local n, P, C, c, ks, qs;
  n := RowDimension(A);
  P := permute(n);
  C := seq(add(A[i,p[i]],i=1 .. n), p in P);
  c := max(C);
  if pos then
    ks := SearchAll(c,[C]);
    qs := seq(P[k], k = ks);
    return c, qs;
  fi;
  return C;
 end:
> # Idd makes a max-plus identity matrix of a given size.
> Idd := proc(n::posint)
   local en;
   en := proc(i,j)
     if i <> j then
       -infinity;
     else
       0;
     fi;
   end:
   Matrix(n,n,(i,j) -> en(i,j));
 end:
> # mppow produces a max-plus power of a matrix up to a given degree.
> mppow := proc(A,n::posint)
   local i, C;
   C[0] := A;
     for i from 1 to n do
       C[i] := mpmul(A, C[i-1])
     od;
   return C[n];
 end:
> # wtc produces a weak transitive closure of a matrix.
> wtc := proc(A,n::posint)
   local m, i, C;
   m := RowDimension(A);
   C[0] := A;
     for i from 1 to n do
       C[i] := mpsum(mppow(A,i), C[i-1])
     od;
   return C[n];
 end:
> # stc produces a strong transitive closure of a matrix.
> stc := proc(A,n::posint)
```

```
   local m, i, C;
   m := RowDimension(A);
   C[0] := Idd(m);
     for i from 1 to n do
       C[i] := mpsum(mppow(A,i), C[i-1])
     od;
   return C[n];
 end:
> # mcm returns the maximum cycle mean of a matrix. This method is given
> # at page 19 [5].
> mcm := proc (A)
   local i, j, m, cm;
   m := RowDimension(A);
     for i from 1 to m do
       for j from 1 to m do
         cm := max((mppow(A, i))(j, j))
       od;
     od;
   return cm;
 end:
```

9 Acknowledgments

The authors would like to thank anonymous referees for valuable comments on the paper, and Hoon Hong for helping us to write maple codes.

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