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NORMAL WEIGHTED BERGMAN TYPE OPERATORS ON MIXED NORM SPACES OVER THE BALL IN $\mathbb{C}^n$

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Abstract. The paper studies some new $\mathbb{C}^n$-generalizations of Bergman type operators introduced by Shields and Williams depending on a normal pair of weight functions. We find the values of parameter β for which these operators are bounded on mixed norm spaces $L(p,q,\beta)$ over the unit ball in $\mathbb{C}^n$. Moreover, these operators are bounded projections as well, and the images of $L(p,q,\beta)$ under the projections are found.

1. Introduction and notation

Let $B = B_n := \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : \ |z| < 1\}$ be the open unit ball in $\mathbb{C}^n$, and $S := \partial B$ its boundary, the unit sphere. The inner product in $\mathbb{C}^n$ will be denoted as $\langle z, w\rangle := z_1\overline{w}_1 + \cdots + z_n\overline{w}_n$, $z, w \in \mathbb{C}^n$. Throughout the paper, it is assumed that $z = r\zeta$, $w = \rho\eta \in B$, $0 \le r, \rho < 1$, $\zeta, \eta \in S$, $r = |z| = \sqrt{\langle z, z\rangle}$.

Denote by $H(B)$ the set of all holomorphic functions in the ball B. The pth integral mean of a function $f(z) = f(r\zeta)$ given in B is denoted as usual by

$$M_p(f;r) = \big\|f(r\cdot)\big\|_{L^p(S;d\sigma)}, \qquad 0 \le r < 1,\ 0 < p \le \infty,$$

where $d\sigma$ is the $(2n-1)$-dimensional Lebesgue measure on the sphere S normalized so that $\sigma(S) = 1$. The class of holomorphic functions $f \in H(B)$, for which $\|f\|_{H^p} = \sup\limits_{0<r<1} M_p(f;r) < +\infty$, is the usual Hardy space $H^p(B)$.

The Banach space $L(p,q,\beta)$ $\big(1 \le p, q \le \infty,\ \beta \in \mathbb{R}\big)$ is the set of those functions $f(z) = f(r\zeta)$ given in B, for which the norm

$$\|f\|_{L(p,q,\beta)} := \begin{cases} \left(\displaystyle\int_0^1 (1-r)^{\beta q - 1} M_p^q(f;r)\,dr\right)^{1/q}, & 1 \le q < \infty, \\ \operatorname*{ess\,sup}\limits_{0<r<1} (1-r)^{\beta} M_p(f;r), & q = \infty, \end{cases}$$

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is finite. For the subspaces consisting of holomorphic functions let $H(p,q,\beta) := H(B)\cap L(p,q,\beta)$, $\beta>0$. For $p=q<\infty$, the spaces $H(p,p,\beta)=A^p_{\beta p-1}$ coincide with the well-known weighted Bergman spaces, while for $q=\infty$ they are known as weighted Hardy spaces.

The mixed norm spaces of holomorphic functions in the unit disc $\mathbb{D}$ were introduced by Hardy and Littlewood [9, 10] and developed later by Flett [8]. For more information about weighted Bergman spaces $A^p_\alpha = H\big(p,p,\frac{\alpha+1}{p}\big)$ on the unit disc, we refer to monographs [6, 7, 11]. A lot of works are devoted to the mixed norm and Bergman spaces of holomorphic functions in the ball B, and Bergman operators on them, see, e.g., [12, 18–20, 24]. Bergman type operators on mixed norm spaces of n-harmonic functions on the polydisc can be found in [2].

Throughout the paper, the letters $C(\alpha,\beta,\dots), C_\alpha$ etc. stand for positive different constants depending only on the parameters indicated. Let dV denote the Lebesgue measure on B normalized so that $V(B)=1$. In the polar coordinates, we have $dV(z)=2n\,r^{2n-1}drd\sigma(\zeta)$.

Instead of standard power weight functions, Shields and Williams [21] suggested to use more general normal weight functions. Actually, such weights are those having power majorants and minorants with positive exponents.

Definition 1.1 (Normal weight function [21])**.** A positive continuous function $\varphi(r)$, $0\le r<1$, is called normal if there are constants $0<a<b$ and $0\le r_0<1$ such that

$$(1)\qquad \frac{\varphi(r)}{(1-r)^a}\searrow 0 \qquad\text{and}\qquad \frac{\varphi(r)}{(1-r)^b}\nearrow +\infty \quad\text{as}\quad r\to 1^-,\quad r_0\le r<1.$$

Note that indices a and b for a normal function φ are not uniquely determined. Here and throughout this paper, the monotonicity is assumed in the essential sense. Recall that a function $\omega(r)>0$ is essentially (or almost) increasing on $[0,1)$ if there exists a constant $C>0$ such that $\omega(r_1)\le C\omega(r_2)$ for all $0\le r_1<r_2<1$. Essentially decreasing functions are defined similarly.

The typical and simple examples of normal functions are of type

$$\varphi_{c,d}(r)=(1-r)^c\left(\log\frac{e}{1-r}\right)^d,\qquad c>0,\ d\in\mathbb{R},$$

while for $c=0$, the function $\varphi_{0,d}=\left(\log\frac{e}{1-r}\right)^d$ is not normal.

Definition 1.2 (Normal pair [21])**.** Functions $\{\varphi,\psi\}$ form a normal pair if the function φ is normal and there exists a number α (the index of the pair), $\alpha>b-1$, such that

$$(2)\qquad \varphi(r)\,\psi(r)=(1-r^2)^\alpha,\qquad 0\le r<1.$$

The second function ψ is integrable on the interval $(0,1)$ because of the condition $\alpha>b-1$. As shown in [21], a normal function φ always has its pair, moreover under a stronger condition $\alpha>b$ the function ψ itself is also normal with indices $\alpha - b$ and $\alpha - a$. We extend the domain of weight functions up to the whole ball B by $\varphi(z) := \varphi(|z|) = \varphi(r),\ \psi(z) := \psi(|z|) = \psi(r)$.

Normal weight functions are closely related to other similar classes of weights such as Békollé–Bonami weights [4, 5], admissible weights [1, 22], secure and regular weights [13–15]. A lot of relevant information on nonstandard weight functions and corresponding Bergman spaces and operators can be found in the recently published works of Peláez and Rättyä [14, 15]. By means of normal weights Shields and Williams [21] suggested generalizations for Bergman operators in the unit disc $\mathbb{D} = B_1$. Their higher dimensional extensions to the unit ball B in $\mathbb{C}^n$ are defined in A. I. Petrosyan's papers [16, 17] in the form

$$P_{\varphi,\psi}(f)(z) := \int_B \frac{\varphi(z)\,\psi(w)}{(1-\langle z,w\rangle)^{n+1+\alpha}}\, f(w)\,dV(w), \qquad z \in B, \tag{3}$$

$$\widetilde{P}_{\varphi,\psi}(f)(z) := \int_B \frac{\varphi(z)\,\psi(w)}{|1-\langle z,w\rangle|^{n+1+\alpha}}\, f(w)\,dV(w), \qquad z \in B, \tag{4}$$

$$Q_{\varphi,\psi}(f)(z) := \int_B \frac{\psi(z)\,\varphi(w)}{(1-\langle z,w\rangle)^{n+1+\alpha}}\, f(w)\,dV(w), \qquad z \in B, \tag{5}$$

$$\widetilde{Q}_{\varphi,\psi}(f)(z) := \int_B \frac{\psi(z)\,\varphi(w)}{|1-\langle z,w\rangle|^{n+1+\alpha}}\, f(w)\,dV(w), \qquad z \in B. \tag{6}$$

Although the operators $P_{\varphi,\psi}, \widetilde{P}_{\varphi,\psi}$ seem to be very similar to $Q_{\varphi,\psi}, \widetilde{Q}_{\varphi,\psi}$, they differ in the fact that the second function ψ in general is not normal. Operators (3) and (4) in the limit case $\varphi \equiv 1,\ \psi(r) = (1-r^2)^\alpha$, as well as operators (5) and (6) in particular case $\varphi(r) = (1-r^2)^\alpha,\ \psi \equiv 1$ reduce to the classical Bergman projection P_α (see [6, 7, 11, 12, 18–20, 24]),

$$P_\alpha(f)(z) := \gamma_{\alpha,n} \int_B \frac{(1-|w|^2)^\alpha}{(1-\langle z,w\rangle)^{n+1+\alpha}}\, f(w)\,dV(w), \quad z \in B, \quad \alpha > -1, \tag{7}$$

where $\gamma_{\alpha,n} := \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\,\Gamma(\alpha+1)}$. For $\varphi(r) = (1-r^2)^\lambda,\ \psi(r) = (1-r^2)^\gamma,\ \lambda+\gamma = \alpha$, Bergman type operators (3)–(6) are well-known, too, see [2, 12, 18, 20, 24]. For projections P_α in the ball B, a representation holds,

$$f(z) = P_\alpha(f)(z), \qquad z \in B, \qquad \alpha > -1, \tag{8}$$

which is valid for all holomorphic functions f in the class $H(1, 1, \alpha+1) = A^1_\alpha$, see, for example, [6, Thm 6.1] or [24, Thm 2.2].

It is well known that Bergman (type) operators are widely applicable in many areas such as duality, complex interpolation, Toeplitz and Hankel operators, weighted Besov, Bloch and other spaces. So, it is natural to ask whether the general operators (3)–(6) are bounded on mixed norm spaces. In the present paper, given $1 \le p, q \le \infty$, we find conditions on the parameter β for general operators (3)–(6) to be bounded on the mixed norm spaces $L(p, q, \beta)$ in the ball B.

The main result of the paper is the following theorem of Forelli-Rudin type.

Theorem 1.3. *Suppose* $1 \le p, q \le \infty$, $\beta \in \mathbb{R}$, *and let* $\{\varphi, \psi\}$ *be a normal pair with indices* a *and* b $(0 < a < b)$, *and with the index of the pair* α $(\alpha > b-1)$ *in the sense of Definitions 1.1–1.2.*

(i) *If* $-a < \beta < 1 + \alpha - b$, *then the operators* $P_{\varphi,\psi}$ *and* $\widetilde{P}_{\varphi,\psi}$ *are bounded from the space* $L(p,q,\beta)$ *into itself, that is,*

$$P_{\varphi,\psi} : L(p,q,\beta) \longrightarrow L(p,q,\beta), \tag{9}$$

$$\widetilde{P}_{\varphi,\psi} : L(p,q,\beta) \longrightarrow L(p,q,\beta). \tag{10}$$

(ii) *If* $b - \alpha < \beta < 1 + a$, *then the operators* $Q_{\varphi,\psi}$ *and* $\widetilde{Q}_{\varphi,\psi}$ *are bounded from the space* $L(p,q,\beta)$ *into itself, that is,*

$$Q_{\varphi,\psi} : L(p,q,\beta) \longrightarrow L(p,q,\beta), \tag{11}$$

$$\widetilde{Q}_{\varphi,\psi} : L(p,q,\beta) \longrightarrow L(p,q,\beta). \tag{12}$$

Remark 1.4. In particular case $p = q = \infty$, $\beta = 0$, that is, for the space $L(\infty,\infty,0) = L^\infty(B)$ of essentially bounded functions in the ball, the relations (9) and (10) are proved in [16]. For $1 \le p = q = 1/\beta < \infty$, that is, for the unweighted space $L(p,p,1/p)$, the relations (11) and (12) are proved in [16,17] by a different method with the use of so called Schur test ([6,11,19,24]) which is not applicable in our case. More particular Bergman type operators with power weights are studied in [2,7,11,12,18–20,24]. Relations (11) and (12) are essentially proved in the recent paper [3].

Remark 1.5. In fact, in Theorem 1.3, we generalize the main result of [16,17] into three directions: first, we suppose all the values $1 \le p \le \infty$, second, weighted spaces are considered, and third, we consider more general mixed norm spaces $L(p,q,\beta)$.

2. Hardy and other integral inequalities

Classical Hardy's inequalities are well-known, see, for instance, [8,23],

$$\int_0^1 x^{-\beta-1}\left(\int_0^x h(t)\,dt\right)^p dx \le C(p,\beta)\int_0^1 x^{p-\beta-1}\,h^p(x)\,dx, \tag{13}$$

$$\int_0^1 (1-r)^{\beta-1}\left(\int_0^r h(t)\,dt\right)^p dr \le C(p,\beta)\int_0^1 (1-r)^{p+\beta-1}\,h^p(r)\,dr, \tag{14}$$

$$\int_0^1 (1-r)^{-\beta-1}\left(\int_r^1 h(t)\,dt\right)^p dr \le C(p,\beta)\int_0^1 (1-r)^{p-\beta-1}\,h^p(r)\,dr, \tag{15}$$

where $1 \le p < \infty$, $\beta > 0$, $h(r) \ge 0$.

Note that inequality (15) follows directly from (13) by a linear change of integration variables. For further proofs we need some generalizations of (14) and (15).

Lemma 2.1. *Suppose* $1 \le p < \infty$, $h(r) \ge 0$, *and for a positive and continuous function* $\varphi(r)$, $0 \le r < 1$, *there exist constants* $a, \gamma \in \mathbb{R}$, $\gamma + pa > 0$, *and* $0 \le r_0 < 1$ *such that*

$$\frac{\varphi(r)}{(1-r)^a} \searrow \qquad \text{as} \quad r_0 \le r < 1. \tag{16}$$

Then

$$\int_0^1 (1-r)^{\gamma-1}\, \varphi^p(r) \left(\int_0^r h(t)\, dt \right)^p dr \le C \int_0^1 (1-r)^{p+\gamma-1}\, \varphi^p(r)\, h^p(r)\, dr, \tag{17}$$

where the constant $C = C(p, \gamma, a, r_0) > 0$ *depends only on the parameters indicated.*

Proof. Applying Hardy's inequality (14) to the function $\frac{\varphi(r)}{(1-r)^a} h(r)$ and with the index $\beta = \gamma + pa > 0$, we have

$$\begin{aligned} &\int_0^1 (1-r)^{\gamma+pa-1} \left(\int_0^r \frac{\varphi(t)}{(1-t)^a} h(t)\, dt \right)^p dr \\ &\le C \int_0^1 (1-r)^{p+\gamma+pa-1} \left(\frac{\varphi(r)}{(1-r)^a} h(r) \right)^p dr, \end{aligned}$$

where the constant C depends only on p, γ, a.

Since the function $\frac{\varphi(r)}{(1-r)^a}$ is continuous on $[0, 1)$ and, by (16), essentially decreasing on $(r_0, 1)$,

$$\int_0^1 (1-r)^{\gamma+pa-1} \frac{\varphi^p(r)}{(1-r)^{pa}} \left(\int_0^r h(t)\, dt \right)^p dr \le C \int_0^1 (1-r)^{p+\gamma-1}\, \varphi^p(r)\, h^p(r)\, dr,$$

that coincides with (17). □

Remark 2.2. An inequality of Hardy type for normal functions can be found in [20].

We also need another version of inequality (17).

Lemma 2.3. *Suppose* $1 \le p < \infty$, $h(r) \ge 0$, *and for a positive and continuous function* $\varphi(r)$, $0 \le r < 1$, *there exist the constants* $b, \gamma \in \mathbb{R}$, $\gamma + pb < 0$, *and* $0 \le r_0 < 1$ *such that*

$$\frac{\varphi(r)}{(1-r)^b} \nearrow \qquad r_0 \le r < 1. \tag{18}$$

Then there exists a constant $C = C(p, \gamma, b, r_0) > 0$ *such that*

$$\int_0^1 (1-r)^{\gamma-1}\, \varphi^p(r) \left(\int_r^1 h(t)\, dt \right)^p dr \le C \int_0^1 (1-r)^{p+\gamma-1}\, \varphi^p(r)\, h^p(r)\, dr. \tag{19}$$

Proof. Applying Hardy's inequality (15) to the function $\frac{\varphi(r)}{(1-r)^b}\,h(r)$ and with the index $-\beta=\gamma+pb<0$, we have

$$\int_0^1 (1-r)^{\gamma+pb-1}\left(\int_r^1 \frac{\varphi(t)}{(1-t)^b}\,h(t)\,dt\right)^p dr$$
$$\le C\int_0^1 (1-r)^{p+\gamma+pb-1}\left(\frac{\varphi(r)}{(1-r)^b}\,h(r)\right)^p dr,$$

where the constant C depends only on p,γ,b. Since the function $\frac{\varphi(r)}{(1-r)^b}$ is continuous on $[0,1)$ and, by (18), essentially increasing on the interval $(r_0,1)$,

$$\int_0^1 (1-r)^{\gamma+pb-1}\frac{\varphi^p(r)}{(1-r)^{pb}}\left(\int_r^1 h(t)\,dt\right)^p dr\le C\int_0^1 (1-r)^{p+\gamma-1}\,\varphi^p(r)\,h^p(r)\,dr,$$

that coincides with (19). □

Lemma 2.4 ([19,24])**.** *For $\alpha>0$, there holds the inequality*

$$\int_S \frac{d\sigma(\xi)}{|1-\langle z,\xi\rangle|^{n+\alpha}}\le\frac{C(\alpha,n)}{(1-|z|)^\alpha},\qquad z\in B.$$

Lemma 2.5 ([21])**.** *For $m>\beta>0$, there holds the inequality*

$$\int_0^1 \frac{(1-\rho)^{\beta-1}}{(1-r\rho)^m}\,d\rho\le\frac{C(\beta,m)}{(1-r)^{m-\beta}},\qquad 0\le r<1.$$

Various variants of the next lemma can be found in [16–18, 21].

Lemma 2.6. *Let $\{\varphi,\psi\}$ be a normal pair with indices a and b $(0<a<b)$ and the index of the pair α $(\alpha>b-1)$ in the sense of Definitions 1.1–1.2.*

If $-a<\beta<1+\alpha-b$, then

$$\int_0^1 \frac{\psi(\rho)}{(1-r\rho)^{1+\alpha}(1-\rho)^\beta}\,d\rho\le C(\alpha,\beta,a,b,r_0)\,\frac{\psi(r)}{(1-r)^{\alpha+\beta}},\quad 0\le r<1. \tag{20}$$

Proof. The condition $\beta<1+\alpha-b$ guarantees the convergence of the integral in (20). It suffices to prove inequality (20) for r close to 1. Observe that the normality condition (1) for φ implies that the second function ψ satisfies similar conditions, namely there exists a constant $0\le r_0<1$ such that

$$\frac{\psi(r)}{(1-r)^{\alpha-b}}\searrow 0\quad\text{and}\quad\frac{\psi(r)}{(1-r)^{\alpha-a}}\nearrow+\infty\quad\text{as}\quad r\to 1^-,\ r_0\le r<1. \tag{21}$$

Despite conditions (21), the function ψ in general is not normal since the exponent $\alpha-b>-1$ can be non-positive.

We now split the integral in (20) into three parts,

$$J:=\int_0^1 \frac{\psi(\rho)}{(1-r\rho)^{1+\alpha}(1-\rho)^\beta}\,d\rho$$
$$=\left(\int_0^{r_0}+\int_{r_0}^r+\int_r^1\right)\frac{\psi(\rho)}{(1-r\rho)^{1+\alpha}(1-\rho)^\beta}\,d\rho=:J_1+J_2+J_3.$$

Integral J_1 is bounded by a constant $C(\alpha, \beta, r_0)$. Conditions (21) combined with Lemma 2.5 lead to estimations for the integrals J_2 and J_3,

$$\begin{aligned} J_2 &= \int_{r_0}^{r} \frac{\psi(\rho)}{(1-\rho)^{\alpha-a}} \, \frac{(1-\rho)^{\alpha-a}}{(1-r\rho)^{1+\alpha}(1-\rho)^{\beta}} \, d\rho \\ &\leq C \frac{\psi(r)}{(1-r)^{\alpha-a}} \int_{r_0}^{r} \frac{(1-\rho)^{\alpha-a-\beta}}{(1-r\rho)^{1+\alpha}} \, d\rho \\ &\leq C(\alpha, \beta, a) \, \frac{\psi(r)}{(1-r)^{\alpha+\beta}}, \end{aligned}$$

since $-a < \beta < 1+\alpha-b < 1+\alpha-a$. By analogy with the above, an estimation of the integral J_3 gives

$$\begin{aligned} J_3 &= \int_{r}^{1} \frac{\psi(\rho)}{(1-\rho)^{\alpha-b}} \, \frac{(1-\rho)^{\alpha-b}}{(1-r\rho)^{1+\alpha}(1-\rho)^{\beta}} \, d\rho \\ &\leq C \frac{\psi(r)}{(1-r)^{\alpha-b}} \int_{r}^{1} \frac{(1-\rho)^{\alpha-b-\beta}}{(1-r\rho)^{1+\alpha}} \, d\rho \\ &\leq C(\alpha, \beta, b) \, \frac{\psi(r)}{(1-r)^{\alpha+\beta}}, \end{aligned}$$

and this completes the proof of Lemma 2.6. □

The proofs of the following three lemmas below are similar to those of Lemmas 2.1, 2.3 and 2.6, and therefore they will be omitted.

Lemma 2.7. *Suppose* $1 \leq p < \infty$, $h(r) \geq 0$, *and for a positive and continuous function* $\varphi(r)$, $0 \leq r < 1$, *there exist constants* $b, \gamma \in \mathbb{R}$, $\gamma - pb > 0$, *and* $0 \leq r_0 < 1$ *such that*

$$\frac{\varphi(r)}{(1-r)^{b}} \nearrow \qquad \text{as} \quad r_0 \leq r < 1.$$

Then

$$\int_0^1 \frac{(1-r)^{\gamma-1}}{\varphi^p(r)} \left(\int_0^r h(t) \, dt \right)^p dr \leq C(p, \gamma, b, r_0) \int_0^1 \frac{(1-r)^{p+\gamma-1}}{\varphi^p(r)} \, h^p(r) \, dr.$$

Lemma 2.8. *Suppose* $1 \leq p < \infty$, $h(r) \geq 0$, *and for a positive and continuous function* $\varphi(r)$, $0 \leq r < 1$, *there exist constants* $a, \gamma \in \mathbb{R}$, $\gamma - pa < 0$, *and* $0 \leq r_0 < 1$ *such that*

$$\frac{\varphi(r)}{(1-r)^{a}} \searrow \qquad \text{as} \quad r_0 \leq r < 1.$$

Then

$$\int_0^1 \frac{(1-r)^{\gamma-1}}{\varphi^p(r)} \left(\int_r^1 h(t) \, dt \right)^p dr \leq C(p, \gamma, a, r_0) \int_0^1 \frac{(1-r)^{p+\gamma-1}}{\varphi^p(r)} \, h^p(r) \, dr.$$

Lemma 2.9. *Let φ be a normal function with indices a and b $(0 < a < b)$ and the index of the pair α $(\alpha > b-1)$ in the sense of Definitions 1.1–1.2.*

If $b-\alpha < \beta < 1+a$, then

$$\int_0^1 \frac{\varphi(\rho)}{(1-r\rho)^{1+\alpha}(1-\rho)^{\beta}}\, d\rho \le C(\alpha,\beta,a,b,r_0)\, \frac{\varphi(r)}{(1-r)^{\alpha+\beta}}, \qquad 0 \le r < 1.$$

3. Boundedness of Bergman type operators on mixed norm spaces

We proceed to the proof of the main theorem of this paper.

Lemma 3.1. *Let $1 \le p \le \infty$, $\alpha > -1$, and $\{\varphi,\psi\}$ be a couple of positive weight functions. Then there exists a constant $C = C(p,n,\alpha) > 0$ such that*

$$M_p\big(\widetilde{P}_{\varphi,\psi}(f); r\big) \le C\, \varphi(r) \int_0^1 \frac{\psi(\rho)}{(1-r\rho)^{1+\alpha}}\, M_p(f;\rho)\, d\rho, \quad 0 \le r < 1, \tag{22}$$

$$M_p\big(\widetilde{Q}_{\varphi,\psi}(f); r\big) \le C\, \psi(r) \int_0^1 \frac{\varphi(\rho)}{(1-r\rho)^{1+\alpha}}\, M_p(f;\rho)\, d\rho, \quad 0 \le r < 1. \tag{23}$$

Proof. A passage to the polar coordinates in the integral representation of $\widetilde{P}_{\varphi,\psi}(f)(z)$ yields

$$\begin{aligned} \big|\widetilde{P}_{\varphi,\psi}(f)(r\zeta)\big| &\le \varphi(z) \int_B \frac{\psi(w)}{|1-\langle r\zeta, w\rangle|^{n+1+\alpha}}\, |f(w)|\, dV(w) \\ &= 2n\, \varphi(r) \int_0^1 \left[\int_S \frac{|f(\rho\eta)|}{|1-\langle r\zeta, \rho\eta\rangle|^{n+1+\alpha}}\, d\sigma(\eta) \right] \psi(\rho)\, \rho^{2n-1}\, d\rho \\ &= 2n\, \varphi(r) \int_0^1 g(r,\rho,\zeta)\, \psi(\rho)\, \rho^{2n-1}\, d\rho, \end{aligned} \tag{24}$$

with the temporary notation

$$g(r,\rho,\zeta) := \int_S \frac{|f(\rho\eta)|}{|1-\langle r\zeta, \rho\eta\rangle|^{n+1+\alpha}}\, d\sigma(\eta).$$

For $p = \infty$, by (24) and Lemma 2.4, we immediately obtain

$$\begin{aligned} &M_\infty\big(\widetilde{P}_{\varphi,\psi}(f); r\big) \\ &\le 2n\, \varphi(r) \int_0^1 M_\infty(f;\rho) \sup_{\zeta\in S} \left[\int_S \frac{d\sigma(\eta)}{|1-\langle r\zeta, \rho\eta\rangle|^{n+1+\alpha}} \right] \psi(\rho)\, \rho^{2n-1}\, d\rho \\ &\le C(n,\alpha)\, \varphi(r) \int_0^1 \frac{\psi(\rho)}{(1-r\rho)^{1+\alpha}}\, M_\infty(f;\rho)\, d\rho. \end{aligned}$$

For $p = 1$, by integrating (24) with the use of Fubini's theorem and Lemma 2.4, we get the desired inequality (22).

For $1 < p < \infty$, an application of Hölder's inequality and Lemma 2.4 implies

$$g(r,\rho,\zeta) \le \left(\int_S \frac{|f(\rho\eta)|^p\, d\sigma(\eta)}{|1-\langle r\zeta, \rho\eta\rangle|^{n+1+\alpha}} \right)^{1/p} \left(\int_S \frac{d\sigma(\eta)}{|1-\langle r\zeta, \rho\eta\rangle|^{n+1+\alpha}} \right)^{1/p'}$$

$$\leq \frac{C(p,n,\alpha)}{(1-r\rho)^{(1+\alpha)/p'}} \left(\int_S \frac{|f(\rho\eta)|^p \, d\sigma(\eta)}{|1-\langle r\zeta, \rho\eta\rangle|^{n+1+\alpha}} \right)^{1/p},$$

where p' is the exponent conjugate to p, $1/p + 1/p' = 1$. Next, we integrate in ζ over the sphere S and again apply Lemma 2.4,

$$\begin{aligned} &\big\|g(r,\rho,\cdot)\big\|^p_{L^p(S;d\sigma)} \\ &\leq \frac{C(p,n,\alpha)}{(1-r\rho)^{(1+\alpha)p/p'}} \int_S \left(\int_S \frac{d\sigma(\zeta)}{|1-\langle r\zeta,\rho\eta\rangle|^{n+1+\alpha}} \right) |f(\rho\eta)|^p \, d\sigma(\eta) \\ &\leq \frac{C(p,n,\alpha)}{(1-r\rho)^{(1+\alpha)p/p'}(1-r\rho)^{1+\alpha}} \int_S |f(\rho\eta)|^p \, d\sigma(\eta) \\ &= \frac{C(p,n,\alpha)}{(1-r\rho)^{(1+\alpha)p}} M_p^p(f;\rho). \end{aligned} \tag{25}$$

Now we can return to inequality (24) and apply to it Minkowski's inequality and estimate (25) to get

$$\begin{aligned} M_p\big(\widetilde{P}_{\varphi,\psi}(f);r\big) &\leq 2n\,\varphi(r) \int_0^1 \big\|g(r,\rho,\cdot)\big\|_{L^p(S;d\sigma)} \psi(\rho)\,\rho^{2n-1}\,d\rho \\ &\leq C(p,n,\alpha)\,\varphi(r) \int_0^1 \frac{\psi(\rho)}{(1-r\rho)^{1+\alpha}} M_p(f;\rho)\,d\rho. \end{aligned}$$

This completes the proof of (22). Inequality (23) can be proved in the same manner. □

3.1. Proof of Theorem 1.3

Since $\big|P_{\varphi,\psi}(f)(z)\big| \leq \widetilde{P}_{\varphi,\psi}(|f|)(z)$, it suffices to prove the boundedness of the operator $\widetilde{P}_{\varphi,\psi}(|f|)$, that is, the relation (10).

First, suppose $1 \leq q < \infty$. Having proved the estimate (22) in Lemma 3.1, we can integrate (22) in the radial variable to obtain the mixed norm

$$\begin{aligned} \big\|\widetilde{P}_{\varphi,\psi}(f)\big\|^q_{L(p,q,\beta)} &= \int_0^1 (1-r)^{\beta q-1} M_p^q\big(\widetilde{P}_{\varphi,\psi}(f);r\big)\,dr \\ &\leq C \int_0^1 (1-r)^{\beta q-1}\,\varphi^q(r) \left[\int_0^1 \frac{\psi(\rho)}{(1-r\rho)^{1+\alpha}} M_p(f;\rho)\,d\rho \right]^q dr. \end{aligned}$$

We now break up the integral into its two components

$$\begin{aligned} &\big\|\widetilde{P}_{\varphi,\psi}(f)\big\|^q_{L(p,q,\beta)} \\ &\leq C \int_0^1 (1-r)^{\beta q-1}\,\varphi^q(r) \left[\left(\int_0^r + \int_r^1 \right) \frac{\psi(\rho)\,M_p(f;\rho)}{(1-r\rho)^{1+\alpha}}\,d\rho \right]^q dr \\ &\leq C \int_0^1 (1-r)^{\beta q-1}\,\varphi^q(r) \left[\int_0^r \frac{\psi(\rho)}{(1-r\rho)^{1+\alpha}} M_p(f;\rho)\,d\rho \right]^q dr \\ &\quad + C \int_0^1 (1-r)^{\beta q-1}\,\varphi^q(r) \left[\int_r^1 \frac{\psi(\rho)}{(1-r\rho)^{1+\alpha}} M_p(f;\rho)\,d\rho \right]^q dr \end{aligned}$$

$$=: I_1 + I_2. \tag{26}$$

Integrals I_1 and I_2 will be estimated separately by using Hardy's type inequalities in Lemmas 2.3 and 2.1. Since the assumption $\beta q + aq > 0$ is equivalent to $\beta > -a$, we may apply (17) to the integral I_1,

$$\begin{aligned} I_1 &\le C \int_0^1 (1-r)^{\beta q - 1 + q} \varphi^q(r) \left[\frac{\psi(r)}{(1-r^2)^{1+\alpha}} M_p(f;r) \right]^q dr \\ &\le C \int_0^1 (1-r)^{\beta q - 1} M_p^q(f;r)\, dr \\ &= C(n,p,q,\beta,\alpha,a,r_0)\, \|f\|^q_{L(p,q,\beta)}. \end{aligned} \tag{27}$$

Since the assumption $\beta q - \alpha q - q + bq < 0$ is equivalent to $\beta < 1 + \alpha - b$, we may apply (19) to the integral I_2 to get

$$\begin{aligned} I_2 &\le C \int_0^1 (1-r)^{\beta q - \alpha q - q - 1} \varphi^q(r) \left[\int_r^1 \psi(\rho)\, M_p(f;\rho)\, d\rho \right]^q dr \\ &\le C \int_0^1 (1-r)^{\beta q - \alpha q - q - 1 + q} \varphi^q(r) \Big[\psi(r)\, M_p(f;r) \Big]^q dr \\ &\le C \int_0^1 (1-r)^{\beta q - 1} M_p^q(f;r)\, dr \\ &= C(n,p,q,\beta,\alpha,b,r_0)\, \|f\|^q_{L(p,q,\beta)}. \end{aligned} \tag{28}$$

Inequalities (26), (27) and (28) together yield

$$\big\| \widetilde{P}_{\varphi,\psi}(f) \big\|_{L(p,q,\beta)} \le C\, \|f\|_{L(p,q,\beta)},$$

where the constant $C = C(n,p,q,\beta,\alpha,a,b,r_0) > 0$ depends only on the parameters indicated.

Suppose now $q = \infty$. From inequality (22) with the use of Lemma 2.9, we conclude

$$\begin{aligned} M_p\big(\widetilde{P}_{\varphi,\psi}(f);r\big) &\le C(p,n,\alpha)\, \varphi(r) \int_0^1 \frac{\psi(\rho)}{(1-r\rho)^{1+\alpha}(1-\rho)^{\beta}} (1-\rho)^{\beta} M_p(f;\rho)\, d\rho \\ &\le C\, \varphi(r)\, \|f\|_{L(p,\infty,\beta)} \int_0^1 \frac{\psi(\rho)}{(1-r\rho)^{1+\alpha}(1-\rho)^{\beta}}\, d\rho \\ &\le C\, \varphi(r)\, \|f\|_{L(p,\infty,\beta)} \frac{\psi(r)}{(1-r)^{\alpha+\beta}} \\ &\le C(p,n,\alpha,\beta,a,b,r_0)\, \|f\|_{L(p,\infty,\beta)} \frac{1}{(1-r)^{\beta}}. \end{aligned}$$

Therefore

$$\big\| \widetilde{P}_{\varphi,\psi}(f) \big\|_{L(p,\infty,\beta)} \le C\, \|f\|_{L(p,\infty,\beta)},$$

where the constant $C = C(p,n,\alpha,\beta,a,b,r_0) > 0$ depends only on the parameters indicated.

This completes the proof of (9) and (10). The last two relations (11) and (12) can be proved in the same manner and so omitted. □

4. Bounded projections

In this section, we show that $P_{\varphi,\psi}$ and $Q_{\varphi,\psi}$ are not only bounded operators in $L(p,q,\beta)$, but bounded projections as well. We also find the images of $L(p,q,\beta)$ under the projections $P_{\varphi,\psi}$ and $Q_{\varphi,\psi}$.

To this end, define more general mixed norm spaces

$$H^{p,q}_{\beta}(\varphi):=\left\{f\in H(B):\|f\|^q_{p,q,\beta,\varphi}:=\int_0^1(1-r)^{\beta q-1}\,\varphi^q(r)M_p^q(f;r)dr<+\infty\right\},$$

where $1\le p,q\le\infty,\ \beta\in\mathbb{R}$ and φ is a (normal) weight function. In the case $q=\infty$, as usual, the integral norm is to be interpreted as a supremum

$$\|f\|_{p,\infty,\beta,\varphi}:=\sup_{0<r<1}(1-r)^{\beta}\,\varphi(r)\,M_p(f;r),\qquad 1\le p\le\infty,\ q=\infty,\ \beta\in\mathbb{R}.$$

For $\varphi\equiv 1$, evidently we have $H^{p,q}_{\beta}(1)=H(p,q,\beta)$.

Introduce a multiplication operator Π_φ on $H^{p,q}_{\beta}(\varphi)$ with symbol φ and consider the resulting class,

$$\Pi_\varphi(g):=\varphi g\quad\text{for}\quad g\in H^{p,q}_{\beta}(\varphi),\qquad \Pi_\varphi H^{p,q}_{\beta}(\varphi)=\varphi\cdot H^{p,q}_{\beta}(\varphi).$$

It is not hard to see that

$$\Pi_\varphi H^{p,q}_{\beta}(\varphi)\subset L(p,q,\beta).$$

Indeed, by the above definitions, we have

(29)

$$f\in\Pi_\varphi H^{p,q}_{\beta}(\varphi)\iff f=\varphi g,\ g\in H^{p,q}_{\beta}(\varphi)\iff f=\varphi g\in L(p,q,\beta),\ g\in H(B).$$

Lemma 4.1. *For $1\le p,q\le\infty,\ 0<\delta<\alpha+1$, the inclusion*

$$H(p,q,\delta)\subset H(1,1,\alpha+1)$$

is continuous.

Proof. The widest mixed norm space here appears when $p=1$ and $q=\infty$ so that $H(p,q,\delta)\subset H(1,\infty,\delta)$ with the norm inequality. It remains to note only that

$$\|f\|_{L(1,1,\alpha+1)}=\int_0^1(1-r)^{\alpha-\delta}(1-r)^{\delta}M_1(f;r)\,dr\le\frac{1}{1+\alpha-\delta}\,\|f\|_{L(1,\infty,\delta)}.$$ □

Theorem 4.2. *Let $1\le p,q\le\infty,\ \beta\in\mathbb{R}$, and $\{\varphi,\psi\}$ be a normal pair of functions with indices a and b $(0<a<b)$ and with the index of the pair α $(\alpha>b-1)$ in the sense of Definitions 1.1–1.2.*

(i) *If $-a<\beta<1+\alpha-b$, then the operator $P_{\varphi,\psi}$ projects the space $L(p,q,\beta)$ boundedly onto $\Pi_\varphi H^{p,q}_{\beta}(\varphi)$,*

$$P_{\varphi,\psi}:L(p,q,\beta)\xrightarrow{\text{onto}}\Pi_\varphi H^{p,q}_{\beta}(\varphi).$$

(ii) *If* $b-\alpha<\beta<1+a$, *then the operator* $Q_{\varphi,\psi}$ *projects the space* $L(p,q,\beta)$ *boundedly onto* $\Pi_\psi H^{p,q}_\beta(\psi)$,

$$Q_{\varphi,\psi} : L(p,q,\beta) \xrightarrow{\text{onto}} \Pi_\psi H^{p,q}_\beta(\psi).$$

Proof. (i) According to Theorem 1.3 and (9), $P_{\varphi,\psi}\big(L(p,q,\beta)\big) \subset L(p,q,\beta)$. Let us show that $P_{\varphi,\psi} f \in \Pi_\varphi H^{p,q}_\beta(\varphi)$ for any function $f \in L(p,q,\beta)$. Indeed,

$$P_{\varphi,\psi}(f)(z) = \varphi(z) \int_B \frac{\psi(w)}{(1-\langle z,w\rangle)^{n+1+\alpha}}\, f(w)\, dV(w) = \varphi(z)\, P_{1,\psi}(f)(z),$$

where $P_{1,\psi}(f)(z)$ is a holomorphic function, and $P_{\varphi,\psi}(f)(z) \in L(p,q,\beta)$. In view of (29), this means that $P_{\varphi,\psi} f \in \Pi_\varphi H^{p,q}_\beta(\varphi)$.

Let us now prove the projection is onto, specifically $f = P_{\varphi,\psi} f \quad \forall\ f \in \Pi_\varphi H^{p,q}_\beta(\varphi)$, that is, $P_{\varphi,\psi}$ is the identity on $\Pi_\varphi H^{p,q}_\beta(\varphi)$. Indeed, assume that f is an arbitrary function in the class $\Pi_\varphi H^{p,q}_\beta(\varphi)$, or, in view of (29), equivalently $f = \varphi g$ $g \in H^{p,q}_\beta(\varphi)$. Then

$$\begin{aligned} P_{\varphi,\psi}(f)(z) &= P_{\varphi,\psi}(\varphi g)(z) \\ &= \varphi(z) \int_B \frac{\varphi(w)}{(1-\langle z,w\rangle)^{n+1+\alpha}}\, \psi(w)\, g(w)\, dV(w) \\ &= \varphi(z) \int_B \frac{(1-|w|^2)^\alpha}{(1-\langle z,w\rangle)^{n+1+\alpha}}\, g(w)\, dV(w) \\ &= \varphi(z)\, P_\alpha(g)(z) = \varphi(z)\, g(z) = f(z). \end{aligned}$$

Here P_α is the classical projection (7), while the identity $P_\alpha(g) = g$, see (8), follows from Lemma 4.1 and the inclusion

$$g \in H^{p,q}_\beta(\varphi) \subset H^{p,q}_\beta\big((1-r^2)^b\big) = H(p,q,\beta+b) \subset H(1,1,\alpha+1) \tag{30}$$

with the assumption $0 < b-a < \beta+b < 1+\alpha$. Thus, Part (i) of Theorem 4.2 is proved.

Part (ii) can be proved in the same way, the only difference is that instead of (30), we apply another inclusion

$$g \in H^{p,q}_\beta(\psi) \subset H^{p,q}_\beta\big((1-r^2)^{\alpha-a}\big) = H(p,q,\beta+\alpha-a), \tag{31}$$

with the assumption $0 < b-a < \beta+\alpha-a < 1+\alpha$. This completes the proof of Theorem 4.2. □

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