J. Korean Math. Soc. **55** (2018), No. 2, pp. 253–292
https://doi.org/10.4134/JKMS.j170037
pISSN: 0304-9914 / eISSN: 2234-3008

A GENERAL RICCI FLOW SYSTEM

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Abstract. In this paper, we introduce a general Ricci flow system, which is closely linked with the Ricci flow and the renormalization group flow, etc. We prove the short-time existence, the entropy functionals, the higher derivatives estimates and the compactness theorem for this general Ricci flow system on closed Riemannian manifolds. These basic results are useful tools to understand the singularities of this system.

1. Introduction

Let (M^n, g) be a closed n-dimensional Riemannian manifold. Consider B be a local 2-form defined up to the addition of an exact 2-form on M, and $H = dB$ denotes a well-defined 3-form on M. Then the B-fields can be introduced, see [8, 15, 17]. Furthermore, let u be a smooth function on M, sometimes the function u is also called the Lapse function. We say that a family $(M, g(t), H(t), u(t))$ is a solution to a general Ricci flow system (GRF system for short) if

$$\begin{aligned} \partial_t g &= -2Rc + h/2 + 2\alpha_n du \otimes du \\ \partial_t H &= \Delta_{LB} H \qquad \text{and} \quad \partial_t u = \Delta u, \end{aligned} \tag{1.1}$$

where Rc is the Ricci curvature of the manifold M, h a two-form, written in a local coordinates as $h_{ij} = g^{kl} g^{mn} H_{ikm} H_{jln}$, α_n a constant depending only on n, Δ_{LB} the usual Laplace-Beltrami operator on forms associated to $g(x,t)$ and $\Delta = g^{ij}\nabla_i\nabla_j$.

The coupled geometric flow (1.1) is related to the Ricci flow [6] and the so-called renormalization group flow [12]. Note that the idea of coupling the Ricci flow with another flow also appeared in [9–11, 16]. In this paper, we mainly discuss the short-time existence, the entropy functionals, the higher derivatives estimates and the compactness theorem for this general Ricci flow system on closed Riemannian manifolds. Many of our results obviously extend previous results in [6, 11, 16].

Received January 7, 2017; Revised October 13, 2017; Accepted December 15, 2017.
2010 *Mathematics Subject Classification.* Primary 53C44; Secondary 35K55, 35P15.
Key words and phrases. Ricci flow, renormalization group flow, B-field, static Einstein vacuum equations, entropy, gradient estimates, compactness theorem.

If $H \equiv 0$ and $u \equiv 0$, the GRF system (1.1) reduces to the Hamilton's Ricci flow [6], which is an effective tool to solve the Poincaré and Geometrization Conjectures.

The system (1.1) when function u identically equals to constant naturally arises in physics, which can be interpreted as a certain nonlinear sigma model. We regard the special case as the B-field flow. The B-field flow was studied in [12], where they introduced an interesting functional, similar to Perelman's $\mathcal{F}$-functional. Note that B-field flow was also interpreted as the connection Ricci flow [16].

Also, the case $H = 0$ preserved by (1.1) was studied in [11] and the system can be reduced to the static Einstein vacuum equations. A main motivation to study (1.1) with $H = 0$ stems from its connection to general relativity. An important issue in the numerical evolution of the Einstein equations is the construction of good initial data sets which have to satisfy the so-called constraint equations.

The structure of this paper is organized as follows. In Section 2, we prove the short time existence of the GRF system (1.1). In Section 3, we give interesting entropy formulas, similar to the Perelman's entropy functional. As applications, we prove the nonexistence of the nontrivial steady or expanding breathers for the GRF system. In Section 4, we compute the evolution equations for many geometric quantities and their derivatives along the GRF system. In Section 5, we obtain the Bernstein-Bando-Shi type derivative estimates for geometric solutions of the GRF system. In Section 6, we prove the compactness theorems for the GRF system, which is a useful tool to understand the singularities of the GRF system.

The proofs in this paper will often involve local computations. Therefore, we assume a coordinate system $\{x^1, \ldots, x^n\}$ is fixed in a neighborhood of every point $x \in M$. In order to facilitate the computations, we often implicitly assume that $\{x^1, \ldots, x^n\}$ are normal coordinates. We use the standard shorthand: Given a real-valued function f on M, the notation f_i stands for $\frac{\partial f}{\partial x^i}$, the notation f_{ij} refers to the Hessian of f applied to $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial x^j}$, and similar for higher derivatives. The notation R_{ij} refers to the corresponding components of the Ricci tensor, i.e., $R_{ij} = Rc(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$. For a real-valued function u on M, $u_i u_j = du \otimes du(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$. The subscript t designates the differentiation in $t \in [0, T)$. Throughout this paper we use the Einstein summation convention, meaning that we sum over a repeated lower and upper index from 1 to n. In normal coordinates, the summation can be over two lower indices. We also write $A * B$ for a linear combination of contractions of components of the two tensors A and B when the precise form and number of these terms is irrelevant for the computation.

Acknowledgements. This work was initiated during the author's study at ECNU in 2009. The first version of the work was finished in 2010. For some reason, the author didn't submit it at that time. The author would like to thank the referee for pointing out many grammatical errors and making valuable suggestions for the earlier version of this paper. This work is partially supported by the NSFC (No.11671141) and the Natural Science Foundation of Shanghai (No.17ZR1412800).

2. Short time existence

In this section, we will use the DeTurck's trick to prove short time existence and uniqueness of the GRF system (1.1) on closed manifolds.

Theorem 2.1. *Let M be a closed n-dimensional manifold and the initial data $(\tilde{g}, \tilde{H}, \tilde{u})$ be given. Then the initial value problem*

$$\begin{aligned} \partial_t g_{ij} &= -2R_{ij} + 1/2 h_{ij} + 2\alpha_n u_i u_j \\ \partial_t H &= \Delta_{LB} H \quad \text{and} \quad \partial_t u = \Delta u \end{aligned} \tag{2.1}$$

with the initial data $(g(x,0), H(x,0), u(x,0)) = (\tilde{g}, \tilde{H}, \tilde{u})$ has a unique smooth solution on some time interval $[0,T)$, where $h_{ij} = g^{kl} g^{mn} H_{ikm} H_{jln}$.

Since the second and third terms on the right hand side of the first equation of (2.1) have not any second-order expression on the metric g, they do not affect the principal symbol. So the principal symbol of (2.1) is the same as the principal symbol of the Ricci operator. Therefore the considerations used by R. Hamilton [6] concerning the Ricci flow are also true for the system (2.1).

In the following, we shall find a strongly parabolic system which is equivalent to the system (2.1) by the application of a diffeomorphism. This is referred to as DeTurck's trick [4]. We shall employ Shi's method [14] and first calculate evolution equations for solutions pulled back by such a diffeomorphism.

Let $V \in \mathcal{X}(M \times [0,T))$ be a smooth time dependent vector field and denote the induced 1-parameter family of diffeomorphisms by φ_t. Then the diffeomorphisms satisfy at every $x \in M$ the following ordinary differential equation:

$$\frac{d}{dt}\varphi_t(x) = V(\varphi_t(x)) \quad \text{with} \quad \varphi_0(x) = x.$$

Lemma 2.2. *Suppose $(\bar{g}, \bar{H}, \bar{u})$ is a solution to the system (2.1) on closed manifolds on $[0,T)$. Let $\varphi_t : M \to M$ be the 1-parameter family of diffeomorphisms generated by V. Then the pullbacks satisfy the system*

$$\begin{aligned} \partial_t g_{ij} &= -2R_{ij} + 1/2 h_{ij} + 2\alpha_n u_i u_j + \nabla_i V_j + \nabla_j V_i \\ \partial_t H &= \Delta_{LB} H - d\langle H, V\rangle \quad \text{and} \quad \partial_t u = \Delta u + du(V) \end{aligned} \tag{2.2}$$

with the same initial values as $(\bar{g}, \bar{H}, \bar{u})$ on $M \times [0,T)$, where $\{V_i\}$ is the associated 1-form to V.

Proof. Denote by $\{y^\alpha\}_{\alpha=1,\dots,n}$ the coordinates where $\bar{g}$, $\bar{H}$ and $\bar{u}$ are represented by $\bar{g}_{\alpha\beta}$, $\bar{H}_{\alpha\beta\gamma}$ and $\bar{u}$. Define new coordinates by $x^i := (y \circ \varphi)^i$ for

$i = 1, \ldots, n$. We follow the same calculation of [14] without any unchanged except the following extra two terms of the system in the new coordinates $\{x^i\}$

$$\varphi_t^* \left(\bar{h}_{\alpha\beta}\right) = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\bar{h}_{\alpha\beta} = h_{ij} \text{ and } \varphi_t^* \left(\bar{u}_\alpha \bar{u}_\beta\right)_{ij} = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial \bar{u}}{\partial y^\alpha}\frac{\partial y^\beta}{\partial x^j}\frac{\partial \bar{u}}{\partial y^\beta} = u_i u_j.$$

Meanwhile the evolution of H with respect to new coordinates is

$$\begin{aligned} \partial_t H &= \varphi_t^* \left(\partial_t \bar{H} + L_{\bar{V}} \bar{H}\right) = \varphi_t^* \left(\bar{\Delta}_{LB} \bar{H} + L_{\bar{V}} \bar{H}\right) \\ &= \varphi_t^* \left(\bar{\Delta}_{LB} \bar{H} - d\langle \bar{H}, \bar{V}\rangle\right) = \Delta_{LB} H - d\langle H, V\rangle, \end{aligned}$$

where we used H is a closed 3-form. In the end, the evolution of u satisfies

$$\partial_t u(x,t) = \partial_t \bar{u}(y,t) = \bar{u}_t + \frac{\partial \bar{u}}{\partial y^\alpha}\partial_t y^\alpha = \bar{\Delta}\bar{u} + \frac{\partial \bar{u}}{\partial y^\alpha}\partial_t y^\alpha.$$

Note that

$$\bar{\Delta}\bar{u} = \bar{g}^{\alpha\beta}\bar{\nabla}_\alpha \bar{\nabla}_\beta \bar{u} = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} g^{ij} \frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta}\nabla_i \nabla_j u = \Delta u$$

and

$$\partial_t y^\alpha = \left(\frac{d}{dt}\varphi_t(x)\right)^\alpha = (V\varphi_t(x))^\alpha = (D\varphi_t(V))^\alpha = \frac{\partial y^\alpha}{\partial x^k}V^k.$$

Therefore

$$\partial_t u(x,t) = \Delta u + \frac{\partial u}{\partial x^k}V^k.$$

We also notice that the initial data remain the same under this coordinate change since $\varphi_0 = \mathrm{id}$. Therefore we finish the proof of the lemma. $\square$

If we choose a suitable vector field V, then the system (2.2) is strictly parabolic.

Lemma 2.3. *Let $V^i = g^{mn}(\Gamma^i_{mn} - \tilde{\Gamma}^i_{mn})$. Then the system* (2.2) *is strictly parabolic on $M \times [0,T)$, where $\tilde{\Gamma}^i_{mn}$ denotes the connection with respect to the initial metric $\tilde{g}$, which is time-independent.*

Proof. To check that the system is strictly parabolic, we rewrite the equations such that all derivatives are with respect to the (fixed) initial metric $\tilde{g}$ and examine the leading order terms in coordinates. Here we use the following identity

$$\Gamma^i_{mn} - \tilde{\Gamma}^i_{mn} = 1/2 g^{il}(\tilde{\nabla}_m g_{nl} + \tilde{\nabla}_n g_{ml} - \tilde{\nabla}_l g_{mn})$$

to replace Christoffel symbols of g by derivatives $\tilde{\nabla} g$ and work in normal coordinates for $\tilde{g}$ such that $\tilde{\Gamma}^i_{mn} = 0$ at the base point. From Lemma 2.1 of [14], we have

(2.3)

$$\begin{aligned} \partial_t g_{ij} = {} & g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b g_{ij} - g^{ab} g_{ik}\tilde{g}^{kl}\tilde{R}_{jalb} - g^{ab} g_{jk}\tilde{g}^{kl}\tilde{R}_{ialb} + 1/2 g^{kl} g^{mn} H_{ikm} H_{jln} \\ & + 2\alpha_n u_i u_j + 1/2 g^{ab} g^{kl}\Big(\tilde{\nabla}_i g_{ka}\tilde{\nabla}_j g_{lb} + 2\tilde{\nabla}_a g_{jk}\tilde{\nabla}_l g_{ib} - 2\tilde{\nabla}_a g_{jk}\tilde{\nabla}_b g_{il} \end{aligned}$$

$$-2\tilde{\nabla}_j g_{ka}\tilde{\nabla}_b g_{il}-2\tilde{\nabla}_i g_{ka}\tilde{\nabla}_b g_{jl}\Big).$$

Similarly, by the Bochler formula, the equation for 3-form H is roughly computed as follows (In fact we only need to understand the second-order derivative of H):

$$\begin{aligned}
\partial_t H_{ijk} &= \Delta_{LB} H - d\langle H, V\rangle \\
&= \Delta H_{ijk} + (Rm * H)_{ijk} + (Rm * \nabla H)_{ijk} \\
&= g^{ab}\nabla_a\nabla_b H_{ijk} + (Rm * H)_{ijk} + (Rm * \nabla H)_{ijk} \\
&= g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b H_{ijk} - g^{ab}\Gamma^p_{ab}H_{pjk} - g^{ab}\Gamma^p_{ab}H_{ipk} - g^{ab}\Gamma^p_{ab}H_{ijp} \\
&\quad + (Rm * H)_{ijk} + (Rm * \nabla H)_{ijk}.
\end{aligned} \tag{2.4}$$

The evolution equation of u is given by

$$\begin{aligned}
\partial_t u &= \Delta u + du(V) = g^{ab}\nabla_a\nabla_b u + u_p V^p \\
&= g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b u - g^{ab}\Gamma^p_{ab}u_p + u_p\cdot g^{ab}(\Gamma^p_{ab} - \tilde{\Gamma}^p_{ab}) = g^{ab}\tilde{\nabla}_a\tilde{\nabla}_b u,
\end{aligned} \tag{2.5}$$

where we used $\tilde{\Gamma}^p_{ab} = 0$. The principal symbol of system (2.2) in these coordinates is given by the coefficient matrices of the second derivatives of g, H and u. By (2.3), (2.4) and (2.5), we conclude that the system (2.2) is strictly parabolic. $\square$

Now we can finish the proof of Theorem 2.1.

Proof of Theorem 2.1. From Lemma 2.3, we know that the system (2.2) is strictly parabolic. It is a standard result that for any smooth initial data $(\tilde{g}, \tilde{H}, \tilde{u})$, there exists $\varepsilon > 0$ depending on the initial data such that a unique smooth solution $(g(t), H(t), u(t))$ to the system (2.2) will exist for a short time $0 \le t < \varepsilon$.

We observe that 1-parameter family of diffeomorphisms ϕ_t, which is defined by

$$\frac{d}{dt}\phi_t(x) = -V(\phi_t(x)) \quad \text{with} \quad \phi_0(x) = x. \tag{2.6}$$

Note that this 1-parameter family of diffeomorphisms can be seen as the inverse of φ_t. We claim that ϕ_t exists as long as the solution of (2.2) exists. Because M is compact, it follows from Lemma 3.15 of [2].

In the end, from Lemma 2.2 we have $\bar{g} := \phi_t^* g = (\varphi_t^{-1})^* g$, $\bar{H} := \phi_t^* H = (\varphi_t^{-1})^* H$ and $\bar{u} := \phi_t^* u = (\varphi_t^{-1})^* u$ is a solution of the system (2.1).

To address the uniqueness, we follow the ideas relating harmonic maps to the GRF system. The basic idea is to write the ODE for ϕ_t given in (2.6) in terms of the metric $\bar{g}$. We can rewrite the solution to (2.6) as the harmonic map heat flow

$$\partial_t \phi_t = \Delta_{\bar{g}(t),\tilde{g}}\, \phi_t, \tag{2.7}$$

where $\tilde{g}$ is a fixed background metric on M. We refer the reader to Lemma 3.18 of [2] for the detailed discussions.

We now prove the uniqueness of the solution to (2.1) with the same initial data. Let $(\bar{g}_1(t), \bar{H}_1(t), \bar{u}_1(t))$ and $(\bar{g}_2(t), \bar{H}_2(t), \bar{u}_2(t))$ denote two solutions to (2.1) on some time interval. Let $\phi_i(t)$ denote the solution to (2.7) with respect to $\bar{g}_i$, which exists because (2.7) is strictly parabolic on the underlying manifold is compact. Note that $g_i(t) = (\phi_t^{-1})^* \bar{g}_i(t)$, $H_i(t) = (\phi_t^{-1})^* \bar{H}_i(t)$ and $u_i(t) = (\phi_t^{-1})^* \bar{u}_i(t)$ is a solution to (2.2). Since $(g_1(0), H_1(0), u_1(0)) = (g_2(0), H_2(0), u_2(0))$ and solutions to (2.2) are unique, $(g_1(t), H_1(t), u_1(t)) = (g_2(t), H_2(t), u_2(t))$ as long as they exist. But both $(\phi_1)_t$ and $(\phi_2)_t$ are solutions of (2.6) generated by the same vector field

$$V^i = g^{mn}(\Gamma^i_{mn} - \tilde{\Gamma}^i_{mn}).$$

Hence $(\phi_1)_t = (\phi_2)_t$. This implies $\bar{g}_1(t) = (\phi_1)_t^* g_1(t) = (\phi_2)_t^* g_2(t) = \bar{g}_2(t)$, $\bar{H}_1(t) = (\phi_1)_t^* H_1(t) = (\phi_2)_t^* H_2(t) = \bar{H}_2(t)$ and $\bar{u}_1(t) = (\phi_1)_t^* u_1(t) = (\phi_2)_t^* u_2(t) = \bar{u}_2(t)$ and hence the uniqueness follows. □

3. Energy and monotonicity

In this section, we will generalize some results of Perelman's $\mathcal{F}$-functional under the Ricci flow to the GRF system case.

3.1. Entropy and the gradient flow

Definition 3.1. A solution $(g(t), H(t), u(t))$ to system (1.1) on M^n is called a *GRF system soliton*, if it varies only along a 1-parameter family of diffeomorphisms or by scaling. Therefore it satisfies

$$\begin{aligned} &\partial_t g(t) = \mathcal{L}_{X(t)} g(t) + c(t) g(t) \\ &\partial_t H(t) = \mathcal{L}_{X(t)} H(t) \quad \text{and} \quad \partial_t u(t) = \mathcal{L}_{X(t)} u(t), \end{aligned}$$

where $X \in \mathcal{X}(M \times [0,T))$ is the generator of the diffeomorphisms and $c : [0,T) \to \mathbb{R}$ is the scaling factor, depending on time only. If $X = \nabla h$ is the gradient of a smooth function h, the soliton is called a gradient soliton. We say that the soliton is *shrinking*, *steady* or *expanding*, if $c < 0$, $c = 0$ or $c > 0$, respectively.

Definition 3.2. Let M be a closed n-dimensional manifold and $(g(t)$, $H(t)$, $u(t))$ be given as the system (1.1). Following the idea of Perelman, the entropy of the system (1.1) is defined as follows

$$\mathcal{F}(g(t), H(t), u(t), f(t)) := \int_M \left(R + |\nabla f|^2 - 1/12|H|^2 - \alpha_n |du|^2\right) w d\mu$$

restricted to function w satisfying $\int_M w d\mu = 1$ along this system, where f is defined implicity by $w = e^{-f}$.

Suppose w satisfies the conjugate heat equation

$$w_t = -\Delta w + Rw - 1/4|H|^2 w - \alpha_n |du|^2 w. \tag{3.1}$$

It follows then that $\frac{d}{dt}\int_M w d\mu = 0$ and moreover f satisfies

$$\partial_t f = -\Delta f + |\nabla f|^2 - R + 1/4|H|^2 + \alpha_n |du|^2. \tag{3.2}$$

Theorem 3.3. *Suppose M is a closed n-dimensional manifold. Let $(g(t), H(t), u(t))$ be a solution to the system* (1.1) *and f be the solution of* (3.2)*. Then*

$$\begin{aligned}\frac{d}{dt}\mathcal{F} &= 2\int_M \left|R_{ij} - 1/4 H_{ikm}{H_j}^{km} - \alpha_n u_i u_j + f_{ij}\right|^2 e^{-f} d\mu \\ &\quad + \frac{1}{2}\int_M \left|(d^*H)_{ij} - H_{kij}\nabla^k f\right|^2 e^{-f} d\mu + 2\alpha_n \int_M |\Delta u - du(\nabla f)|^2 e^{-f} d\mu,\end{aligned} \tag{3.3}$$

where $(d^*H)_{ij} = -\nabla^k H_{kij}$. In particular this entropy is non-decreasing. Equality holds if and only if the solution is a steady gradient soliton. In this case $(g, H, u)(t)$ satisfies at all times t:

$$\begin{aligned} R_{ij} - 1/4 H_{ikm}{H_j}^{km} - \alpha_n u_i u_j + f_{ij} = 0,& \\ (d^*H)_{ij} - H_{kij}\nabla^k f = 0 \quad \textit{and} \quad \Delta u - du(\nabla f) = 0.& \end{aligned}$$

Theorem 3.3 is still true on complete manifolds as long as the integration by parts can be justified. To prove Theorem 3.3, we start with the following two lemmas.

Lemma 3.4. *Given (M^n, g) a manifold and H a closed three-form on M, then*

(1) $\nabla^i h_{ij} = \frac{1}{6}\nabla_j |H|^2 + (d^*H)^{mn} H_{jmn}$.

(2) $\nabla^j \nabla^i h_{ij} = \frac{1}{3}\langle \Delta_{LB} H, H\rangle + \frac{1}{6}\Delta |H|^2 + |d^*H|^2$.

Proof. We refer the reader to Lemma A.4 of [16] for detailed discussions. □

Lemma 3.5. *Let (g, H, u) be a solution to system* (1.1) *and let f be a solution to the equation* (3.2)*. Define $V := 2\Delta f - |\nabla f|^2 + R - 1/12|H|^2 - \alpha_n |du|^2$. Then*

$$\begin{aligned}(\partial_t + \Delta) V = 2\left|R_{ij} - 1/4 h_{ij} - \alpha_n u_i u_j + f_{ij}\right|^2 + 1/2\left|(d^*H)_{ij} - H_{kij}\nabla^k f\right|^2 \\ + 2\alpha_n |\Delta u - du(\nabla f)|^2 + 2\langle \nabla V, \nabla f\rangle.\end{aligned} \tag{3.4}$$

Proof. The proof is straightforward by direct computation. First, we have

$$\begin{aligned} &(\partial_t + \Delta) 2\Delta f \\ =\ & 2\langle 2R_{ij} - 1/2 h_{ij} - 2\alpha_n u_i u_j, f_{ij}\rangle - 2\left(1/2\nabla^i h_{ij} - 1/4\nabla^j |H|^2\right)\nabla^j f \\ &+ 2\Delta\big(-\Delta f + |\nabla f|^2 - R + 1/4|H|^2 + \alpha_n |du|^2\big) - 4\alpha_n \Delta u \cdot du(\nabla f) + 2\Delta\Delta f \\ =\ & \langle 4R_{ij} - h_{ij} - 4\alpha_n u_i u_j, f_{ij}\rangle + 2\Delta|\nabla f|^2 - 2\Delta R + 1/2\Delta |H|^2 + 2\alpha_n \Delta |du|^2 \\ &+ \left\langle 1/2\nabla^j |H|^2 - \nabla^i h_{ij}, \nabla^j f\right\rangle - 4\alpha_n \Delta u \cdot du(\nabla f).\end{aligned}$$

Second,

$$(\partial_t+\Delta)(-2|\nabla f|^2)=2\left\langle\nabla(\Delta f-|\nabla f|^2+R-1/4|H|^2-\alpha_n|du|^2),\nabla f\right\rangle \\ +\langle 1/2h_{ij}+2\alpha_n u_iu_j-2R_{ij},f_if_j\rangle-\Delta|\nabla f|^2.$$

Third,

$$(\partial_t+\Delta)R=2\Delta R+2|R_{ij}|^2-1/2\Delta|H|^2+1/2\nabla_j\nabla_i h_{ij}-1/2\langle R_{ij},h_{ij}\rangle \\ -2\alpha_n\Delta|du|^2+2\alpha_n\nabla_j\nabla_i(u_iu_j)-2\alpha_nR_{ij}u_iu_j.$$

Fourth,

$$\begin{aligned}&(\partial_t+\Delta)(-1/12|H|^2)\\ &=-1/12[\langle 6R_{ij}-3/2h_{ij}-6\alpha_nu_iu_j,h_{ij}\rangle+2\langle H_t,H\rangle+\Delta|H|^2]\\ &=\langle 1/8h_{ij}+\alpha_n/2u_iu_j-1/2R_{ij},h_{ij}\rangle-1/6\langle\Delta_{LB}H,H\rangle-1/12\Delta|H|^2.\end{aligned}$$

For the last term of V, we have

$$\begin{aligned}&(\partial_t+\Delta)(-\alpha_n|du|^2)\\ &=-\alpha_n\langle 2R_{ij}-1/2h_{ij}-2\alpha_nu_iu_j,u_iu_j\rangle-2\alpha_n\langle\nabla u,\nabla\Delta u\rangle-\alpha_n\Delta|du|^2\\ &=-2\alpha_nR_{ij}u_iu_ju+\alpha_n/2h_{ij}u_iu_j+2\alpha_n^2|du|^4-2\alpha_n\langle\nabla u,\nabla\Delta u\rangle-\alpha_n\Delta|du|^2.\end{aligned}$$

Combining the above five calculations gives

$$\begin{aligned}(\partial_t+\Delta)\,V=&\,\Delta|\nabla f|^2-1/12\Delta|H|^2-1/6\langle\Delta_{LB}H,H\rangle+1/2\nabla_j\nabla_ih_{ij}\\ &+\left\langle 1/2\nabla^j|H|^2-\nabla^ih_{ij},\nabla^jf\right\rangle+\langle 1/2h_{ij}-2R_{ij},f_if_j\rangle\\ &+2\left|R_{ij}-1/4h_{ij}+f_{ij}\right|^2-2|f_{ij}|^2\\ &+2\left\langle\nabla\left(\Delta f-|\nabla f|^2+R-1/4|H|^2-\alpha_n|du|^2\right),\nabla f\right\rangle\\ &-4\alpha_n\left\langle u_iu_j,f_{ij}\right\rangle-4\alpha_nR_{ij}u_iu_j+\alpha_nh_{ij}u_iu_j+2\alpha_n^2|du|^4\\ &-4\alpha_n\Delta u\cdot du(\nabla f)+2\alpha_n|du(\nabla f)|^2\\ &+2\alpha_n\nabla_j\nabla_i(u_iu_j)-2\alpha_n\langle\nabla u,\nabla\Delta u\rangle-\alpha_n\Delta|du|^2.\end{aligned}\tag{3.5}$$

Note that

$$\begin{aligned}&2|R_{ij}-1/4h_{ij}+f_{ij}|^2-4\alpha_n\langle u_iu_j,f_{ij}\rangle-4\alpha_nR_{ij}u_iu_j+\alpha_nh_{ij}u_iu_j+2\alpha_n^2|du|^4\\ &=2\left|R_{ij}-1/4h_{ij}-\alpha_nu_iu_j+f_{ij}\right|^2,\\ &2\alpha_n\nabla_j\nabla_i(\partial_iu\partial_ju)-2\alpha_n\langle\nabla u,\nabla\Delta u\rangle-\alpha_n\Delta|du|^2=2\alpha_n(\Delta u)^2\end{aligned}$$

and

$$\Delta|\nabla f|^2-2|f_{ij}|^2-2\langle R_{ij},f_if_j\rangle=2\langle\nabla\Delta f,\nabla f\rangle.$$

Hence (3.5) reduces to

$$
\begin{aligned}
(\partial_t+\Delta)\, V = & -1/12\Delta|H|^2 - 1/6\langle \Delta_{LB}H, H\rangle + 1/2\nabla_j\nabla_i h_{ij} \\
& + \langle 1/2\nabla^j|H|^2 - \nabla^i h_{ij}, \nabla^j f\rangle + \langle 1/2 h_{ij}, f_i f_j\rangle \\
& + 2\left|R_{ij} - 1/4h_{ij} - \alpha_n u_i u_j + f_{ij}\right|^2 + 2\langle \nabla V, \nabla f\rangle \\
& - 1/3\langle \nabla|H|^2, \nabla f\rangle + 2\alpha_n\left|\Delta u - du(\nabla f)\right|^2 .
\end{aligned} \tag{3.6}
$$

Now using Lemma 3.4 we know that

$$
-1/12\Delta|H|^2 - 1/6\langle \Delta_{LB}H, H\rangle + 1/2\nabla_j\nabla_i h_{ij} = 1/2|d^*H|^2
$$

and

$$
\left\langle 1/2\nabla^j|H|^2 - \nabla^i h_{ij}, \nabla^j f\right\rangle = 1/3\left\langle \nabla|H|^2, \nabla f\right\rangle - \left\langle (d^*H)_{ij}, H_{kij}\nabla^k f\right\rangle .
$$

Substituting the above two equalities into (3.6) gives (3.4). $\square$

Now we can give the proof of Theorem 3.3.

Proof of Theorem 3.3. Since $w\nabla f = -\nabla w$, then

$$
\begin{aligned}
(\partial_t+\Delta)(Vw) &= (\partial_t+\Delta)V\cdot w + V\cdot(\partial_t+\Delta)w + 2\langle \nabla V, \nabla w\rangle \\
&= 2\left|R_{ij} - 1/4h_{ij} - \alpha_n u_i u_j + f_{ij}\right|^2 w + 1/2\left|(d^*H)_{ij} - H_{kij}\nabla^k f\right|^2 w \\
&\quad + 2\alpha_n\left|\Delta u - du(\nabla f)\right|^2 w + V\cdot(\partial_t+\Delta)\, w,
\end{aligned}
$$

where we used Lemma 3.5. Therefore

$$
\begin{aligned}
\frac{d}{dt}\int_M (Vw)d\mu &= \int_M \frac{d}{dt}(Vw)\, d\mu + \int_M (Vw)\cdot\left[-R + 1/4|H|^2 + \alpha_n|du|^2\right] d\mu \\
&= 2\int_M \left[\left|R_{ij} - 1/4h_{ij} - \alpha_n u_i u_j + f_{ij}\right|^2 + \alpha_n|\Delta u - du(\nabla f)|^2\right] w d\mu \\
&\quad + 1/2\int_M \left|(d^*H)_{ij} - H_{kij}\nabla^k f\right|^2 w d\mu + \int_M V\cdot(\partial_t+\Delta) w d\mu \\
&\quad + \int_M V\cdot\left[-Rw + 1/4|H|^2 w + \alpha_n|du|^2 w\right] d\mu.
\end{aligned} \tag{3.7}
$$

Since w satisfies (3.1), the theorem follows from (3.7). $\square$

Similar to Ricci flow, we can also consider the following system

$$
\begin{aligned}
&\partial_t g_{ij} = -2R_{ij} + 1/2h_{ij} + 2\alpha_n u_i u_j - 2\nabla_i\nabla_j f \\
&\partial_t H = \Delta_{LB}H - d\langle H, \nabla f\rangle \qquad \text{and} \qquad \partial_t u = \Delta u - du(\nabla f),
\end{aligned} \tag{3.8}
$$

where the restricted function f satisfies the following conjugate heat-type equation

$$
\partial_t f = -\Delta f - R + 1/4|H|^2 + \alpha_n|du|^2. \tag{3.9}
$$

It follows that $\frac{d}{dt}\int_M e^{-f} d\mu = 0$. From this, we can easily obtain the same monotonicity formula (3.3) for the modified GRF system (3.8) coupled with

(3.9). More importantly, (twice) positive gradient flow of the functional $\mathcal{F}(g(t), H(t), u(t), f(t))$ is (3.8) coupled with (3.9).

Below we will see that the GRF system (1.1) coupled with (3.2) is equivalent to the modified GRF system (3.8) coupled with (3.9) by diffeomorphism.

Proposition 3.6. *Let $(\bar{g}(t), \bar{H}(t), \bar{u}(t), \bar{f}(t))$ be a solution of the modified GRF system (3.8) coupled with (3.9) on $[0, T]$. We define a 1-parameter family of diffeomorphisms $\Psi(t) : M \to M$ by*

$$\frac{d}{dt}\Psi(t) = \nabla_{\bar{g}(t)}\bar{f}(t) \quad \textit{with} \quad \Psi(t) = \mathrm{id}_M. \tag{3.10}$$

Then the pull back metric $g(t) = \Psi(t)^*\bar{g}(t)$, the pull back 3-form $H(t) = \Psi(t)^*\bar{H}(t)$, the pull back $u(t) = \Psi(t)^*\bar{u}(t)$ and the dilation $f(t) = \bar{f}(t) \circ \Psi(t)$ satisfy the GRF system (1.1) coupled with (3.2).

Proof. By Lemma 3.15 of [2] the system of (3.10) is always solvable. Now we compute

$$\begin{aligned} \partial_t g &= \Psi(t)^*(\partial_t \bar{g}) + \Psi(t)^* \left(L_{\nabla_{\bar{g}}\bar{f}}\bar{g}\right) \\ &= \Psi(t)^* \left(-2Rc(\bar{g}) + 1/2\bar{h}_{ij} + 2\alpha_n \bar{u}_i \bar{u}_j\right) \\ &= -2R_{ij} + 1/2h_{ij} + 2\alpha_n u_i u_j. \end{aligned}$$

Since H is a closed 3-form, we also have

$$\begin{aligned} \partial_t H &= \Psi(t)^* \left(\partial_t \bar{H} + L_{\nabla_{\bar{g}}\bar{f}}\bar{H}\right) \\ &= \Psi(t)^* \left(\bar{\Delta}_{LB}\bar{H} - d\langle \bar{H}, \nabla_{\bar{g}}\bar{f}\rangle + L_{\nabla_{\bar{g}}\bar{f}}\bar{H}\right) \\ &= \Psi(t)^* \left(\bar{\Delta}_{LB}\bar{H}\right) = \Delta_{LB}H. \end{aligned}$$

To obtain the formula for $\frac{\partial u}{\partial t}$, we compute

$$\partial_t u = \Psi(t)^* \left(\partial_t \bar{u} + L_{\nabla_{\bar{g}}\bar{f}}\bar{u}\right) = \Psi(t)^* \left(\bar{\Delta}\bar{u} - d\bar{u}(\nabla\bar{f}) + L_{\nabla_{\bar{g}}\bar{f}}\bar{u}\right) = \Delta u.$$

In the end,

$$\begin{aligned} \partial_t f &= \partial_t(\bar{f} \circ \Psi) = \partial_t \bar{f} \circ \Psi + \left\langle (\bar{\nabla}\bar{f}) \circ \Psi, \partial_t \Psi \right\rangle_{\tilde{g}} \\ &= \left(-\bar{\Delta}\bar{f} - \bar{R} + 1/4|\bar{H}|^2_{\bar{g}} + \alpha_n |d\bar{u}|^2_{\bar{g}}\right) \circ \Psi + \left|(\bar{\nabla}\bar{f}) \circ \Psi\right|^2_{\tilde{g}} \\ &= -\Delta f - R + 1/4|H|^2 + \alpha_n |du|^2 + |\nabla f|^2. \end{aligned}$$

□

From the above property, we see that the system (1.1) coupled with (3.2) is a *gradient-like flow.* In other words, the system (1.1) is not a gradient flow of a functional on the space of smooth metrics on a manifold with respect to the standard L^2-inner product, but its modified system (3.8) coupled with (3.9) is a gradient flow. This phenomenon also appears in the Ricci flow, found by G. Perelman [13].

3.2. Expanding entropy

It is worth noting that, due to the lack of scale-invariance of the evolution of g, the obvious generalization of the Perelman's shrinking entropy does not generalize to the our case. However, the Feldman-Ilmanen-Ni expanding entropy [5] can be extended to the case of our system. In particular we define

$$\mathcal{W}_+(g,H,u,f,\tau) := \int_M \left[\tau\left(R+|\nabla f|^2 - 1/12|H|^2 - \alpha_n|du|^2\right) - f_+ + n\right] w d\mu$$

restricted to function w satisfying $\int_M w d\mu = 1$ along the GRF system (1.1), where $\tau := t - T > 0$ and f_+ is defined by $w = \frac{e^{-f_+}}{(4\pi\tau)^{n/2}}$. Suppose w satisfies the equation

$$w_t = -\Delta w + Rw - 1/4|H|^2 w - \alpha_n |du|^2 w.$$

It follows then that $\frac{d}{dt}\int_M w d\mu = 0$ and moreover f_+ satisfies

$$\partial_t f_+ = -\Delta f_+ + |\nabla f_+|^2 - R + 1/4|H|^2 + \alpha_n |du|^2 - \frac{n}{2(t-T)}. \tag{3.11}$$

Theorem 3.7. *Let* $(g(t), H(t), u(t))$ *be a solution to the system* (1.1) *and* f_+ *be the solution of* (3.11). *Then*

(3.12)

$$\begin{aligned}\frac{d}{dt}\mathcal{W}_+ = {} & 2\tau \int_M \left| R_{ij} - \frac{1}{4}H_{ikm}{H_j}^{km} - \alpha_n u_i u_j + (f_+)_{ij} + \frac{g_{ij}}{2(t-T)}\right|^2 \frac{e^{-f_+}}{(4\pi\tau)^{n/2}} d\mu \\ & + \frac{\tau}{2}\int_M \left|(d^*H)_{ij} - H_{kij}\nabla^k f_+\right|^2 \frac{e^{-f_+}}{(4\pi\tau)^{n/2}} d\mu \\ & + 2\alpha_n\tau \int_M |\Delta u - du(\nabla f_+)|^2 \frac{e^{-f_+}}{(4\pi\tau)^{n/2}} d\mu + \frac{1}{6}\int_M |H|^2 \frac{e^{-f_+}}{(4\pi\tau)^{n/2}} d\mu,\end{aligned}$$

where $(d^*H)_{ij} = -\nabla^k H_{kij}$. *In particular this expanding entropy is non-decreasing. Equality holds if and only if the solution is an expanding gradient soliton with* $H \equiv 0$. *In this case* $(g, H, u)(t)$ *satisfies at all times* t:

$$\begin{aligned} R_{ij} - \alpha_n u_i u_j + (f_+)_{ij} + \frac{g_{ij}}{2(t-T)} &= 0, \\ \Delta u - du(\nabla f_+) = 0 \quad \textit{and} \quad H &\equiv 0.\end{aligned}$$

Proof. We assume $\tau = t$. Define $V_+ := 2\Delta f_+ - |\nabla f_+|^2 + R - \frac{1}{12}|H|^2 - \alpha_n |du|^2$. Let $W := tV_+ - f_+ + n$. Note that Lemma 3.5 still holds if f is replaced by f_+. Hence

$$\begin{aligned} & (\partial_t + \Delta)W \\ = {} & V_+ + 2t\left|R_{ij} - 1/4h_{ij} - \alpha_n u_i u_j + (f_+)_{ij}\right|^2 + t/2\left|(d^*H)_{ij} - H_{kij}\nabla^k f_+\right|^2 \\ & + 2\alpha_n t\left|\Delta u - du(\nabla f_+)\right|^2 + 2t\left\langle \nabla V_+, \nabla f_+\right\rangle \\ & - |\nabla f_+|^2 + R - 1/4|H|^2 - \alpha_n|du|^2 + \frac{n}{2t}\end{aligned}$$

$$\begin{aligned}
&= 2\Delta f_+ + 2R - 1/3|H|^2 - 2\alpha_n|du|^2 + \frac{n}{2t} + 2\langle\nabla W, \nabla f_+\rangle \\
&\quad + 2t|R_{ij} - 1/4h_{ij} - \alpha_n u_i u_j + (f_+)_{ij}|^2 + t/2\left|(d^*H)_{ij} - H_{kij}\nabla^k f_+\right|^2 \\
&\quad + 2\alpha_n t\left|\Delta u - du(\nabla f_+)\right|^2 \\
&= 2t\left|R_{ij} - 1/4h_{ij} - \alpha_n u_i u_j + (f_+)_{ij} + \frac{g_{ij}}{2t}\right|^2 + \frac{t}{2}\left|(d^*H)_{ij} - H_{kij}\nabla^k f_+\right|^2 \\
&\quad + 2\alpha_n t\left|\Delta u - du(\nabla f_+)\right|^2 + 1/6|H|^2 + 2\langle\nabla W, \nabla f_+\rangle.
\end{aligned}$$

If we let $v_+ := \left[\tau\left(2\Delta f_+ - |\nabla f_+|^2 + R - \frac{1}{12}|H|^2 - \alpha_n|du|^2\right) - f_+ + n\right]w$, then

$$\begin{aligned}
&(\partial_t + \Delta)v_+ \\
&= (\partial_t + \Delta)(Ww) \\
&= w(\partial_t + \Delta)W + W(\partial_t + \Delta)w + 2\langle\nabla W, \nabla w\rangle \\
&= 2t\left|R_{ij} - 1/4h_{ij} - \alpha_n u_i u_j + (f_+)_{ij} + \frac{g_{ij}}{2t}\right|^2 w + \frac{t}{2}\left|(d^*H)_{ij} - H_{kij}\nabla^k f_+\right|^2 w \\
&\quad + 2\alpha_n t\left|\Delta u - du(\nabla f_+)\right|^2 w + 1/6|H|^2 w + Rv_+ - 1/4|H|^2 v_+ - \alpha_n|du|^2 v_+.
\end{aligned}$$

In the end, Theorem 3.7 follows by the above equality and the relation

$$\frac{d}{dt}\mathcal{W}_+ = \int_M \left(\partial_t + \Delta - R + 1/4|H|^2 + \alpha_n|du|^2\right)v_+ d\mu. \qquad \square$$

From the definition of $\mathcal{W}_+$, we see that $\mathcal{W}_+$ has the following property:

Proposition 3.8. *If $\varphi : M \to M$ is a diffeomorphism and $c > 0$ is a constant, then we have*

(1) $\mathcal{W}_+(\varphi^*g, \varphi^*H, \varphi^*u, \varphi^*f, \tau) = \mathcal{W}_+(g, H, u, f, \tau)$.
(2) $\mathcal{W}_+(cg, H, u, f, c\tau) = \mathcal{W}_+(g, H, u, f, \tau)$.

3.3. Steady and expanding breathers

Definition 3.9. A solution $(g(t), H(t), u(t))$ to the system (1.1) on a manifold M^n is called a breather if there exist times $t_1 < t_2$, a constant $c > 0$ and a diffeomorphism $\varphi : M \to M$ such that $g(t_2) = c\varphi^*g(t_1)$, $H(t_2) = c\varphi^*H(t_1)$ and $u(t_2) = c\varphi^*u(t_1)$. When $c < 1$, $c = 1$ or $c > 1$, we call $(g, H, u)(t)$ a shrinking, steady or expanding breather, respectively.

The soliton is a special of the breather. If we consider the GRF system as a dynamical system on the space of $\mathcal{M}(M) \times \bigwedge^3(M) \times C^\infty(M)$ modulo diffeomorphisms and homotheties, the breathers correspond to the periodic orbits whereas the solitons correspond to the fixed points. Since the GRF system is a heat-type system, we expect that there are no periodic orbits except fixed points. In the following we will confirm it for the steady or expanding case.

To study the steady breather, we introduce the λ-functional, which is similar to the Ricci flow case.

Definition 3.10.

$$\lambda(g,H,u) := \inf\left\{\mathcal{F}\left(g(t),H(t),u(t),f(t)\right)\Big| f\in C^\infty(M), \int_M e^{-f}d\mu = 1\right\}.$$

Taking $v = e^{-f/2}$, we have

$$\lambda(g,H,u) = \inf\left\{\mathcal{G}\left(g(t),H(t),u(t),v(t)\right)\Big| \int_M v^2 d\mu = 1, v>0\right\}, \tag{3.13}$$

where

$$\mathcal{G}\left(g(t),H(t),u(t),v(t)\right) := \int_M \left(4|\nabla v|^2 + Rv^2 - 1/12|H|^2v^2 - \alpha_n|du|^2v^2\right)d\mu.$$

It is easy to see that the Euler-Lagrange equation for (3.13) is

$$Lv := -4\Delta v + Rv - 1/12|H|^2 v - \alpha_n |du|^2 v = \lambda(g,H,u)v.$$

Meanwhile we have the following existence and regularity of minimizer of $\mathcal{G}$.

Lemma 3.11. *The* inf *in the definition of $\lambda(g,H,u)$ is attained by a unique positive and smooth minimizer v_0. Moreover,*

(1) *the minimum value $\lambda(g,H,u)$ of $\mathcal{G}$ is equal to $\lambda_1(g,H,u)$, where $\lambda_1(g, H, u)$ is the lowest eigenvalue of the operator $-4\Delta + R - 1/12|H|^2 - \alpha_n|du|^2$, and*

(2) *v_0 is the unique positive eigenfunction of*

$$-4\Delta v_0 + Rv_0 - 1/12|H|^2 v_0 - \alpha_n|du|^2 v_0 = \lambda(g,H,u)v_0$$

with L^2-norm equal to 1.

Proof. The proof involves the Sobolev embedding theorem and the standard regularity theory for the second-order linear elliptic equations, similar to the Ricci flow case. See Lemma 5.22 of Chapter 5 in [1] for detailed discussions. □

Remark 3.12. In Lemma 3.11, (2) can be stated that: The minimizer $f_0 = -2\log v_0$ of $\mathcal{F}(g(t),H(t),u(t),\cdot)$ is unique, C^∞, and a solution to

$$\lambda(g,H,u) = 2\Delta f_0 - |\nabla f_0|^2 + R - 1/12|H|^2 - \alpha_n|du|^2.$$

We summarize the properties of the functional λ on closed manifolds.

Proposition 3.13. (1) (*Bounds for λ*) *Let λ be defined as above.*

$$R_{\min} - 1/12|H|^2_{\max} - \alpha_n|du|^2_{\max} \le \lambda(g,H,u) \le \frac{\int_M R d\mu}{\mathrm{Vol}(M)}.$$

(2) (*Diffeomorphism invariance*) *If $\varphi: M\to M$ is a diffeomorphism, then*

$$\lambda(\varphi^*g,\varphi^*H,\varphi^*u) = \lambda(g,H,u).$$

(3) (*Existence of a smooth minimizer*) *There exists a minimizer $f\in C^\infty(M)$ with $\int_M e^{-f}d\mu = 1$ such that $\lambda(g,H,u) = \mathcal{F}(g,H,u,f)$.*

(4) (*Scaling*) $\lambda(cg,H,u) = c^{-1}\lambda(g,H,u)$.

In the following we claim that $\lambda(g,H,u)$ is also a monotonic quantity.

Proposition 3.14. *If $(g(\cdot), H(\cdot), u(\cdot))$ is a solution to the GRF system* (1.1), *then $\lambda(g(t), H(t), u(t))$ is non-decreasing in time.*

Proof. Given $t_0 \in [0, T]$, let f_0 be the minimizer of $\mathcal{F}(g(t), H(t), u(t), f(t))$ at t_0. Then $\lambda(g(t_0), H(t_0), u(t_0)) = \mathcal{F}(g(t_0), H(t_0), u(t_0), f(t_0))$. Let f solve the backwards heat-type equation

$$\partial_t f = -\Delta f + |\nabla f|^2 - R + 1/4|H|^2 + \alpha_n |du|^2 \qquad \text{with} \quad f(t_0) = f_0$$

on the time interval $[0, t_0]$. Then $\frac{d}{dt}\mathcal{F}(g(t), H(t), u(t), f(t)) \geq 0$ for all $t \leq t_0$. Also the backwards heat-type equation preserves the constraint $\int_M e^{-f} d\mu = 1$. So $\lambda(g(t), H(t), u(t)) \leq \mathcal{F}(g(t), H(t), u(t), f(t))$ for $t \leq t_0$. Thus

$$\begin{aligned}\lambda(g(t), H(t), u(t)) &\leq \mathcal{F}(g(t), H(t), u(t), f(t))\\ &\leq \mathcal{F}(g(t_0), H(t_0), u(t_0), f(t_0)) = \lambda(g(t_0), H(t_0), u(t_0)).\end{aligned}$$

Hence $\lambda(g(t), H(t), u(t))$ is non-decreasing in time. □

As an application of the functional λ we prove that:

Theorem 3.15. *Let $(g(t), H(t), u(t))$ to be a solution to the GRF system* (1.1) *on a closed manifold M^n. If there exist $t_1 < t_2$ with $\lambda(g(t_1),\ H(t_1),\ u(t_1)) = \lambda(g(t_2),\ H(t_2), u(t_2))$ (i.e., the solution is a steady breather), then $(g(t), H(t), u(t))$ is a steady gradient GRF system soliton.*

Proof. Suppose there exist $t_1 < t_2$ with $\lambda(g(t_1),\ H(t_1),\ u(t_1)) = \lambda(g(t_2),\ H(t_2),\ u(t_2))$. Let f_2 be the minimizer for $\mathcal{F}$ at t_2, so that $\mathcal{F}(g(t_2),\ H(t_2),\ u(t_2)) = \lambda(g(t_2),\ H(t_2),\ u(t_2))$. Take $f(t)$ be to the solution to (3.2) on $[t_1, t_2]$ with the initial data $f(t_2) = f_2$. By (3.3) and the definition of λ, we have

$$\begin{aligned}\lambda(g(t_1), H(t_1), u(t_1)) &\leq \mathcal{F}(g(t), H(t), u(t), f(t))\\ &\leq \mathcal{F}(g(t_2), H(t_2), u(t_2), f(t_2))\\ &= \lambda(g(t_2), H(t_2), u(t_2))\end{aligned}$$

for $t \in [t_1, t_2]$. Since $\lambda(g(t_1),\ H(t_1),\ u(t_1)) = \lambda(g(t_2),\ H(t_2),\ u(t_2))$ and $\lambda(g(t),\ H(t),\ u(t))$ is monotone, then $\mathcal{F}(g(t), H(t), u(t), f(t)) = \lambda(g(t), H(t), u(t)) \equiv$ const for $t \in [t_1, t_2]$. Therefore, by (3.3), we have

$$\begin{aligned}&R_{ij} - 1/4 H_{ikm} {H_j}^{km} - \alpha_n u_i u_j + f_{ij} = 0,\\ &(d^* H)_{ij} - H_{kij} \nabla^k f = 0 \quad \text{and} \quad \Delta u - du(\nabla f) = 0\end{aligned}$$

for $t \in [t_1, t_2]$. Hence $(g(t), H(t), u(t))$ is a steady gradient GRF system soliton. □

Below we focus on the expanding breather. First we define the functionals μ_+ and ν_+ as follows:

Definition 3.16.

$$\mu_+(g, H, u, \tau) := \inf \left\{ \mathcal{W}_+(g, H, u, f, \tau) \middle| f \in C^\infty(M), \int_M \frac{e^{-f_+}}{(4\pi\tau)^{n/2}} d\mu = 1 \right\},$$

$$\nu_+(g,H,u) := \inf\left\{\mu_+(g,H,u,\tau)\Big|\tau\in\mathbb{R}^+\right\}.$$

By the definitions of functionals μ_+ and ν_+, we have the following property:

Proposition 3.17. *If $\varphi : M \to M$ is a diffeomorphism and $c > 0$ is a constant, then*

(1) $\mu_+(\varphi^*g,\varphi^*H,\varphi^*u,\tau)=\mu_+(g,H,u,\tau)$, $\nu_+(\varphi^*g,\varphi^*H,\varphi^*u)=\nu_+(g,H,u)$.
(2) $\mu_+(cg,H,u,c\tau)=\mu_+(g,H,u,\tau)$, $\nu_+(cg,H,u)=\nu_+(g,H,u)$.

Now we study the existence, smoothness, and monotonicity of these quantities.

Lemma 3.18. *Let M^n be a closed manifold. If $(g(\cdot),H(\cdot),u(\cdot))$ is a solution to the GRF system* (1.1) *on M^n, then we have*

(1) *the* inf *in the definition of μ_+ is achieved by a unique w. Moreover, $\mu_+(g(t),H(t),u(t),t-T)$ is non-decreasing in time, and is constant only on an expanding soliton with $H\equiv 0$.*
(2) *If $\nu_+<0$, then the* sup *in the definition of ν_+ is achieved by a unique τ. Moreover, $\nu_+(g(t),H(t),u(t))$ is non-decreasing in time, and is constant only on an expanding soliton with $H\equiv 0$.*

Proof. This proof is identical to that of Theorem 1.7 of [5]. In particular, we have the following two formulas

$$\frac{d}{dt}\mu_+(g(t),H(t),u(t),t-T)=\frac{d}{dt}\mathcal{W}_+(g(t),H(t),u(t),w,t-T),$$

where w realizes the minimum at time t, and

$$\frac{d}{dt}\nu_+(g(t),H(t),u(t))=\frac{d}{dt}\mu_+(g(t),H(t),u(t),\tau),$$

where (w,τ) realizes the minimax at time t, hold. If μ_+ and ν_+ are fixed constants, by the monotonicity formula (3.12), the results follow. □

In the same way, using the functional μ_+, we prove the nonexistence of the nontrivial expanding breathers.

Theorem 3.19. *Let $(g(t),H(t),u(t))$ to be a solution to the GRF system* (1.1) *on a closed manifold M^n. If $(g(t),H(t),u(t))$ is a expanding breather, then it in fact is a steady gradient GRF system soliton with $H\equiv 0$.*

Proof. Let $(M,g(t),H(t),u(t))$ be a expanding breather with

$$g(t_2)=c\varphi^*g(t_1),\quad H(t_2)=c\varphi^*H(t_1),\quad u(t_2)=c\varphi^*u(t_1), \tag{3.14}$$

where $t_2>t_1$ and $c>1$. Define $\tau(t):=t+\frac{t_2-ct_1}{c-1}$, so that $\frac{d\tau}{dt}=1$, $\tau(t_1)=\frac{t_2-t_1}{c-1}$, $\tau(t_2)=\frac{c(t_2-t_1)}{c-1}$ and $\tau(t_2)=c\tau(t_1)$. Let $\hat{f}_2$ be the minimizer for $\mathcal{W}_+$ at t_2, so that

$$\mathcal{W}_+(g(t_2),H(t_2),u(t_2),\hat{f}_2,\tau(t_2))=\mu_+(g(t_2),H(t_2),u(t_2),\tau(t_2)).$$

Take $f_+(t)$ be to the solution to the equation (3.11) on the time interval $[t_1, t_2]$ with the initial data $f_+(t_2) = \hat{f}_2$. By (3.12) and the definition of μ_+ we obtain

$$\begin{aligned}\mu_+(g(t_1), H(t_1), u(t_1), \tau(t_1)) &\leq \mathcal{W}_+(g(t_1), H(t_1), u(t_1), f_+(t_1), \tau(t_1))\\ &\leq \mathcal{W}_+(g(t), H(t), u(t), f_+(t), \tau(t))\\ &\leq \mathcal{W}_+(g(t_2), H(t_2), u(t_2), \hat{f}_2, \tau(t_2))\\ &= \mu_+(g(t_2), H(t_2), u(t_2), \tau(t_2))\end{aligned}$$

for all $t\in[t_1, t_2]$. Since (3.14) and $\tau(t_2) = c\tau(t_1)$, by Proposition 3.17, we know that

$$\mu_+(g(t_1), H(t_1), u(t_1), \tau(t_1)) = \mu_+(g(t_2), H(t_2), u(t_2), \tau(t_2)).$$

Furthermore using the fact that $\mathcal{W}_+$ is increasing, therefore we have

$$\mathcal{W}_+(g(t), H(t), u(t), f_+(t), \tau(t)) = \mu_+(g(t), H(t), u(t), \tau(t)) \equiv \text{const}$$

for $t \in [t_1, t_2]$. Thus $f_+(t)$ is the minimizer for $\mathcal{W}_+(g(t), H(t), u(t), f_+(t), \tau(t))$ and $\frac{d}{dt}\mathcal{W}_+ \equiv 0$, so by (3.12), we have

$$\begin{gathered}R_{ij} - \alpha_n u_i u_j + (f_+)_{ij} + \frac{g_{ij}}{2(t-T)} = 0,\\ \Delta u - du(\nabla f_+) = 0 \quad \text{and} \quad H \equiv 0\end{gathered}$$

for $t \in [t_1, t_2]$, where $T = \frac{ct_1 - t_2}{c-1}$. Therefore the result follows. □

4. Curvature evolution equations

In this section we compute evolution equations for curvature tensors, 3-forms, the Lapse functions and their derivatives under the GRF system (1.1). We first recall the following result for a general flow (see [2], Lemma 6.5).

Lemma 4.1. *Given $(M^n, g(t))$ with $\partial_t g_{ij} = v_{ij}$, then*

$$\begin{aligned}\partial_t \Gamma^k_{ij} &= 1/2 g^{kl}(\nabla_i v_{jl} + \nabla_j v_{il} - \nabla_l v_{ij}),\\ \partial_t R_{ijkl} &= 1/2[\nabla_i\nabla_l v_{jk} + \nabla_j\nabla_k v_{il} - \nabla_i\nabla_k v_{jl} - \nabla_j\nabla_l v_{ik}]\\ &\quad + 1/2(R_{ijpl}v_{kp} - R_{ijpk}v_{lp}),\\ \partial_t R_{ij} &= -1/2\left[\Delta_L v_{ij} + \nabla_i\nabla_j v - g^{pq}\left(\nabla_i\nabla_p v_{jq} + \nabla_j\nabla_p v_{iq}\right)\right],\\ \partial_t R &= -\Delta v + g^{pq}g^{rs}(\nabla_p\nabla_r v_{qs} - R_{pr}v_{qs}),\end{aligned}$$

where $\Delta_L := \Delta v_{ij} + 2R_{ikjl}v_{kl} - R_{ip}v_{jp} - R_{jp}v_{ip}$, $R_{ijkl} := g_{pk}R^p_{ijl}$ and $v := g^{ij}v_{ij}$.

Using the above formulas, we have:

Proposition 4.2. *Under the GRF system* (1.1)*, we have*

$$\begin{aligned}\partial_t R_{ijkl} &= \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl})\\ &\quad - R_{pjkl}R_{pi} - R_{ipkl}R_{pj} - R_{ijpl}R_{pk} - R_{ijkp}R_{pl}\\ &\quad + 1/4(R_{ijpl}h_{kp} - R_{ijpk}h_{lp})\\ &\quad + 1/4(\nabla_i\nabla_l h_{jk} + \nabla_j\nabla_k h_{il} - \nabla_i\nabla_k h_{jl} - \nabla_j\nabla_l h_{ik})\\ &\quad + 2\alpha_n(\nabla_i\nabla_k u\nabla_j\nabla_l u - \nabla_i\nabla_l u\nabla_j\nabla_k u),\end{aligned} \tag{4.1}$$

where $B_{ijkl} := R_{piqj}R_{pkql}$, *and*
(4.2)

$$\begin{aligned}\partial_t\left(\nabla^k Rm\right) = \Delta(\nabla^k Rm) &+ \sum_{\alpha+\beta=k}\nabla^\alpha Rm*\nabla^\beta Rm + \sum_{\alpha+\beta+\gamma=k}\nabla^\alpha H*\nabla^\beta H*\nabla^\gamma Rm\\ &+\sum_{\alpha+\beta=2+k}\nabla^\alpha H*\nabla^\beta H + \sum_{\alpha+\beta=k}\nabla^{2+\alpha}u*\nabla^{2+\beta}u\\ &+\sum_{\alpha+\beta=k-1} du*\nabla^{2+\alpha}u*\nabla^\beta Rm + \sum_{\alpha+\beta+\gamma=k-2}\nabla^{2+\alpha}u*\nabla^{2+\beta}u*\nabla^\gamma Rm.\end{aligned}$$

Proof. Let $v_{ij} = -2R_{ij} + 1/2h_{ij} + 2\alpha_n\partial_i u\partial_j u$. We observe that quantities $-2R_{ij}$, $1/2h_{ij}$ and $2\alpha_n\partial_i u\partial_j u$ are independent in some sense. So we can compute the evolution of curvature under the metric evolving by those three quantities separately.

If $\partial_t g_{ij} = -2R_{ij}$, by Hamilton's Ricci flow result, the evolution equation of curvature is the first and second lines on the right hand side of (4.1).

If $\partial_t g_{ij} = 1/2h_{ij}$, using Lemma 4.1, we see that the evolution equation of curvature is the third and fourth lines on the right hand side of (4.1).

If $\partial_t g_{ij} = 2\alpha_n\partial_i u\partial_j u$, using Lemma 4.1, we have

$$\begin{aligned}\partial_t R_{ijkl} &= \alpha_n\left[\nabla_i\nabla_l(u_ju_k) + \nabla_j\nabla_k(u_iu_l) - \nabla_i\nabla_k(u_ju_l) - \nabla_j\nabla_l(u_iu_k)\right]\\ &\quad + \alpha_n\left[R_{ijpl}(u_ku_p) - R_{ijpk}(u_lu_p)\right]\\ &= u_{jli}u_k + u_{jl}u_{ki} + u_{ji}u_{kl} + u_ju_{kli} + u_{ikj}u_l + u_{ik}u_{lj} + u_{ij}u_{lk} + u_iu_{lkj}\\ &\quad - u_{jki}u_l - u_{jk}u_{li} - u_{ji}u_{lk} - u_ju_{lki} - u_{ilj}u_k - u_{il}u_{kj} - u_{ij}u_{kl}\\ &\quad - u_iu_{klj} + \alpha_n(R_{ijpl}u_ku_p - R_{ijpk}u_lu_p)\\ &= 2\alpha_n(u_{ki}u_{lj} - u_{jk}u_{li}),\end{aligned}$$

where $u_{jli} := \nabla_i\nabla_l\nabla_j u$. Note that the above computation involves the communication formula $\nabla_i\nabla_j\nabla_l u - \nabla_j\nabla_i\nabla_l u = R_{ijlp}\nabla_p u$. Hence (4.1) follows by adding the above three evolution formulas.

Below we shall prove (4.2). Note that under a general geometric flow $\partial_t g_{ij} = v_{ij}$, for any tensor A, we have $\partial_t\nabla A = \nabla\partial_t A + A*\nabla v$ and

$$[\nabla,\Delta]A = \nabla\Delta A - \Delta\nabla A = Rm*\nabla A + \nabla Rm*A.$$

Therefore, under the GRF system (1.1), we have

$$\begin{aligned}\partial_t\nabla Rm &= \nabla\partial_t Rm + Rm*\nabla\left(Rm + H*H + du*du\right)\\ &= \nabla\Big(\Delta Rm + Rm*Rm + H*H*Rm + \sum_{\alpha+\beta=2}\nabla^\alpha H*\nabla^\beta H + \nabla^2u*\nabla^2u\Big)\\ &\quad + Rm*\nabla(Rm + H*H + du*du)\\ &= \Delta\nabla Rm + \nabla Rm*Rm + \sum_{\alpha+\beta+\gamma=1}\nabla^\alpha H*\nabla^\beta H*\nabla^\gamma Rm\\ &\quad + \sum_{\alpha+\beta=3}\nabla^\alpha H*\nabla^\beta H + \nabla^2u*\nabla^3u + \nabla u*\nabla^2u*Rm.\end{aligned}$$

This finishes the proof of the case $k=1$. Now by induction, we assume that we have gotten the evolution formula of $\nabla^j Rm$ for all $1\le j<k$. We begin by computing

$$
\begin{aligned}
&\partial_t(\nabla^k Rm)\\
=\;&\partial_t\nabla(\nabla^{k-1}Rm)\\
=\;&\nabla\partial_t(\nabla^{k-1}Rm)+(\nabla^{k-1}Rm)*\nabla(Rm+H*H+du*du)\\
=\;&\nabla\Big[\Delta(\nabla^{k-1}Rm)+\sum_{\alpha+\beta=k-1}\nabla^\alpha Rm*\nabla^\beta Rm+\sum_{\alpha+\beta+\gamma=k-1}\nabla^\alpha H*\nabla^\beta H*\nabla^\gamma Rm\\
&+\sum_{\alpha+\beta=1+k}\nabla^\alpha H*\nabla^\beta H+\sum_{\alpha+\beta=k-1}\nabla^{2+\alpha}u*\nabla^{2+\beta}u+\sum_{\alpha+\beta=k-2}du*\nabla^{2+\alpha}u*\nabla^\beta Rm\\
&+\sum_{\alpha+\beta+\gamma=k-3}\nabla^{2+\alpha}u*\nabla^{2+\beta}u*\nabla^\gamma Rm\Big]+(\nabla^{k-1}Rm)*\nabla(Rm+H*H+du*du)\\
=\;&\Delta(\nabla^k Rm)+\sum_{\alpha+\beta=k}\nabla^\alpha Rm*\nabla^\beta Rm+\sum_{\alpha+\beta+\gamma=k}\nabla^\alpha H*\nabla^\beta H*\nabla^\gamma Rm\\
&+\sum_{\alpha+\beta=2+k}\nabla^\alpha H*\nabla^\beta H+\sum_{\alpha+\beta=k}\nabla^{2+\alpha}u*\nabla^{2+\beta}u+\sum_{\alpha+\beta=k-1}du*\nabla^{2+\alpha}u*\nabla^\beta Rm\\
&+\sum_{\alpha+\beta+\gamma=k-2}\nabla^{2+\alpha}u*\nabla^{2+\beta}u*\nabla^\gamma Rm.
\end{aligned}
$$

This completes the inductive step. Hence we prove (4.2). $\square$

By the above curvature evolution equation under the system (1.1), we have:

Corollary 4.3. *Under the GRF system* (1.1), *we have*

$$
\partial_t Rm=\Delta Rm+Rm*Rm+H*H*Rm+\sum_{\alpha+\beta=2}\nabla^\alpha H*\nabla^\beta H+\nabla^2u*\nabla^2u; \tag{4.3}
$$

$$
\begin{aligned}
\partial_t|Rm|^2\le\;&\Delta|Rm|^2-2|\nabla Rm|^2+C|Rm|^3+C(|H|^2+|du|^2)\cdot|Rm|^2\\
&+C\sum_{\alpha+\beta=2}|\nabla^\alpha H|\cdot|\nabla^\beta H|\cdot|Rm|+C|\nabla^2u|^2\cdot|Rm|;
\end{aligned}\tag{4.4}
$$

$$
\begin{aligned}
&\partial_t\left|\nabla^k Rm\right|^2\\
\le\;&\Delta\left|\nabla^k Rm\right|^2-2\left|\nabla^{k+1}Rm\right|^2+C\left|\nabla^k Rm\right|\cdot\sum_{\alpha+\beta=k}|\nabla^\alpha Rm|\cdot|\nabla^\beta Rm|\\
&+C\left|\nabla^k Rm\right|\cdot\sum_{\alpha+\beta+\gamma=k}|\nabla^\beta H|\cdot|\nabla^\gamma H|\cdot|\nabla^\alpha Rm|\\
&+C\left|\nabla^k Rm\right|\cdot\sum_{\alpha+\beta=2+k}|\nabla^\alpha H|\cdot|\nabla^\beta H|+C\left|\nabla^k Rm\right|\cdot\sum_{\alpha+\beta=k}|\nabla^{2+\alpha}u|\cdot|\nabla^{2+\beta}u|
\end{aligned}\tag{4.5}
$$

$$+ C \left|\nabla^k Rm\right| \cdot \sum_{\alpha+\beta=k-1} |du|\cdot|\nabla^{2+\alpha}u|\cdot|\nabla^\beta Rm|$$
$$+ C \left|\nabla^k Rm\right| \cdot \sum_{\alpha+\beta+\gamma=k-2} |\nabla^{2+\alpha}u|\cdot|\nabla^{2+\beta}u|\cdot|\nabla^\gamma Rm| + C|du|^2\cdot \left|\nabla^k Rm\right|^2.$$

Proof. Evolution formula (4.3) is obvious. Next we will prove the second evolution formula. By the evolution equation (4.1), we have

$$\begin{aligned}
\partial_t|Rm|^2 &= 2Rm * (\partial_t Rm) + Rm * Rm * (Rc + H * H + du * du) \\
&= 2Rm * \Big(\Delta Rm + Rm*Rm + Rm*H*H + \sum_{\alpha+\beta=2} \nabla^\alpha H * \nabla^\beta H + \nabla^2 u * \nabla^2 u\Big) \\
&\quad + Rm * Rm * (Rc + H * H + du * du) \\
&\leq \Delta|Rm|^2 - 2|\nabla Rm|^2 + C|Rm|^3 + C\left(|H|^2 + |du|^2\right)\cdot|Rm|^2 \\
&\quad + C\sum_{\alpha+\beta=2} |\nabla^\alpha H|\cdot|\nabla^\beta H|\cdot|Rm| + C|\nabla^2 u|^2\cdot|Rm|.
\end{aligned}$$

Hence (4.4) follows. In the end we will prove (4.5). Using

$$\partial_t \left|\nabla^k Rm\right|^2 = 2\left\langle \partial_t(\nabla^k Rm), \nabla^k Rm\right\rangle + (Rm + H*H + du*du) * (\nabla^k Rm)^2$$

and combining with (4.2), then (4.5) follows. □

Proposition 4.4. *Under the GRF system* (1.1)*, we have*

$$\partial_t(\nabla H) = \Delta(\nabla H) + \sum_{\alpha+\beta=1} \nabla^\alpha Rm * \nabla^\beta H + H*H*\nabla H + du*\nabla^2 u*H \tag{4.6}$$

and for all $k \geq 2$

$$\begin{aligned}
\partial_t(\nabla^k H) &= \Delta(\nabla^k H) + \sum_{\alpha+\beta=k} \nabla^\alpha Rm * \nabla^\beta H + \sum_{\alpha+\beta+\gamma=k} \nabla^\alpha H * \nabla^\beta H * \nabla^\gamma H \\
&\quad + \sum_{\alpha+\beta=k-1} du * \nabla^{2+\alpha} u * \nabla^\beta H + \sum_{\alpha+\beta+\gamma=k-2} \nabla^{2+\alpha} u * \nabla^{2+\beta} u * \nabla^\gamma H.
\end{aligned} \tag{4.7}$$

Proof. Recall that $\partial_t H = \Delta H + H * Rm$. Hence

$$\begin{aligned}
\partial_t(\nabla H) &= \nabla \partial_t H + H * \nabla(Rm + H * H + du \otimes du) \\
&= \nabla(\Delta H + H * Rm) + H * \nabla(Rm + H * H + du \otimes du) \\
&= \Delta\nabla H + Rm*\nabla H + \nabla Rm * H + H*H*\nabla H + du*\nabla^2 u*H
\end{aligned}$$

and (4.6) follows. In the following, we assume the evolution equation $\nabla^j H$ for all $1 \leq j < k$ holds as in (4.7) and we want to prove this also holds for the case

$j = k$.

$$\begin{aligned}\partial_t\left(\nabla^k H\right) &= \nabla\left(\partial_t \nabla^{k-1} H\right) + \nabla^{k-1} H * \nabla(Rm + H*H + du\otimes du)\\
&= \nabla\Big[\Delta(\nabla^{k-1}H) + \sum_{\alpha+\beta=k-1}\nabla^\alpha Rm*\nabla^\beta H + \sum_{\alpha+\beta+\gamma=k-1}\nabla^\alpha H*\nabla^\beta H*\nabla^\gamma H\\
&\quad + \sum_{\alpha+\beta=k-2} du*\nabla^{2+\alpha}u*\nabla^\beta H + \sum_{\alpha+\beta+\gamma=k-3}\nabla^{2+\alpha}u*\nabla^{2+\beta}u*\nabla^\gamma H\Big]\\
&\quad + \nabla^{k-1}H*\nabla(Rm + H*H + du\otimes du)\\
&= \Delta(\nabla^k H) + \sum_{\alpha+\beta=k}\nabla^\alpha Rm*\nabla^\beta H + \sum_{\alpha+\beta+\gamma=k}\nabla^\alpha H*\nabla^\beta H*\nabla^\gamma H\\
&\quad + \sum_{\alpha+\beta=k-1} du*\nabla^{2+\alpha}u*\nabla^\beta H + \sum_{\alpha+\beta+\gamma=k-2}\nabla^{2+\alpha}u*\nabla^{2+\beta}u*\nabla^\gamma H.\end{aligned}$$

This completes the proposition. □

By Proposition 4.4, we have:

Corollary 4.5. *Under the GRF system* (1.1)*, we have*

$$\begin{aligned}\partial_t|\nabla H|^2 &\le \Delta|\nabla H|^2 - 2|\nabla^2 H|^2 + C|\nabla H|\cdot\sum_{\alpha+\beta=1}|\nabla^\alpha Rm|\cdot|\nabla^\beta H|\\
&\quad + C\left(|H|^2 + |du|^2\right)\cdot|\nabla H|^2 + C|du|\cdot|\nabla^2 u|\cdot|H|\cdot|\nabla H|\end{aligned}$$

and for all $k \ge 2$

(4.8)

$$\begin{aligned}&\partial_t|\nabla^k H|^2\\
&\le \Delta|\nabla^k H|^2 - 2|\nabla^{k+1}H|^2 + C|\nabla^k H|\cdot\sum_{\alpha+\beta=k}|\nabla^\alpha H|\cdot|\nabla^\beta Rm|\\
&\quad + C|\nabla^k H|\cdot\sum_{\alpha+\beta+\gamma=k}|\nabla^\alpha H|\cdot|\nabla^\beta H|\cdot|\nabla^\gamma H| + C|\nabla^k H|\cdot\sum_{\alpha+\beta=k-1}|du|\cdot|\nabla^{2+\alpha}u|\cdot|\nabla^\beta H|\\
&\quad + C|du|^2\cdot|\nabla^k H|^2 + C|\nabla^k H|\cdot\sum_{\alpha+\beta+\gamma=k-2}|\nabla^{2+\alpha}u|\cdot|\nabla^{2+\beta}u|\cdot|\nabla^\gamma H|.\end{aligned}$$

Proof. The proof of this corollary is similar to that of Corollary 4.3. □

Proposition 4.6. *Under the GRF system* (1.1)*, we also have*

$$\begin{aligned}\partial_t(u_{ij}) &= \Delta(u_{ij}) + 2R_{ipjq}u_{pq} - R_{ip}u_{jp} - R_{jp}u_{ip} - 2\alpha_n|du|^2 u_{ij}\\
&\quad - 1/4(\nabla_i h_{jk} + \nabla_j h_{ik} - \nabla_k h_{ij})u_k;\end{aligned} \tag{4.9}$$

$$\begin{aligned}\partial_t\left(\nabla^{2+k}u\right) &= \Delta(\nabla^{2+k}u) + \sum_{\alpha+\beta=k}\nabla^{2+\alpha}u*\nabla^\beta Rm\\
&\quad + \sum_{\alpha+\beta=k-1} du*\nabla^{2+\alpha}u*\nabla^{2+\beta}u\end{aligned} \tag{4.10}$$

$$+\sum_{\alpha+\beta+\gamma=k-2}\nabla^{2+\alpha}u*\nabla^{2+\beta}u*\nabla^{2+\gamma}u+|du|^2\cdot\nabla^{2+k}u$$
$$+\sum_{\alpha+\beta+\gamma=k}\nabla^{1+\alpha}H*\nabla^{\beta}H*\nabla^{1+\gamma}u.$$

Proof. We calculate

$$\partial_t(u_{ij})=\nabla_i\nabla_j\left(\partial_t u\right)-\left(\partial_t\Gamma^k_{ij}\right)u_k. \tag{4.11}$$

Note that

$$\begin{aligned}\nabla_i\nabla_j(u_t)&=\nabla_i\nabla_j(\nabla_k\nabla_k u)\\&=\nabla_i(\nabla_k\nabla_j\nabla_k u+R_{jkkp}\nabla_p u)\\&=\nabla_k\nabla_i\nabla_k\nabla_j u+R_{ikjp}\nabla_p\nabla_k u+R_{ikkp}\nabla_j\nabla_p u-\nabla_i R_{jp}\nabla_p u\\&\quad-R_{jp}\nabla_i\nabla_p u\\&=\nabla_k\nabla_k\nabla_i\nabla_j u+\nabla_k R_{ikjp}\nabla_p u+R_{ikjp}\nabla_k\nabla_p u+R_{ikjp}\nabla_p\nabla_k u\\&\quad-R_{ip}\nabla_j\nabla_p u-\nabla_i R_{jp}\nabla_p u-R_{jp}\nabla_i\nabla_p u\\&=\Delta(u_{ij})+2R_{ipjq}u_{pq}-R_{ip}u_{jp}-R_{jp}u_{ip}+\nabla_k R_{ikjp}\nabla_p u-\nabla_i R_{jp}\nabla_p u\end{aligned}$$

and

$$\begin{aligned}-\left(\partial_t\Gamma^k_{ij}\right)u_k&=g^{kl}(\nabla_i R_{jl}+\nabla_j R_{il}-\nabla_l R_{ij})u_k\\&\quad-1/4g^{kl}(\nabla_i h_{jl}+\nabla_j h_{il}-\nabla_l h_{ij})u_k+g^{kl}(-2\alpha_n\nabla_i\nabla_j u_l)u_k.\end{aligned}$$

Substituting the above two formulas into (4.11), then (4.9) follows from the second Bianchi identity.

For the second part, we prove (4.10) by induction where the claim for $\nabla^{2+0}u$ is proven in (4.9). Plugging in the induction hypotheses for $\nabla^{2+k}u$, we compute

$$\begin{aligned}\partial_t(\nabla^{2+k+1}u)&=\nabla\left(\partial_t\nabla^{2+k}u\right)+\nabla^{2+k}u*\nabla(Rm+H*H+du*du)\\&=\nabla\Big[\Delta(\nabla^{2+k}u)+\sum_{\alpha+\beta=k}\nabla^{2+\alpha}u*\nabla^{\beta}Rm+\sum_{\alpha+\beta=k-1}du*\nabla^{2+\alpha}u*\nabla^{2+\beta}u\\&\quad+\sum_{\alpha+\beta+\gamma=k-2}\nabla^{2+\alpha}u*\nabla^{2+\beta}u*\nabla^{2+\gamma}u+|du|^2\cdot\nabla^{2+k}u\\&\quad+\sum_{\alpha+\beta+\gamma=k}\nabla^{1+\alpha}H*\nabla^{\beta}H*\nabla^{1+\gamma}u\Big]\\&\quad+\nabla^{2+k}u*\nabla(Rm+H*H+du*du).\end{aligned}$$

This can be rearranged to yield the claim for $\nabla^{2+(k+1)}u$. □

In the same way, we have:

Corollary 4.7. *Under the GRF system* (1.1)*, we have*

$$\begin{aligned}\partial_t|\nabla^2u|^2&\le\Delta|\nabla^2u|^2-2|\nabla^3u|^2+C(|Rm|+|H|^2+|du|^2)|\nabla^2u|^2\\&\quad+C|H|\cdot|\nabla H|\cdot|du|\cdot|\nabla^2u|\end{aligned}$$

and for all $k \geq 0$,

(4.12)

$$\begin{aligned}\partial_t|\nabla^{2+k}u|^2 \leq \Delta|\nabla^{2+k}u|^2 &- 2|\nabla^{3+k}u|^2 + C|\nabla^{2+k}u| \cdot \sum_{\alpha+\beta=k}|\nabla^{2+\alpha}u|\cdot|\nabla^{\beta}Rm|\\ &+ C|\nabla^{2+k}u|\cdot\sum_{\alpha+\beta=k-1}|du|\cdot|\nabla^{2+\alpha}u|\cdot|\nabla^{2+\beta}u|\\ &+ C|\nabla^{2+k}u|\cdot\sum_{\alpha+\beta+\gamma=k-2}|\nabla^{2+\alpha}u|\cdot|\nabla^{2+\beta}u|\cdot|\nabla^{2+\gamma}u|\\ &+ C(H^2+|du|^2)\cdot|\nabla^{2+k}u|^2\\ &+ C|\nabla^{2+k}u|\cdot\sum_{\alpha+\beta+\gamma=k}|\nabla^{1+\alpha}H|\cdot|\nabla^{\beta}H|\cdot|\nabla^{1+\gamma}u|.\end{aligned}$$

5. Estimates of Bernstein-Bando-Shi type

In this section we will prove derivative estimates for geometric solutions of the system (1.1). These estimates are generalizations of the Bernstein-Bando-Shi (BBS for short) estimates for Ricci flow. Our proof follows that in [14], see also [2,3].

The evolution equations for u and $|du|^2$ give us good control on the behavior of the derivative of the Lapse function.

Proposition 5.1. *Let $(g(x,t), H(x,t), u(x,t))$ be a solution to the system* (1.1) *on a closed manifold M^n on $0 \leq t \leq T$. Then we have*

$$\inf_{x\in M^n} u(x,0) \leq u(x,t) \leq \sup_{x\in M^n} u(x,0); \tag{5.1}$$

$$\sup_{x\in M^n} |du|^2(x,t) \leq \max_{x\in M^n} |du|^2(x,0); \tag{5.2}$$

$$\sup_{x\in M^n} |du|^2(x,t) \leq \frac{1}{2\alpha_n t} \tag{5.3}$$

for all $(x,t) \in M^n \times (0,T]$.

Proof. Since M^n is closed, we can obtain (5.1) by applying the maximum principle to $u_t = \Delta u$. Similarly, using the maximum principle to the following equation

$$\partial_t|du|^2 = \Delta|du|^2 - 1/2|H_{ikl}u_i|^2 - 2|\nabla^2 u|^2 - 2\alpha_n|du|^4$$

to obtain (5.2). In the end, we will prove (5.3). Note that

$$\partial_t|du|^2 = \Delta|du|^2 - 1/2|H_{ikl}u_i|^2 - 2|\nabla^2 u|^2 - 2\alpha_n|du|^4 \leq \Delta|du|^2 - 2\alpha_n|du|^4.$$

Applying maximum principle, we have $|du|^2 \leq \frac{1}{2\alpha_n t}$ for all $(x,t) \in M^n \times (0,T]$. ∎

We also have the following result.

Theorem 5.2. *Let $(g(x,t), H(x,t), u(x,t))$ be a solution to the system* (1.1) *on a closed manifold M^n on $0 \le t \le T$ and C_1 and C_2 are arbitrary given nonnegative constants. Then there exists a constant $C(n)$ depending only on n such that if*

$$|Rm(x,t)|_{g(x,t)} \le C_1 \quad \text{and} \quad |H(x)|_{g(x,0)} \le C_2$$

for all $(x,t) \in M^n \times [0,T]$, then

$$|H(x,t)|_{g(x,t)} \le C_2 \cdot e^{C(n)\cdot C_1 t}$$

for all $(x,t) \in M^n \times [0,T]$.

Proof. Note that

$$\begin{aligned}\partial_t H_{ijk} &= \Delta H_{ijk} - R_{ip}H_{pjk} - R_{pijq}H_{pqk} - R_{pikq}H_{pjq} \\ &\quad + R_{jp}H_{pik} + R_{pjiq}H_{pqk} + R_{pjkq}H_{piq} \\ &\quad - R_{kp}H_{pij} - R_{pkiq}H_{pqj} - R_{pkjq}H_{piq} \\ &\le \Delta H_{ijk} + C(n)\cdot C_1 \cdot |H|.\end{aligned}$$

Hence

$$\begin{aligned}\partial_t |H|^2 &\le \Delta|H|^2 - 2|\nabla H|^2 + C(n)\cdot C_1\cdot |H|^2 \\ &\quad + 2H_{ikp}H_{jkp}(2R_{ij} - 1/2h_{ij} - 2\alpha_n u_i u_j) \\ &\le \Delta |H|^2 + C(n)\cdot C_1 \cdot |H|^2.\end{aligned}$$

Let $\rho(t)$ be the solution to the corresponding ODE: $\partial_t \rho^2 = C(n)\cdot C_1 \cdot \rho^2$ with $\rho(0) = C_2$. By the maximum principle, we have $|H(x,t)|^2 \le C_2^2 \cdot e^{C(n)\cdot C_1 t}$. $\square$

Note that the above two theorems may not hold on non-compact manifolds. Because we may not apply the maximum principle directly on non-compact manifolds.

In the following, we will derive the Bernstein-Bando-Shi derivative estimates of the GRF system (1.1) on complete manifolds. First we give the maximum principle on non-compact manifolds under the GRF system (1.1).

Theorem 5.3. *Let $(g(x,t), H(x,t), u(x,t))$ be a solution to the system* (1.1) *on a complete manifold M^n on $0 \le t \le T$. Assume that $g(t) \ge g_*$, where g_* is a complete metric, and that $R_*(t) := \inf_M \left[R - \frac{1}{4}|H|^2 - \alpha_n |du|^2\right]$ is finite and integrable on $[0,T]$. Let $v(x,t)$ be a Lipschitz weak subsolution to the heat equation: $v_t \le \Delta_{g(t)} v$ such that there exist a constant $b > 0$ and $O \in M^n$ where*

$$\int_0^T \int_M e^{-bd_*^2(O,x)} v_+^2(x,t)\, d\mu_t(x)\, dt < \infty,$$

where $d_*(O,x)$ denotes the distance with respect to g_* and $v_+ := \max\{v, 0\}$. If $v(x,0) \le 0$, then $v(x,t) \le 0$ on $M \times [0,T]$.

Proof. See Theorem 7.42 in [3]. $\square$

Using the above theorem, we have our main result in this section.

Theorem 5.4. *Let $(g(x,t), H(x,t), u(x,t))$ be a solution to the system* (1.1) *on a complete manifold M^n on $0 \le t \le T$ and K_i $(i = 1,2,3)$ be an arbitrary given positive constant. Then for each $\beta > 0$ and each integer $m \ge 1$ there exists a constant C_m depending on $n, K_1, K_2, K_3, \max\{\beta, 1\}$ and m such that if*

$$(5.4)\quad |Rm(x,t)|_{g(x,t)} \le K_1, \quad |H(x,t)|_{g(x,t)} \le K_2 \quad \text{and} \quad |du(x,t)|_{g(x,t)} \le K_3$$

for all $(x,t) \in M^n \times [0, \frac{\beta}{K}]$, then

$$|\nabla^{m-1} Rm(x,t)|_{g(x,t)} + |\nabla^m H(x,t)|_{g(x,t)} + |\nabla^{m+1} u(x,t)|_{g(x,t)} \le \frac{C_m}{t^{m/2}}$$

for all $(x,t) \in M^n \times (0, \frac{\beta}{K}]$.

Proof. The proof is by complete induction on m. First consider $m = 1$. In the discussion below the constant C may change from line to line and depends on some or all of $n, K_1, K_2, K_3, \max\{\beta, 1\}$, and m. By Corollaries 4.3, 4.5 and 4.7, we have

$$\partial_t |Rm|^2 \le \Delta |Rm|^2 - 2|\nabla Rm|^2 + C \sum_{\alpha+\beta=2} |\nabla^\alpha H| \cdot |\nabla^\beta H| + C|\nabla^2 u|^2 + C;$$

$$\begin{aligned}\partial_t |\nabla H|^2 \le\ & \Delta |\nabla H|^2 - 2|\nabla^2 H|^2 + C|\nabla H| \cdot \sum_{\alpha+\beta=1} |\nabla^\alpha Rm| \cdot |\nabla^\beta H| \\ & + C|\nabla H|^2 + C|\nabla H| \cdot |\nabla^2 u|;\end{aligned}$$

$$\partial_t |\nabla^2 u|^2 \le \Delta |\nabla^2 u|^2 - 2|\nabla^3 u|^2 + C|\nabla^2 u|^2 + C|\nabla H| \cdot |\nabla^2 u|;$$

$$\partial_t |H|^2 \le \Delta |H|^2 - 2|\nabla H|^2 + C; \qquad \partial_t u^2 = \Delta(u^2) - 2|\nabla u|^2;$$

$$\begin{aligned}\partial_t |du|^2 &= \Delta |du|^2 - 1/2|H_{ikl} u_i|^2 - 2|\nabla^2 u|^2 - 2\alpha_n |du|^4 \\ &\le \Delta |du|^2 - 2|\nabla^2 u|^2 - 2\alpha_n |du|^4.\end{aligned}$$

Now we consider the quantity $v := t\left[|Rm|^2 + |\nabla H|^2 + |\nabla^2 u|^2\right] + A|H|^2 + B|du|^2$, where A and B are both positive constants, to be determined later. Using the above evolution formulas, we compute

(5.5)

$$\begin{aligned} v_t \le\ & t\Big[\Delta |Rm|^2 - 2|\nabla Rm|^2 + C|\nabla^2 u|^2 + C|\nabla^2 H| + C|\nabla H|^2 + C \\ & + \Delta |\nabla H|^2 - 2|\nabla^2 H|^2 + C|\nabla H| \cdot |\nabla Rm| + C|\nabla H| \cdot |\nabla^2 u| \\ & + \Delta |\nabla^2 u|^2 - 2|\nabla^3 u|^2 + C|\nabla H| \cdot |\nabla^2 u|\Big] + |\nabla H|^2 + |\nabla^2 u|^2 + C \\ & + A\Big[\Delta |H|^2 - 2|\nabla H|^2 + C\Big] + B\Big[\Delta |du|^2 - 2|\nabla^2 u|^2 - 2\alpha_n |du|^4\Big] \\ \le\ & \Delta v + t\Big[C|\nabla^2 u|^2 + C|\nabla^2 H| + C - 2|\nabla^2 H|^2 + C|\nabla H|^2 - 2|\nabla^3 u|^2\Big] \\ & - 2A|\nabla H|^2 - 2B|\nabla^2 u|^2 - 2\alpha_n B|du|^4 + |\nabla H|^2 + |\nabla^2 u|^2 + C \end{aligned}$$

$$\leq \Delta v + t\left[C|\nabla^2 u|^2 + C + C|\nabla H|^2\right] - 2A|\nabla H|^2 - 2B|\nabla^2 u|^2 + |\nabla H|^2 + |\nabla^2 u|^2 + C$$
$$= \Delta v + (Ct + 1 - 2B)|\nabla^2 u|^2 + (Ct + 1 - 2A)|\nabla H|^2 + Ct + C,$$

where for the third equality we used $C|\nabla^2 H| - 2|\nabla^2 H|^2 + C \leq 0$.

As along as A and B are chosen large enough such that $Ct + 1 - 2B \leq 0$ and $Ct + 1 - 2A \leq 0$, by (5.5), we have $v_t \leq \Delta v + C$. Since $v(0) \leq C$, by Theorem 5.3, we apply the maximum principle to above inequality $v(x,t) \leq C$ for all $(x,t) \in M^n \times [0, \frac{\beta}{K}]$. This proves the theorem in the case $m = 1$.

Next we shall prove the theorem for $m = 2$. Consider the following quantity

$$\begin{aligned} v := {} & t^2\left[|\nabla Rm|^2 + |\nabla^2 H|^2 + |\nabla^3 u|^2\right] + tE\left[|Rm|^2 + |\nabla H|^2 + |\nabla^2 u|^2\right] \\ & + A|H|^2 + B|du|^2, \end{aligned}$$

where A, B and E are all positive constants, to be determined later.

Note that by (4.5), (4.8) and (4.12), using the fact: $|\nabla H| + |\nabla^2 u| \leq \frac{C}{t^{1/2}}$ (the case $m = 1$), we have

$$\begin{aligned} \partial_t |\nabla Rm|^2 \leq {} & \Delta |\nabla Rm|^2 - 2|\nabla^2 Rm|^2 + C|\nabla Rm|^2 + C|\nabla H|\cdot|\nabla Rm| \\ & + C|\nabla^3 H|\cdot|\nabla Rm| + C|\nabla H|\cdot|\nabla^2 H|\cdot|\nabla Rm| \\ & + |\nabla^2 u|\cdot|\nabla^3 u|\cdot|\nabla Rm| + C|\nabla^2 u|\cdot|\nabla Rm| \\ \leq {} & \Delta |\nabla Rm|^2 - 2|\nabla^2 Rm|^2 + C|\nabla Rm|^2 + \frac{C}{t^{1/2}}|\nabla Rm| + C|\nabla^3 H|\cdot|\nabla Rm| \\ & + \frac{C}{t^{1/2}}|\nabla^2 H|\cdot|\nabla Rm| + \frac{C}{t^{1/2}}|\nabla^3 u|\cdot|\nabla Rm| \\ \leq {} & \Delta |\nabla Rm|^2 - 2|\nabla^2 Rm|^2 + \left(C + \frac{C}{t}\right)|\nabla Rm|^2 \\ & + |\nabla^3 H|^2 + |\nabla^2 H|^2 + |\nabla^3 u|^2 + C, \end{aligned}$$

$$\begin{aligned} \partial_t |\nabla^2 H|^2 \leq {} & \Delta |\nabla^2 H|^2 - 2|\nabla^3 H|^2 + C|\nabla^2 H|\cdot|\nabla^2 Rm| + C|\nabla^2 H|\cdot|\nabla H|\cdot|\nabla Rm| \\ & + C|\nabla^2 H|^2 + C|\nabla^2 H|\cdot|\nabla H|^2 + C|\nabla^2 H|\cdot|\nabla H|\cdot|\nabla^2 u| \\ & + C|\nabla^2 H|\cdot|\nabla^3 u| + C|\nabla^2 H|\cdot|\nabla^2 u|^2 \\ \leq {} & \Delta |\nabla^2 H|^2 - 2|\nabla^3 H|^2 + C|\nabla^2 H|\cdot|\nabla^2 Rm| + \frac{C}{t^{1/2}}|\nabla^2 H|\cdot|\nabla Rm| \\ & + C|\nabla^2 H|^2 + \frac{C}{t}|\nabla^2 H| + C|\nabla^2 H|\cdot|\nabla^3 u| \\ \leq {} & \Delta |\nabla^2 H|^2 - 2|\nabla^3 H|^2 + |\nabla^2 Rm|^2 + \frac{C}{t}|\nabla Rm|^2 + \left(C + \frac{C}{t^2}\right)|\nabla^2 H|^2 \\ & + |\nabla^3 u|^2 + C \end{aligned}$$

and

$$\begin{aligned} \partial_t |\nabla^3 u|^2 \leq {} & \Delta |\nabla^3 u|^2 - 2|\nabla^4 u|^2 + C|\nabla^3 u|\cdot|\nabla^2 u|\cdot|\nabla Rm| + C|\nabla^3 u|^2 \\ & + C|\nabla^3 u|\cdot|\nabla^2 u|^2 + C|\nabla^3 u|\cdot|\nabla^2 H| + C|\nabla^3 u|\cdot|\nabla H|^2 \end{aligned}$$

$$
\begin{aligned}
&+C|\nabla^3 u|\cdot|\nabla H|\cdot|\nabla^2 u|\\
&\le \Delta|\nabla^3 u|^2-2|\nabla^4 u|^2+\frac{C}{t^{1/2}}|\nabla^3 u|\cdot|\nabla Rm|+C|\nabla^3 u|^2+\frac{C}{t}|\nabla^3 u|\\
&+C|\nabla^3 u|\cdot|\nabla^2 H|\\
&\le \Delta|\nabla^3 u|^2-2|\nabla^4 u|^2+\frac{C}{t}|\nabla Rm|^2+\left(C+\frac{C}{t^2}\right)|\nabla^3 u|^2+|\nabla^2 H|^2+C.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&\partial_t|\nabla Rm|^2+\partial_t|\nabla^2 H|^2+\partial_t|\nabla^3 u|^2\\
&\le \Delta|\nabla Rm|^2+\Delta|\nabla^2 H|^2+\Delta|\nabla^3 u|^2-|\nabla^2 Rm|^2-|\nabla^3 H|^2\\
&-2|\nabla^4 u|^2+\left(C+\frac{C}{t}\right)|\nabla Rm|^2+\left(C+\frac{C}{t^2}\right)|\nabla^2 H|^2+\left(C+\frac{C}{t^2}\right)|\nabla^3 u|^2+C.
\end{aligned}
$$

Using the fact: $|\nabla H|+|\nabla^2 u|\le \frac{C}{t^{1/2}}$, we also have

$$
\begin{aligned}
&\partial_t|Rm|^2+\partial_t|\nabla H|^2+\partial_t|\nabla^2 u|^2\\
&\le \Delta|Rm|^2+\Delta|\nabla H|^2+\Delta|\nabla^2 u|^2\\
&-2|\nabla Rm|^2-2|\nabla^2 H|^2-2|\nabla^3 u|^2+\frac{C}{t}+C|\nabla^2 H|+C+C|\nabla H|\cdot|\nabla Rm|\\
&\le \Delta|Rm|^2+\Delta|\nabla H|^2+\Delta|\nabla^2 u|^2-|\nabla Rm|^2-|\nabla^2 H|^2-2|\nabla^3 u|^2+\frac{C}{t}+C.
\end{aligned}
$$

By the above evolution formulas, we obtain

$$
\begin{aligned}
v_t &\le 2t\left[|\nabla Rm|^2+|\nabla^2 H|^2+|\nabla^3 u|^2\right]+E\left[|Rm|^2+|\nabla H|^2+|\nabla^2 u|^2\right]\\
&+t^2\left[\frac{\partial|\nabla Rm|^2}{\partial t}+\frac{\partial|\nabla^2 H|^2}{\partial t}+\frac{\partial|\nabla^3 u|^2}{\partial t}\right]\\
&+tE\left[\frac{\partial|Rm|^2}{\partial t}+\frac{\partial|\nabla H|^2}{\partial t}+\frac{\partial|\nabla^2 u|^2}{\partial t}\right]\\
&+A\left[\Delta|H|^2-2|\nabla H|^2+C\right]+B\left[\Delta|du|^2-2|\nabla^2 u|^2-2\alpha_n|du|^4\right]\\
&\le \Delta v+2t\left[|\nabla Rm|^2+|\nabla^2 H|^2+|\nabla^3 u|^2\right]+E\left[|Rm|^2+|\nabla H|^2+|\nabla^2 u|^2\right]\\
&+[C(t^2+t)|\nabla Rm|^2+C(t^2+t)|\nabla^2 H|^2+C(t^2+1)|\nabla^3 u|^2]+Ct^2\\
&-t[E|\nabla Rm|^2+E|\nabla^2 H|^2+2E|\nabla^3 u|^2]+EC+tEC-2A|\nabla H|^2\\
&-2B|\nabla^2 u|^2+AC.
\end{aligned}
$$

Taking E large enough compared to C, the above inequality reduces to

$$
v_t\le \Delta v+E\left[|Rm|^2+|\nabla H|^2+|\nabla^2 u|^2\right]-2A|\nabla H|^2-2B|\nabla^2 u|^2+C.
$$

Choosing A and B large enough compared to E, we conclude $v_t\le\Delta v+C$. Since $v(0)\le C$, by Theorem 5.3, using the maximum principle, we have $v(x,t)\le C$ for all $(x,t)\in M^n\times[0,\frac{\beta}{K}]$. This proves the theorem in the case $m=2$.

In the following, we shall assume by induction that given $m \in \mathbb{N}$ and $m \geq 3$,

$$|\nabla^{j-1} Rm(x,t)|_{g(x,t)} + |\nabla^j H(x,t)|_{g(x,t)} + |\nabla^{j+1} u(x,t)|_{g(x,t)} \leq \frac{C_j}{t^{j/2}} \tag{5.6}$$

for $j = 1, \dots, m$, and want to prove

$$|\nabla^m Rm(x,t)|_{g(x,t)} + |\nabla^{m+1} H(x,t)|_{g(x,t)} + |\nabla^{m+2} u(x,t)|_{g(x,t)} \leq \frac{C_{m+1}}{t^{(m+1)/2}}. \tag{5.7}$$

Let

$$\begin{aligned} v := {} & t^{m+1}\left[|\nabla^m Rm|^2 + |\nabla^{m+1} H|^2 + |\nabla^{m+2} u|^2\right] \\ & + \sum_{i=1}^m E_i t^i \left[|\nabla^{i-1} Rm|^2 + |\nabla^i H|^2 + |\nabla^{i+1} u|^2\right] + A|H|^2 + B|du|^2, \end{aligned}$$

where A, B and E_i are all positive constants, to be determined later. By Corollary 4.3, Corollary 4.5 and Corollary 4.7, using (5.6), we have

$$\begin{aligned} \partial_t |\nabla^m Rm|^2 \leq {} & \Delta|\nabla^m Rm|^2 - 2|\nabla^{m+1} Rm|^2 + C|\nabla^m Rm|^2 + \frac{C}{t^{\frac{m+2}{2}}}|\nabla^m Rm| \\ & + C\left(\frac{1}{t^{m/2}} + \frac{1}{t^{\frac{m+1}{2}}}\right)|\nabla^m Rm| + C|\nabla^{m+2} H| \cdot |\nabla^m Rm| \\ & + \frac{C}{t^{1/2}}|\nabla^{m+1} H| \cdot |\nabla^m Rm| + \frac{C}{t^{1/2}}|\nabla^{2+m} u| \cdot |\nabla^m Rm| \\ = {} & \Delta|\nabla^m Rm|^2 - 2|\nabla^{m+1} Rm|^2 + C|\nabla^m Rm|^2 \\ & + C\left(\frac{1}{t^{m/2}} + \frac{1}{t^{\frac{m+1}{2}}} + \frac{1}{t^{\frac{m+2}{2}}}\right)|\nabla^m Rm| + C|\nabla^{m+2} H| \cdot |\nabla^m Rm| \\ & + \frac{C}{t^{1/2}}|\nabla^{m+1} H| \cdot |\nabla^m Rm| + \frac{C}{t^{1/2}}|\nabla^{2+m} u| \cdot |\nabla^m Rm| \\ \leq {} & \Delta\left|\nabla^m Rm\right|^2 - 2\left|\nabla^{m+1} Rm\right|^2 + |\nabla^{m+2} H|^2 + C|\nabla^{m+1} H|^2 \\ & + C\left(\frac{1}{t^m} + \frac{1}{t^{m+1}} + \frac{1}{t} + 1\right)|\nabla^m Rm|^2 + C|\nabla^{2+m} u|^2 + \frac{1}{t} + C, \end{aligned} \tag{5.8}$$

where for the last inequality, we used the following Young inequalities:

$$C|\nabla^{m+2} H| \cdot |\nabla^m Rm| \leq |\nabla^{m+2} H|^2 + C|\nabla^m Rm|^2,$$

$$\frac{C}{t^{1/2}}|\nabla^{m+1} H| \cdot |\nabla^m Rm| \leq C|\nabla^{m+1} H|^2 + \frac{C}{t}|\nabla^m Rm|^2,$$

$$\frac{C}{t^{1/2}}|\nabla^{2+m} u| \cdot |\nabla^m Rm| \leq C|\nabla^{2+m} u|^2 + \frac{C}{t}|\nabla^m Rm|^2,$$

$$\frac{C}{t^{\frac{m+2}{2}}}|\nabla^m Rm| \leq \frac{C}{t^{m+1}}|\nabla^m Rm|^2 + \frac{1}{t},$$

and

$$\frac{C}{t^{m/2}}|\nabla^m Rm| \leq \frac{C}{t^m}|\nabla^m Rm|^2 + C, \quad \frac{C}{t^{\frac{m+1}{2}}}|\nabla^m Rm| \leq \frac{C}{t^{m+1}}|\nabla^m Rm|^2 + C.$$

In the same way, we have

(5.9)

$$
\begin{aligned}
\partial_t|\nabla^{m+1}H|^2 \le{}& \Delta|\nabla^{m+1}H|^2-2|\nabla^{m+2}H|^2+C|\nabla^{m+1}H|^2+\frac{C}{t^{\frac{m+2}{2}}}|\nabla^{m+1}H|\\
&+C|\nabla^{m+1}H|\cdot|\nabla^{m+1}Rm|+\frac{C}{t^{1/2}}|\nabla^{m+1}H|\cdot|\nabla^{m}Rm|\\
&+\frac{C}{t^{\frac{m+1}{2}}}|\nabla^{m+1}H|+C|\nabla^{m+1}H|\cdot|\nabla^{2+m}u|\\
\le{}& \Delta|\nabla^{m+1}H|^2-2|\nabla^{m+2}H|^2+C\left(1+\frac{1}{t^{m+1}}\right)\cdot|\nabla^{m+1}H|^2+\frac{1}{t}\\
&+|\nabla^{m+1}Rm|^2+\frac{C}{t}|\nabla^{m}Rm|^2+C|\nabla^{2+m}u|^2
\end{aligned}
$$

and

(5.10)

$$
\begin{aligned}
\partial_t|\nabla^{2+m}u|^2 \le{}& \Delta|\nabla^{2+m}u|^2-2|\nabla^{3+m}u|^2+C|\nabla^{2+m}u|^2+\frac{C}{t^{\frac{m+2}{2}}}|\nabla^{2+m}u|\\
&+\frac{C}{t^{\frac{1}{2}}}|\nabla^{2+m}u||\nabla^{m}Rm|+\frac{C}{t^{\frac{m+1}{2}}}|\nabla^{2+m}u|+C|\nabla^{2+m}u||\nabla^{1+m}H|\\
\le{}& \Delta|\nabla^{2+m}u|^2-2|\nabla^{3+m}u|^2+C\left(1+\frac{1}{t^{m+1}}\right)\cdot|\nabla^{2+m}u|^2+\frac{1}{t}\\
&+C|\nabla^{m}Rm|^2+C|\nabla^{1+m}H|^2.
\end{aligned}
$$

Combining (5.8), (5.9) and (5.10), we get

$$
\begin{aligned}
&\partial_t\left(|\nabla^{m}Rm|^2+|\nabla^{m+1}H|^2+|\nabla^{2+m}u|^2\right)\\
\le{}& \Delta|\nabla^{m}Rm|^2+\Delta|\nabla^{m+1}H|^2+\Delta|\nabla^{2+m}u|^2-2|\nabla^{3+m}u|^2\\
&-|\nabla^{m+1}Rm|^2-|\nabla^{m+2}H|^2+C\left(1+\frac{1}{t^{m+1}}\right)\cdot|\nabla^{m+1}H|^2\\
&+C\left(\frac{1}{t^{m}}+\frac{1}{t^{m+1}}+\frac{1}{t}+1\right)|\nabla^{m}Rm|^2+C\left(1+\frac{1}{t^{m+1}}\right)\cdot|\nabla^{2+m}u|^2+\frac{1}{t}+C.
\end{aligned}
$$

Using formula (5.6), we also have

$$
\begin{aligned}
&\sum_{i=1}^{m}E_i t^i\cdot\partial_t[|\nabla^{i-1}Rm|^2+|\nabla^{i}H|^2+|\nabla^{i+1}u|^2]\\
\le{}& \sum_{i=1}^{m}E_i t^i\Bigg\{\Delta|\nabla^{i-1}Rm|^2+\Delta|\nabla^{i}H|^2+\Delta|\nabla^{i+1}u|^2-2|\nabla^{i}Rm|^2-2|\nabla^{i+1}H|^2\\
&-2|\nabla^{i+2}u|^2+C|\nabla^{i-1}Rm|^2+C\left(\frac{1}{t^{\frac{i-1}{2}}}+\frac{1}{t^{i/2}}+\frac{1}{t^{\frac{i+1}{2}}}\right)\cdot|\nabla^{i-1}Rm|\\
&+C|\nabla^{i+1}H|\cdot|\nabla^{i-1}Rm|+\frac{C}{t^{\frac{1}{2}}}|\nabla^{i}H|\cdot|\nabla^{i-1}Rm|+\frac{C}{t^{\frac{1}{2}}}|\nabla^{i+1}u|\cdot|\nabla^{i-1}Rm|
\end{aligned}
$$

$$
\begin{aligned}
&+C|\nabla^i H|^2+\frac{C|\nabla^i H|}{t^{\frac{i+1}{2}}}+C|\nabla^i H|\cdot|\nabla^i Rm|+\frac{C}{t^{\frac{1}{2}}}|\nabla^i H|\cdot|\nabla^{i-1} Rm|\\
&+\frac{C|\nabla^i H|}{t^{\frac{i}{2}}}+C|\nabla^i H|\cdot|\nabla^{i+1}u|+C|\nabla^{i+1}u|^2+\frac{C}{t^{\frac{i+1}{2}}}|\nabla^{i+1}u|\\
&\left.+\frac{C}{t^{\frac{1}{2}}}|\nabla^{i+1}u|\cdot|\nabla^{i-1}Rm|+\frac{C}{t^{\frac{i}{2}}}|\nabla^{i+1}u|+C|\nabla^{i+1}u|\cdot|\nabla^i H|\right\}\\
\leq\ &\sum_{i=1}^{m}E_i t^i\Big\{\Delta|\nabla^{i-1}Rm|^2+\Delta|\nabla^i H|^2+\Delta|\nabla^{i+1}u|^2-2\left|\nabla^i Rm\right|^2-2|\nabla^{i+1}H|^2\\
&\qquad\quad-2|\nabla^{i+2}u|^2\Big\}\\
&+\sum_{i=1}^{m}E_i t^i\left\{\frac{C}{t^i}+\frac{C}{t^{\frac{i}{2}}}|\nabla^{i+1}H|\right.\\
&\left.+\frac{C}{t^{\frac{i}{2}}}|\nabla^i Rm|+\frac{C}{t^{\frac{i+1}{2}}}(|\nabla^{i-1}Rm|+|\nabla^i H|+|\nabla^{i+1}u|)\right\}.
\end{aligned}
$$

Therefore combining the above evolution formulas, we conclude that

$$
\begin{aligned}
v_t\leq\ &\Delta v+(m+1)t^m\left[|\nabla^m Rm|^2+|\nabla^{m+1}H|^2+|\nabla^{m+2}u|^2\right]\\
&+t^{m+1}\Big\{C\left(1+\frac{1}{t^{m+1}}\right)\cdot|\nabla^{m+1}H|^2\\
&+C\left(\frac{1}{t^m}+\frac{1}{t^{m+1}}+\frac{1}{t}+1\right)|\nabla^m Rm|^2\\
&+C\left(1+\frac{1}{t^{m+1}}\right)\cdot|\nabla^{2+m}u|^2+\frac{1}{t}+C\Big\}\\
&+\sum_{i=1}^{m}iE_i t^{i-1}\left[|\nabla^{i-1}Rm|^2+|\nabla^i H|^2+|\nabla^{i+1}u|^2\right]\\
&+\sum_{i=1}^{m}E_i t^i\Big\{-2|\nabla^i Rm|^2-2|\nabla^{i+1}H|^2-2|\nabla^{i+2}u|^2\Big\}\\
&+\sum_{i=1}^{m}E_i t^i\Big\{\frac{C}{t^i}+\frac{C}{t^{\frac{i}{2}}}|\nabla^{i+1}H|\\
&+\frac{C}{t^{\frac{i}{2}}}|\nabla^i Rm|+\frac{C}{t^{\frac{i+1}{2}}}\left(|\nabla^{i-1}Rm|+|\nabla^i H|+|\nabla^{i+1}u|\right)\Big\}\\
&-2A|\nabla H|^2-2B|\nabla^2 u|^2+AC\\
\leq\ &\Delta v+C(t^{m+1}+t^m+t+1)\cdot\left[|\nabla^m Rm|^2+|\nabla^{m+1}H|^2+|\nabla^{m+2}u|^2\right]\\
&+\sum_{i=0}^{m-1}(i+1)E_{i+1}t^i\left[|\nabla^i Rm|^2+|\nabla^{i+1}H|^2+|\nabla^{i+2}u|^2\right]
\end{aligned}\tag{5.11}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{m} E_i t^i\Big\{-2\left|\nabla^i Rm\right|^2-2|\nabla^{i+1}H|^2-2|\nabla^{i+2}u|^2\Big\}\\
&+C\sum_{i=1}^{m} E_i t^{i/2}\Big\{|\nabla^{i+1}H|+|\nabla^i Rm|\Big\}\\
&+C\sum_{i=1}^{m} E_i t^{\frac{i-1}{2}}\Big\{|\nabla^{i-1}Rm|+|\nabla^i H|+|\nabla^{i+1}u|\Big\}\\
&-2A|\nabla H|^2-2B|\nabla^2 u|^2+\Big(A+\sum_{i=1}^{m}E_i\Big)C.
\end{aligned}
$$

Now choose $(i+1)E_{i+1}=E_i,\ \ E_i=\frac{M}{i!}$, where M is constant which is determined later. We also notice the following estimate

$$
\begin{aligned}
&C\sum_{i=1}^{m} E_i t^{i/2}\Big\{|\nabla^{i+1}H|+|\nabla^i Rm|\Big\}\\
\leq\ &\frac{1}{2}\sum_{i=1}^{m} E_i t^i\Big\{|\nabla^{i+1}H|^2+|\nabla^i Rm|^2\Big\}+\frac{1}{2}C^2\sum_{i=1}^{m}E_i.
\end{aligned}
$$

Hence the above inequality (5.11) reduces to

$$
\begin{aligned}
v_t \leq\ &\Delta v+C(t^{m+1}+t^m+t+1)\cdot\left[|\nabla^m Rm|^2+|\nabla^{m+1}H|^2+|\nabla^{m+2}u|^2\right]\\
&+\sum_{i=1}^{m} E_i t^i\Big\{-\left|\nabla^i Rm\right|^2-|\nabla^{i+1}H|^2-|\nabla^{i+2}u|^2\Big\}\\
&+C\sum_{i=1}^{m} E_i t^{\frac{i-1}{2}}\Big\{|\nabla^{i-1}Rm|+|\nabla^i H|+|\nabla^{i+1}u|\Big\}\\
&-2A|\nabla H|^2-2B|\nabla^2 u|^2+\Big(A+\sum_{i=1}^{m}E_i\Big)C.
\end{aligned}
\tag{5.12}
$$

We also see that

$$
\begin{aligned}
&C\sum_{i=1}^{m} E_i t^{\frac{i-1}{2}}\Big\{|\nabla^{i-1}Rm|+|\nabla^i H|+|\nabla^{i+1}u|\Big\}\\
=\ &CE_1+CE_1|\nabla H|+CE_1|\nabla^2 u|+C\sum_{i=1}^{m-1}E_{i+1}t^{\frac{i}{2}}\Big\{|\nabla^i Rm|+|\nabla^{i+1}H|+|\nabla^{i+2}u|\Big\}\\
\leq\ &CE_1+CE_1|\nabla H|+CE_1|\nabla^2 u|+C(E_i,E_{i+1})\\
&+\frac{1}{2}\sum_{i=1}^{m-1}E_i t^i\Big\{|\nabla^i Rm|^2+|\nabla^{i+1}H|^2+|\nabla^{i+2}u|^2\Big\}.
\end{aligned}
$$

Therefore (5.12) becomes

$$
v_t \leq \Delta v+C(t^{m+1}+t^m+t+1)\cdot\left[|\nabla^m Rm|^2+|\nabla^{m+1}H|^2+|\nabla^{m+2}u|^2\right]
$$

$$
\begin{aligned}
&-\frac{1}{2}E_m t^m\Big\{\,|\nabla^m Rm|^2+|\nabla^{m+1}H|^2+|\nabla^{m+2}u|^2\Big\}\\
&+CE_1+CE_1|\nabla H|+CE_1|\nabla^2 u|+C(E_i,E_{i+1})\\
&-2A|\nabla H|^2-2B|\nabla^2 u|^2+\Big(A+\sum_{i=1}^{m}E_i\Big)C.
\end{aligned}
$$

Keep in mind $E_i=\frac{M}{i!}$. Now we choose M large enough and get

$$
\begin{aligned}
v_t\le \Delta v &+CE_1+CE_1|\nabla H|+CE_1|\nabla^2 u|+C(E_i,E_{i+1})\\
&-2A|\nabla H|^2-2B|\nabla^2 u|^2+\Big(A+\sum_{i=1}^{m}E_i\Big)C.
\end{aligned}
$$

At last, choosing A and B large enough compared to E_1, we have $v_t\le\Delta v+C$. Since $v(0)\le C$, by Theorem 5.3, the maximum principle gives $v(x,t)\le C$ for all $(x,t)\in M^n\times[0,\frac{\beta}{K}]$. This proves the theorem for the case of $m+1$. □

Remark 5.5. We do not know if condition (5.4) guarantee the existence of solutions to (1.1) on complete manifolds. For closed manifolds, if we replace conditions $|H(x,t)|_{g(t)}\le K_2$ and $|du(x,t)|_{g(t)}\le K_3$ by $|H(x,0)|_{g(0)}\le K_2$ and $|du(x,0)|_{g(0)}\le K_3$, then Theorem 5.4 still holds. In fact, by Proposition 5.1 and Theorem 5.2, bounds of two quantities at initial time imply their bounds at any time.

Following the arguments of [2], we extend these estimates to obtain bounds on the curvatures and all of their derivatives on compact manifolds.

Corollary 5.6. *Let $(g(x,t),H(x,t),u(x,t))$ be a solution to system* (1.1) *on closed manifolds M^n on $[0,T]$. If there exist $\gamma>0$ and $K_i>0$ $(i=1,2,3)$ such that*

$$|Rm(x,t)|_{g(x,t)}\le K_1,\quad |H(x,0)|_{g(x,0)}\le K_2\quad\textit{and}\quad |du(x,0)|_{g(x,0)}\le K_3$$

for all $(x,t)\in M^n\times[0,T]$, where $T>\frac{\gamma}{K}$, then for all $m\in\mathbb{N}$, there exists a constant C_m depending only on $n,K_1,K_2,K_3,\max\{\beta,1\}$ and m such that

$$|\nabla^{m-1}Rm(x,t)|_{g(x,t)}+|\nabla^m H(x,t)|_{g(x,t)}+|\nabla^{m+1}u(x,t)|_{g(x,t)}\le C_m$$

for all $(x,t)\in M^n\times[\frac{\min\{\gamma,1\}}{K},T]$.

Proof. Let $\gamma_1:=\min\{\gamma,1\}$ and $t_0\in[\frac{\gamma_1}{K},T]$ be arbitrary. Define $T_0:=t_0-\frac{\gamma_1}{K}$. Additionally, let $\bar t=t-T_0$. We define $(\bar g(x,\bar t),\bar H(x,\bar t),\bar u(x,\bar t))$ to be the solution of the initial value problem

$$
\begin{aligned}
\partial_{\bar t}\bar g_{ij}&=-2\bar R_{ij}+1/2\bar h_{ij}+2\alpha_n\partial_i\bar u\partial_j\bar u\\
\partial_{\bar t}\bar H&=\bar\Delta_{LB}\bar H\quad\text{and}\quad\partial_{\bar t}\bar u=\bar\Delta\bar u
\end{aligned}
$$

with $(\bar g(0),\bar H(0),\bar u(0))=(g(T_0),H(T_0),u(T_0))$. Since solutions to the GRF system (1.1) are unique, $\bar g(\bar t)=g(\bar t+T_0)=g(t)$, $\bar H(\bar t)=H(\bar t+T_0)=H(t)$ and $\bar u(\bar t)=u(\bar t+T_0)=u(t)$ for $\bar t\in[0,\frac{\gamma_1}{K}]$. We assume that $|\bar Rm|_{\bar g}\le K$ and

$|\bar{H}(x)|_{\bar{g}} \leq K$ for $\bar{t} \in [0, \frac{\gamma_1}{K}]$, so that we can apply our BBS estimates with $\beta = \gamma_1$. Then there exists a constant $\bar{C}_m$ such that

$$|\nabla^{m-1}\bar{R}m|_{\bar{g}} + |\nabla^m \bar{H}|_{\bar{g}} + |\nabla^{m+1}\bar{u}|_{\bar{g}} \leq \frac{\bar{C}_m}{\bar{t}^{m/2}}$$

for all $(x, \bar{t}) \in M^n \times [0, \frac{\gamma_1}{K}]$. Note that for $\bar{t} \in [\frac{\gamma_1}{2K}, \frac{\gamma_1}{K}]$, we have $\bar{t}^m \geq \frac{\gamma_1^m}{2^m K^m}$. In particular, taking $\bar{t} = \gamma_1/K$, we see that

$$|\nabla^{m-1}Rm(x,t_0)| + |\nabla^m H(x,t_0)| + |\nabla^{m+1}u(x,t_0)| \leq \frac{2^m K^m \bar{C}_m}{\gamma_1^m}$$

for all $x \in M^n$. Since t_0 was arbitrary, the result follows. □

6. Compactness theorem for the GRF system

Given a sequence of solutions $(M_k, g_k(t))$ to the Ricci flow, Hamilton's Cheeger-Gromov-type compactness theorem in [7] states that in the presence of injectivity radii and curvature bounds we can take a C^∞ limit of a subsequence. The role of the Hamilton's compactness theorem is primarily to understand singularity formation. In this section, we state a similar phenomenon in the GRF system, which is also a useful tool to understand the singularities of the GRF system.

Definition 6.1. Given a sequence of complete Riemannian manifolds M_k with origin O_k, Riemannian metrics g_k, 3-forms H_k and smooth functions u_k, we say that the sequence $(M_k,\ g_k,\ H_k,\ u_k,\ O_k)$ converges to the limit $(M_\infty, g_\infty, H_\infty, u_\infty, O_\infty)$ if there exist a sequence of compact set U_k exhausting M_∞ and a sequence of diffeomorphisms Φ_k of U_k in M_∞ to M_k such that Φ_k takes O_∞ to O_k and the pull-back $(\Phi_k^* g_k, \Phi_k^* H_k, \Phi_k^* u_k)$ converge to $(g_\infty, H_\infty, u_\infty)$ uniformly on compact sets together with all their derivatives in M_∞.

Remark 6.2. The above convergence is the topology of C^∞ convergence on compact sets. If the limit exists, it is unique up to a unique isometry preserving the origin.

Let $\mathrm{inj}_g(O)$ denote the injectivity radius of the metric g at the point O. For sequences of manifolds we have the Hamilton's convergence theorems in [7].

Theorem 6.3 (Hamilton's compactness for metrics)**.** *Let $\{(M_k, g_k, O_k)\}_{k\in\mathbb{N}}$ be a sequence of complete pointed Riemannian manifolds that satisfy*

$$|\nabla_k^p Rm_k|_k \leq C_p \quad \textit{on } M_k$$

for all $p \geq 0$ and k where $C_p < \infty$ is a sequence of constants independent of k and

$$\mathrm{inj}_{g_k}(O_k) \geq \iota_0$$

for some positive constant ι_0. Then there exists a subsequence $\{j_k\}_{k\in\mathbb{N}}$ such that $\{(M_{j_k}, g_{j_k}, O_{j_k})\}_{k\in\mathbb{N}}$ converges to a complete manifold $(M_\infty, g_\infty, O_\infty)$ as $k \to \infty$.

Using above compactness theorems for the fixed metrics (i.e., $t = 0$), we can get the compactness theorems for the GRF system on complete manifolds.

Theorem 6.4. *Let T_A, T_O be given such that $-\infty \le T_A < 0 < T_O \le \infty$. Let $\{(M_k, g_k(t), H_k(t), u_k(t), O_k)\}_{k\in\mathbb{N}}$ be a sequence of complete pointed solutions to the GRF system for $t \in [T_A, T_O)$ satisfying*

$$\sup_{M_k} |Rm_k|_k(t) \le C_0 \quad \textit{for all } t \in (T_A, T_O), \tag{6.1}$$

$$\sup_{M_k} |H_k|_k(T_A) \le C_0' \quad \textit{and} \quad \sup_{M_k} |u_k|(T_A) + \sup_{M_k} |du_k|_k(T_A) \le C_0'', \tag{6.2}$$

where C_0, C_0' and C_0'' are all finite constants independent of k and $\mathrm{inj}_{g_k(0)}(O_k) \ge \iota_0$ for some positive constant ι_0. Then there exists a subsequence $\{j_k\}$ such that $\{(M_{j_k}, g_{j_k}(t), H_{j_k}(t), u_{j_k}(t), O_{j_k})\}$ converges to a complete pointed solution to the GRF system $(M_\infty, g_\infty(t), H_\infty(t), u_\infty(t), O_\infty), t \in (T_A, T_O)$ as $k \to \infty$, where $k \in \mathbb{N}$.

Remark 6.5. Proposition 5.1 and Theorem 5.2 state that uniform bounds on $|Rm_k|_k$ and initial bounds on $|H_k|_k$ and $|u_k| + |du_k|_k$ imply uniform bounds on $|H_k|_k$ and $|u_k| + |du_k|_k$ on $[T_A, T_O)$ on compact sets. Moreover, their bounds also imply the bounds of all their derivatives (see Theorem 5.6 in Section 5).

To prove Theorem 6.4, in fact we only need to extend the convergence at one time to convergence at all times. First, we show that the following key lemma.

Lemma 6.6. *Let M^n be a closed manifold with the background metric g, U a compact subset of M, and $(g_k(t), H_k(t), u_k(t))$ a collection of solutions to the GRF system in neighborhoods of $U \times [\beta, \psi]$, where $\beta < 0 < \psi$. At time $t = 0$ on U, let*

(a) $cg(V, V) \le g_k(V, V) \le Cg(V, V) \quad$ *for all $V \in T_xM$,*
(b) $|\nabla^p g_k| \le \hat{C}_p \quad$ *for all $p \ge 1$,*
(c) $|\nabla^p H_k| \le \hat{C}_p' \quad$ *for all $p \ge 0$,*
(d) $|\nabla^p u_k| \le \hat{C}_p'' \quad$ *for all $p \ge 0$*

and in addition

(e) $\sup_{U\times[\beta,\psi]} |\nabla_k^p Rm_k|_k \le C_p \quad$ *for all $p \ge 0$,*
(f) $\sup_{U\times[\beta,\psi]} |\nabla_k^p H_k|_k \le C_p' \quad$ *for all $p \ge 0$,*
(g) $\sup_{U\times[\beta,\psi]} |\nabla_k^p u_k|_k \le C_p'' \quad$ *for all $p \ge 0$*

with constants c, C, $\hat{C}_p$, $\hat{C}_p'$, $\hat{C}_p''$, C_p, C_p', C_p'' independent of k. Then we have

(i) $\tilde{c}g(V, V) \le g_k(t)(V, V) \le \tilde{C}g(V, V)$,
(ii) $\sup_{U\times[\beta,\psi]} |\nabla^p g_k| \le \tilde{C}_p \quad$ *for all $p \ge 1$,*

(iii) $\sup_{U\times[\beta,\psi]} |\nabla^p H_k| \le \tilde{C}'_p$ *for all* $p \ge 0$,

(iv) $\sup_{U\times[\beta,\psi]} |\nabla^p u_k| \le \tilde{C}''_p$ *for all* $p \ge 0$

on any $(x,t) \in U \times [\beta,\psi]$, *where* $\tilde{c}$, $\tilde{C}$, $\tilde{C}_p$, $\tilde{C}'_p$ *and* $\tilde{C}''_p$ *are all independent of* k.

Proof. Let V be a vector field on M. For all k, using (e), (f) and (g), we have

$$\partial_t g_k(t)(V,V) = -2Rc_k(t)(V,V) + 1/2 h_k(t)(V,V) + 2\alpha_n d_k u(t) d_k u(t)(V,V)$$

and

$$\begin{aligned}|Rc_k(t)(V,V)| &\le C_1(n)C_0 g_k(t)(V,V),\\ |h_k(t)(V,V)| &\le C_2(n)|H_k(t)|_k^2 g_k(t)(V,V) \le C_2(n){C'_0}^2 g_k(t)(V,V),\\ |d_k u(t)\otimes d_k u(t)(V,V)| &\le |d_k u(t)|_k^2 g_k(t)(V,V) \le {C''_1}^2 g_k(t)(V,V)\end{aligned}$$

which gives

$$\begin{aligned}\left|\frac{\partial \log g_k(t)(V,V)}{\partial t}\right| &= \left|\frac{-2Rc_k(t)(V,V)+\frac{h_k(t)}{2}(V,V)+2\alpha_n d_k u(t) d_k u(t)(V,V)}{g_k(t)(V,V)}\right|\\ &\le A_0,\end{aligned}$$

where A_0 depends on n, C_0, C'_0 and C''_1. Throughout the proof of this lemma, we will let $0 < t < \psi$ be arbitrary. Then we integrate to obtain

$$\begin{aligned}A_0\psi &\ge \int_0^t |\partial_\tau \log g_k(\tau)(V,V)|\, d\tau\\ &\ge \left|\int_0^t \partial_\tau \log g_k(\tau)(V,V) d\tau\right| = \left|\log \frac{g_k(t)(V,V)}{g_k(0)(V,V)}\right|.\end{aligned}$$

Hence by the assumption condition (a) we have

$$\tilde{c} g(V,V) =: C^{-1} e^{-A_0\psi} g(V,V) \le g_k(t)(V,V) \le C e^{A_0\psi} g(V,V) := \tilde{C} g(V,V).$$

This completes the proof of (i).

We observe that the difference $\Gamma_k - \Gamma$ is a tensor. Taking Γ to be fixed in time, we then get

$$\begin{aligned}|\partial_t(\Gamma_k - \Gamma)|_k &\le C(n)\cdot[|\nabla_k(Rc_k)|_k + |\nabla_k(h_k)|_k + |\nabla_k(du_k\otimes du_k)|_k]\\ &\le C(n)C_1 + 2C(n)|\nabla_k(H_k)|_k \cdot |H_k|_k + 2C(n)|\nabla_k^2(u_k)|_k \cdot |du_k|_k\\ &\le C(n)C_1 + 2C(n)C'_1C'_0 + 2C(n)C''_2C''_1 := A_1.\end{aligned}$$

Since $\nabla g_k \simeq \Gamma_k(t) - \Gamma \simeq \nabla^k(t) - \nabla$, we deduce

$$|\partial_t \nabla g_k(t)| \le c(n)\,|\partial_t\,(\Gamma_k(t)-\Gamma)| \le c(n)C'\,|\partial_t\,(\Gamma_k(t)-\Gamma)|_k \le c(n)C' A_1,$$

where constant C' comes from (i) which is already proven. Integrating again yields

$$\begin{aligned}|\nabla g_k(t)| &= \left|\nabla g_k(0) + \int_0^t \partial_\tau \nabla g_k(\tau) d\tau\right| \\ &\leq |\nabla g_k(0)| + c(n)C' A_1 \psi \leq \hat{C}_1 + c(n)C' A_1 \psi := \tilde{C}_1.\end{aligned}$$

Now we consider $|H_k(t)|$. By (f) and (i), we have

$$|H_k(t)| \leq \bar{C} \cdot |H_k(t)|_k \leq \bar{C} \cdot C_0' := \tilde{C}_0',$$

where $\bar{C}$ is determined by (i). We also notice that $|\cdot| = |\cdot|_k$ on functions. Therefore by (g)

$$|u_k(t)| = |u_k(t)|_k \leq C_0'' := \tilde{C}_0''.$$

Using the fact that ∇ is independent of time, we have

$$\begin{aligned}\partial_t \nabla H_k &= \nabla \partial_t H_k = \nabla\left[\Delta_k H_k + Rm_k * H_k\right] \\ &= (\nabla - \nabla_k)\Delta_k H_k + \nabla_k \Delta_k H_k + \nabla Rm_k * H_k + Rm_k * \nabla H_k \\ &= \nabla g_k * \Delta_k H_k + \nabla_k \Delta_k H_k + (\nabla - \nabla_k) Rm_k * H_k + \nabla_k Rm_k * H_k \\ &\quad + Rm_k * (\nabla - \nabla_k) H_k + Rm_k * \nabla_k H_k \\ &= \nabla g_k * \Delta_k H_k + \nabla_k \Delta_k H_k + \nabla g_k * Rm_k * H_k + \nabla_k Rm_k * H_k \\ &\quad + Rm_k * \nabla g_k * H_k + Rm_k * \nabla_k H_k,\end{aligned}$$

where we used $\nabla g_k \simeq \nabla^k - \nabla$. Then by (e), (f), (i) and (ii) for $p = 1$, the above equation implies

$$\begin{aligned}|\partial_t \nabla H_k| &\leq C|\nabla g_k| \cdot |\Delta_k H_k| + C|\nabla_k \Delta_k H_k| + C|\nabla g_k| \cdot |Rm_k| \cdot |H_k| \\ &\quad + C|\nabla_k Rm_k| \cdot |H_k| + |Rm_k| \cdot |\nabla_k H_k| \leq B_1.\end{aligned}$$

As above,

$$|\nabla H_k(t)| \leq |\nabla H_k(0)| + \int_0^t |\partial_\tau \nabla H_k(\tau)|\, d\tau \leq \hat{C}_1' + B_1 \psi := \tilde{C}_1'.$$

Similarly using (g) and (i), we calculate for the differential:

$$|\nabla u_k(t)| = |du_k(t)| = |\nabla_k u_k(t)| \leq \bar{C} \cdot |\nabla_k u_k(t)|_k \leq \bar{C} \cdot C_1'' := \tilde{C}_1''.$$

We can estimate $|\nabla^2 u_k(t)|$ by the estimate $|\nabla g_k(t)|$. Since

$$\begin{aligned}\partial_t \nabla^2 u_k &= \nabla^2 \Delta_k u_k = (\nabla - \nabla_k) d(\Delta_k u_k) + \nabla_k d(\Delta_k u_k) \\ &= \nabla g_k * \nabla_k \Delta_k u_k + \nabla_k^2 (\Delta_k u_k),\end{aligned}$$

using (g), (i) and (ii) for $p = 1$, we have

$$\left|\partial_t \nabla^2 u_k\right| \leq C|\nabla g_k| \cdot |\nabla_k \Delta_k u_k| + C|\nabla_k^2 (\Delta_k u_k)| \leq D_2.$$

Using (d), an integration gives

$$|\nabla^2 u_k(t)| \leq |\nabla^2 u_k(0)| + \int_0^t \left|\partial_\tau \nabla^2 u_k(\tau)\right| d\tau \leq \hat{C}_2'' + D_2 \psi := \tilde{C}_2''.$$

Higher derivatives of (g_k, H_k, u_k) with respect to g can be estimated in pairs $(\nabla^p g_k, \nabla^p H_k, \nabla^{p+1}u_k)$ for all $p \geq 2$. The technique is similar for all $p \geq 2$, so we only state the case $p = 2$ as reference. Note that

$$\begin{aligned} \partial_t \nabla^2 g_k &= \nabla^2 \left(-2Rc_k + 1/2h_k + 2\alpha_n du_k \otimes du_k\right) \\ &= \nabla^2 Rc_k + H_k * \nabla^2 H_k + \nabla H_k * \nabla H_k + \nabla^3 u_k \otimes \nabla u_k + \nabla^2 u_k \otimes \nabla^2 u_k. \end{aligned} \tag{6.3}$$

We can rewrite some of these terms to be

$$\begin{aligned} \nabla^2 Rc_k &= (\nabla - \nabla_k)\nabla Rc_k + \nabla_k(\nabla - \nabla_k)Rc_k + \nabla_k^2 Rc_k \\ &= \nabla g_k * [(\nabla - \nabla_k)Rc_k + \nabla_k Rc_k] + \nabla_k(\nabla g_k * Rc_k) + \nabla_k^2 Rc_k \\ &= \nabla g_k * [\nabla g_k * Rc_k + \nabla_k Rc_k] + \nabla_k \nabla g_k * Rc_k + \nabla g_k * \nabla_k Rc_k + \nabla_k^2 Rc_k \\ &= \nabla g_k * [\nabla g_k * Rc_k + \nabla_k Rc_k] + \nabla^2 g_k * Rc_k + \nabla_k^2 Rc_k, \end{aligned} \tag{6.4}$$

where in the last equality we used

$$\nabla_k \nabla g_k = \nabla^2 g_k + (\nabla_k - \nabla)\nabla g_k = \nabla^2 g_k + \nabla g_k * \nabla g_k. \tag{6.5}$$

Substituting (6.4) into (6.3), we have

$$\begin{aligned} \left|\partial_t \nabla^2 g_k\right| &\leq C(n)\bar{C}C_0|\nabla^2 g_k| + C(n)\bar{C}C_0'|\nabla^2 H_k| + \tilde{C}_1''|\nabla^3 u_k| \\ &\quad + (\tilde{C}_1')^2 + (\tilde{C}_2'')^2 + C(n)\bar{C}\left[\tilde{C}_1^2 C_0 + \tilde{C}_1 C_1 + C_2^2\right] \\ &\leq \hat{A}_2\left[|\nabla^2 g_k| + |\nabla^2 H_k| + |\nabla^3 u_k|\right] + C, \end{aligned} \tag{6.6}$$

where we used (a)-(g) and (i)-(iv) for the case $p = 0$, 1.

Doing the same calculation for $\nabla^2 H_k$, we have

$$\begin{aligned} \partial_t \nabla^2 H_k &= \nabla^2[\Delta_k H_k + Rm_k * H_k] \\ &= (\nabla - \nabla_k)\nabla\Delta_k H_k + \nabla_k(\nabla - \nabla_k)\Delta_k H_k + \nabla_k^2 \Delta_k H_k \\ &\quad + \nabla^2 Rm_k * H_k + Rm_k * \nabla^2 H_k + \nabla Rm_k * \nabla H_k \\ &= \nabla g_k * [(\nabla - \nabla_k)\Delta_k H_k + \nabla_k \Delta_k H_k] + \nabla_k(\nabla g_k * \Delta_k H_k) + \nabla_k^2 \Delta_k H_k \\ &\quad + \nabla^2 Rm_k * H_k + Rm_k * \nabla^2 H_k + (\nabla - \nabla_k)Rm_k * \nabla H_k + \nabla_k Rm_k * \nabla H_k \\ &= \nabla g_k * [\nabla g_k * \Delta_k H_k + \nabla_k \Delta_k H_k] + \nabla_k(\nabla g_k * \Delta_k H_k) + \nabla_k^2 \Delta_k H_k \\ &\quad + \nabla^2 Rm_k * H_k + Rm_k * \nabla^2 H_k + \nabla g_k * Rm_k * \nabla H_k + \nabla_k Rm_k * \nabla H_k. \end{aligned}$$

Therefore following the arguments above, we get:

$$\left|\partial_t \nabla^2 H_k\right| \leq \bar{A}_2\left[|\nabla^2 g_k| + |\nabla^2 H_k|\right] + C, \tag{6.7}$$

where we still used (6.5), (a)-(g) and (i)-(iv) for the case $p = 0$, 1.

Compute the evolution equation of $\nabla^3 u$, we have

$$\begin{aligned}\partial_t \nabla^3 u_k &= (\nabla - \nabla_k)\nabla d\Delta_k u_k + \nabla_k(\nabla - \nabla_k)d\Delta_k u_k + \nabla_k^2 d\Delta_k u_k \\ &= \nabla g_k * \left[(\nabla - \nabla_k)d\Delta_k u_k + \nabla_k^2 \Delta_k u_k\right] + \nabla_k(\nabla g_k * d\Delta_k u_k) + \nabla_k^2 d\Delta_k u_k \\ &= \nabla g_k * \nabla g_k * \nabla_k \Delta_k u_k + \nabla g_k * \nabla_k^2 \Delta_k u_k \\ &\quad + \nabla^2 g_k * \nabla_k \Delta_k u_k + \nabla g_k * \nabla g_k * \nabla_k \Delta_k u_k + \nabla_k^3 \Delta_k u_k.\end{aligned}$$

This leads to the estimate

$$\left|\partial_t \nabla^3 u_k\right| \leq \tilde{A}_2 |\nabla^2 g_k| + C. \tag{6.8}$$

Putting (6.6), (6.7) and (6.8) together and realizing that $|\cdot|$ is independent of time, we arrive at

$$\left|\partial_t \left(\left|\nabla^2 g_k\right| + \left|\nabla^2 H_k\right| + \left|\nabla^3 u_k\right|\right)\right| \leq A\left[|\nabla^2 g_k| + |\nabla^2 H_k| + |\nabla^3 u_k|\right] + C.$$

Since $|\nabla^2 g_k(0)| + |\nabla^2 H_k(0)| + |\nabla^3 u_k(0)| \leq \hat{C}_2 + \hat{C}_2' + \hat{C}_3''$, which is bounded, we can integrate in time to obtain

$$|\nabla^2 g_k(t)| + |\nabla^2 H_k(t)| + |\nabla^3 u_k(t)| \leq \tilde{C}_2 + \tilde{C}_2' + \tilde{C}_3'''. \tag{6.9}$$

Hence we have estimated in pairs $(\nabla^p g_k, \nabla^p H_k, \nabla^{p+1} u_k)$ for all $p = 2$.

We would like to derive some recursion formulas for higher derivatives.

$$\begin{aligned}\partial_t \nabla^p g_k &= \nabla^p \left(-2Rc_k + 1/2h_k + 2\alpha_n du_k \otimes du_k\right) \\ &= \nabla^p Rc_k + \sum_{i=1}^{p} \nabla^i H_k * \nabla^{p-i} H_k + \sum_{i=1}^{p} \nabla^{1+i} u_k * \nabla^{1+p-i} u_k.\end{aligned} \tag{6.10}$$

If the estimates hold for $p < N$ with $N \geq 2$, then we shall estimate them for $p = N$. First we have

$$\begin{aligned}\left|\nabla^N Rc_k(t)\right| &= \Big|\sum_{i=1}^{N} \nabla^{N-i}(\nabla - \nabla_k)\nabla^{i-1} Rc_k + \nabla_k^N Rc_k\Big| \\ &\leq \sum_{i=1}^{N} \left|\nabla^{N-i}(\nabla - \nabla_k)\nabla^{i-1} Rc_k\right| + \left|\nabla_k^N Rc_k\right|.\end{aligned} \tag{6.11}$$

Note that we can rewrite $\nabla - \nabla_k = \Gamma - \Gamma_k$ as a sum of terms of the form ∇g_k.

When $i = 1$, we can bound $\left|\nabla^{N-1}(\nabla - \nabla_k)Rc_k\right|$ by a sum of terms of the form $\left|\nabla^{N-j} g_k\right| \cdot \left|\nabla^j Rc_k\right|$, $0 \leq j \leq N-1$.

When $2 \leq i \leq N$, we can bound $\left|\nabla^{N-i}(\nabla - \nabla_k)\nabla_k^{i-1} Rc_k\right|$ by a sum of terms of the form $\left|\nabla^{N-i-j+1} g_k\right| \cdot \left|\nabla^j \nabla^{i-1} Rc_k\right|$, $0 \leq j \leq N-i$. Furthermore, we can also bound $\left|\nabla^j \nabla^{i-1} Rc_k\right| = \left|((\nabla - \nabla_k) + \nabla_k)^j \nabla^{i-1} Rc_k\right|$ by a sum of terms which are products of $\left|\nabla_k^{\ell+i-1} Rc_k\right|$, $0 \leq \ell \leq j$, and $|\nabla^\ell g_k|$, $1 \leq \ell \leq j$.

Hence by the assumption of Lemma 6.6, the induction assumption and the equivalence of $|\cdot|$ and $|\cdot|_k$, we get from (6.11)

$$|\nabla^N Rc_k(t)| \leq C_N' |\nabla^N g_k| + C_N''.$$

By induction, we have that $|\nabla^p H_k|$ and $|\nabla^p u_k|$ bounded for all $p<N$. This allows us to estimate from (6.10)

$$\begin{aligned}\left|\partial_t\nabla^N g_k\right| &\le C'_N|\nabla^N g_k|+C''_N|\nabla^N H_k|+ +C'''_N|\nabla^{N+1}u_k|+C\\ &\le \hat{A}_N\left[|\nabla^N g_k|+|\nabla^N H_k|+|\nabla^{N+1}u_k|\right]+C.\end{aligned}\tag{6.12}$$

Doing the same calculation for $\nabla^p H_k$, we have

$$\partial_t\nabla^p H_k=\nabla^p\left[\Delta_k H_k+Rm_k*H_k\right]=\nabla^p\Delta_k H_k+\sum_{i=1}^{p}\nabla^i Rm_k*\nabla^{p-i}H_k.\tag{6.13}$$

We assume the cases hold for all $p<N$ with $N\ge 2$, then we shall estimate them for $p=N$. As above, in the right hand side of (6.13), we find

$$|\nabla^N\Delta_k H_k|\le \hat{C}'_N|\nabla^N g_k|+\hat{C}''_N,\quad |\nabla^N Rm_k(t)|\le \hat{C}'_N|\nabla^N g_k|+\hat{C}''_N$$

and the others are bounded by induction.

Hence from (6.13), we get

$$\left|\partial_t\nabla^N H_k\right|\le \bar{A}_N\left[|\nabla^N g_k|+|\nabla^N H_k|\right]+C.\tag{6.14}$$

Compute the evolution equation for $\nabla^{p+1}u$, we have

$$\begin{aligned}\partial_t\nabla^{N+1}u_k=\nabla_k(\Delta_k u_k)*\nabla^N g_k&+\sum_{i=2}^{N+1}\nabla_k^i(\Delta_k u_k)*P(\nabla^0 g_k,\dots,\nabla^{N+1-i}g_k)\\ &+\nabla_k(\Delta_k u_k)*P(\nabla^0 g_k,\dots,\nabla^{N-1}g_k),\end{aligned}$$

where $N\ge 2$ and P is a polynomial in the components of the derivatives of g_k of the designated order. We have the following estimate by induction.

$$\left|\partial_t\nabla^{N+1}u_k\right|\le \tilde{A}_N|\nabla^N g_k|+C.\tag{6.15}$$

Combining (6.12), (6.14) and (6.15), we have

$$\left|\partial_t\left(|\nabla^N g_k|+|\nabla^N H_k|+|\nabla^{N+1}u_k|\right)\right|\le A\left[|\nabla^N g_k|+|\nabla^N H_k|+|\nabla^{N+1}u_k|\right]+C.$$

Note that at time $t=0$, $|\nabla^N g_k|+|\nabla^N H_k|+|\nabla^{N+1}u_k|$ is bounded by the assumption of Lemma 6.6. Then we integrate in time to give this quantity bounded for all time. Hence this finishes the proof of the lemma. □

Now we use Theorem 6.3 and Lemma 6.6 to prove Theorem 6.4.

Proof of Theorem 6.4. We want to use the convergence theorem as stated in Theorem 6.3. Assume for the proof that $T_A, T_O<\infty$. We also assume that the injectivity radius is bounded below by some positive constant at time $t=0$. Using Proposition 5.1, Theorem 5.2, Theorem 5.4 and Corollary 5.6, the assumptions (6.1) and (6.2) of Theorem 6.4 imply that their uniform bounds and uniform bounds on all derivatives of $Rm_k(t)$, $H_k(t)$ and $u_k(t)$ on all compact sets; namely,

$$|H_k|_k+|u_k|+|du_k|_k\le C(n,C_0,C'_0,C''_0,T)\tag{6.16}$$

and

$$|\nabla_k^{m-1} Rm_k|_k + |\nabla_k^m H_k|_k + |\nabla_k^{m+1} u_k|_k \le C_m(n, m, C_0, C_0', C_0'', T) \tag{6.17}$$

for all $m \ge 1$, where C and C_m are both independent of k. Now we apply Theorem 6.3 to get a convergent subsequence of $(M_k, g_k(0), x_k)$, also denoted $(M_k, g_k(0), x_k)$, at time $t = 0$ to a limit (M_∞, h, x_∞) in the sense of Definition 6.1, i.e.,

$$\lim_{k\to\infty} |\nabla_h^m(\Phi_k^* g_k(0)) - \nabla_h^m h|_h = 0 \qquad \text{for all } m \ge 0. \tag{6.18}$$

We will apply Lemma 6.6 at time $t = 0$, h as the background metric, and $\Phi_k^* g_k(t)$, $\Phi_k^* H_k(t)$ and $\Phi_k^* u_k(t)$, $t \in (T_A, T_O)$ as the sequence. Let $[\beta, \psi] \subset (T_A, TO)$, $0 \in [\beta, \psi]$, and $U \subset M_\infty$ be compact. Since we have convergence of $\Phi_k^* g_k(0)$ to the limit metric h at $t = 0$, the following is true:

(a) $\Phi_k^* g_k(0)$ is equivalent to h on U, that is, $ch \le \Phi_k^* g_k(0) \le Ch$ holds for all k big enough and some constants c and C independent of k.

(b) The covariant derivatives of $\Phi_k^* g_k(0)$ with respect to h are uniformly bounded on $U \times \{0\}$. From (6.18) we have $|\nabla_h^m(\Phi_k^* g_k(0))|_h \le \hat{C}_m$ for all $m \ge 1$ independent of k.

(c) By assumption and the equivalence of metrics, by (6.16), $|\Phi_k^* H_k(0)|_h \le C|H_k(0)|_k \le \hat{C}_0'$. Moreover for $m \ge 1$, we can use the equivalence of metrics and the fact that $\nabla^m_{\Phi_k^* g_k(0)} \to \nabla_h^m$ at $t = 0$ to obtain

$$\begin{aligned} |\nabla_h^m(\Phi_k^* H_k(0))|_h &\le C|\nabla_h^m(\Phi_k^* H_k(0))|_{\Phi_k^* g_k(0)} \\ &\le C|\nabla^m_{\Phi_k^* g_k(0)}(\Phi_k^* H_k(0))|_{\Phi_k^* g_k(0)} \le C|\nabla_k^m(H_k(0))|_k \le \hat{C}_m' \end{aligned}$$

for k big enough independent of k, where in the last line we used (6.17).

(d) Using the similar method of (c), we can obtain $|\nabla_h^m(\Phi_k^* u_k(0))|_h \le \hat{C}_m''$ for k big enough independent of k, where $m \ge 0$.

And in addition (6.16) and (6.17) imply (e), (f) and (g) in Lemma 6.6 are satisfied.

This allow us to apply Lemma 6.6. Then we have

(i) $\tilde{c}h \le \Phi_k^* g_k(t) \le \tilde{C}h \quad \text{on } U \times [\beta, \psi]$,

(ii) $\sup_{U\times[\beta,\psi]} |\nabla_h^m \Phi_k^* g_k(t)|_h \le \tilde{C}_m \quad \text{for all } m \ge 1$,

(iii) $\sup_{U\times[\beta,\psi]} |\nabla_h^m \Phi_k^* H_k(t)|_h \le \tilde{C}_m' \quad \text{for all } m \ge 0$,

(iv) $\sup_{U\times[\beta,\psi]} |\nabla_h^m \Phi_k^* u_k(t)|_h \le \tilde{C}_m'' \quad \text{for all } m \ge 0$,

where $\tilde{c}$, $\tilde{C}$, $\tilde{C}_m$, $\tilde{C}_m'$ and $\tilde{C}_m''$ are all independent of k.

At last by Arzela-Ascoli theorem, we find a subsequence converging uniformly on every compact subset of $M_\infty \times (T_A, T_O)$. In addition the limit $g_\infty(t) := \lim_{k\to\infty} \Phi_k^* g_k(t)$ agrees at time $t = 0$ with h since it already converged there by construction. Let $H_\infty(t) := \Phi_k^* H_k(t)$ and $u_\infty(t) := \Phi_k^* u_k(t)$.

Since the convergence is smooth and taking the limit commutes with all derivatives, we see that $\{g_\infty(t), H_\infty(t), u_\infty(t), O_\infty(t)\}$ is also a solution of the GRF system (1.1) and it satisfies the same bounds on derivatives and the injectivity radius. This finishes the proof of Theorem 6.4. □

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