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AN OVERLAPPING SCHWARZ METHOD FOR SINGULARLY PERTURBED THIRD ORDER CONVECTION-DIFFUSION TYPE

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Abstract. In this paper, an almost second order overlapping Schwarz method for singularly perturbed third order convection-diffusion type problem is constructed. The method splits the original domain into two overlapping subdomains. A hybrid difference scheme is proposed in which on the boundary layer region we use the combination of classical finite difference scheme and central finite difference scheme on a uniform mesh while on the non-layer region we use the midpoint difference scheme on a uniform mesh. It is shown that the numerical approximations which converge in the maximum norm to the exact solution. We proved that, when appropriate subdomains are used, the method produces convergence of second order. Furthermore, it is shown that, two iterations are sufficient to achieve the expected accuracy. Numerical examples are presented to support the theoretical results. The main advantages of this method used with the proposed scheme are it reduce iteration counts very much and easily identifies in which iteration the Schwarz iterate terminates.

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1. Introduction

Singular Perturbation Problems (SPPs) appear in many branches of applied mathematics, like fluid dynamics, quantum mechanics, turbulent interaction of waves and currents, electrodes theory, etc. The convergence of the numerical approximations generated by standard numerical methods applied to such problems depends adversely on the singular perturbation parameter. Most of these works have concentrated on second order single differential equations (see [3]

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and the references there in), but for third order equations only few results are reported in the literature (see [1], [16]-[19] and the references there in).

The classical numerical methods fail to produce good approximations for singularly perturbed problems (SPPs). Several non-classical approaches are used to design the numerical methods for Singularly Perturbed Problems. Such approaches can be either iterative or non-iterative. With an iterative approach numerical methods for SPPs comprising domain decomposition and Schwarz iterative techniques have been examined by various authors, for example, in [5]-[7], [12], [14]. In [7], Miller et al. examined a continuous overlapping Schwarz method for a singularly perturbed convection-diffusion equation with arbitrary fixed interface positions and found it to be uniformly convergent with respect to the perturbation parameter. In [15], an analysis of overlapping domain decomposition methods for singularly perturbed reaction-diffusion problems with distinct small positive parameters is presented. The authors of [15] found a flaw in the analysis of domain decomposition methods explored in [4], [12], [14]. The authors observation is that the constant C is not independent of the iteration number k and it is growing at each induction step in their proof of [4], [12], [14]. But in [15] the authors present an alternate analysis of overlapping domain decomposition methods for singularly perturbed reaction-diffusion problems with two parameters and problems in [14].

As far as the authors knowledge goes third order SPPs have not yet been examined for higher order convergence [1], [16]-[19]. Therefore, we are interested in constructing a numerical method for third order SPPs which produce higher order convergence. Of primary interest we have been proved that when appropriate subdomains are used the method produce convergence of almost second order.

Motivated by the works of [7], [16]-[19], we examined experimentally the performance of Schwarz method to the singularly perturbed third order BVPs described as below.

$$-\varepsilon y^{'''}(x) + a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x), \quad x \in \Omega = (0,1), \quad (1)$$

$$y(0) = q_1, \quad y'(0) = q_2, \quad y'(1) = q_3 \quad (2)$$

with $y \in C^{(3)}(\Omega) \cap C^{(1)}(\bar{\Omega})$. The functions $a(x)$, $b(x)$, $c(x)$ and $f(x)$ are sufficiently smooth functions satisfying the following conditions:

$$a(x) \geq \alpha', \quad \alpha' > 0, \quad (3)$$

$$b(x) \geq 0, \quad (4)$$

$$0 \geq c(x) \geq -\gamma, \quad \gamma > 0, \quad (5)$$

$$\alpha' > 8\gamma. \quad (6)$$

The SPBVPs (1)-(2) can be transformed into an equivalent weakly coupled system of two ODEs subject to suitable initial and boundary conditions of the form:

$$\begin{cases} L_1\boldsymbol{y}(x) \equiv y_1'(x) - y_2(x) = 0, \quad x \in \Omega^0 = (0,1], \\ L_2\boldsymbol{y}(x) \equiv -\varepsilon y_2''(x) + a(x)y_2'(x) + b(x)y_2(x) + c(x)y_1(x) \\ \qquad\qquad = f(x), \quad x \in \Omega = (0,1), \end{cases} \tag{7}$$

$$y_1(0) = q_1, \quad y_2(0) = q_2, \quad y_2(1) = q_3. \tag{8}$$

Where $\boldsymbol{y} = (y_1, y_2)^T$ and $a(x)$, $b(x)$, $c(x)$ and $f(x)$ are sufficiently smooth functions satisfying the above conditions (3)-(6). The above weakly coupled system can be written in the matrix-vector form as

$$\mathbf{L}\boldsymbol{y} \equiv \begin{pmatrix} L_1\boldsymbol{y} \\ L_2\boldsymbol{y} \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 \\ 0 & -\varepsilon\frac{d^2}{dx^2} \end{pmatrix}\boldsymbol{y} + \mathbf{A}(x)\boldsymbol{y}' + \mathbf{B}(x)\boldsymbol{y} = \mathbf{f}(x), \quad x \in \Omega, \tag{9}$$

$$\boldsymbol{y}(0) = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \; y_2(1) = q_3, \tag{10}$$

where $\boldsymbol{y}(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$, $\mathbf{f}(x) = \begin{pmatrix} 0 \\ f(x) \end{pmatrix}$, $\mathbf{A}(x) = \begin{pmatrix} 1 & 0 \\ 0 & a(x) \end{pmatrix}$ and $\mathbf{B}(x) = \begin{pmatrix} 0 & -1 \\ c(x) & b(x) \end{pmatrix}$. Let $\alpha = \min\{1, a(x)\}$ and $\beta = \min\{-1, b(x) + c(x)\}$.

In this paper, we have proved that discrete Schwarz method converge to the solution of the continuous problem. The method is shown to be of almost second order convergence. Iteration counts for the method are presented.

Remark 1.1. *The solution of the problem (1)-(2) exhibits a boundary layer at $x = 1$ which is less severe because the boundary conditions are prescribed for the derivative of the solution* [11]. *The condition (3) says that (1)-(2) is a non-turning point problem. The condition (5) is known as the quasi-monotonicity condition* [11]. *The maximum principle theorem for the above system (1)-(2) and for the corresponding discrete problem are established using the conditions (3)-(6) and using this principle, we can establish a stability result.*

An outline of the rest of the paper is as follows. In Section 2, the continuous Schwarz method is described . The derivative estimates are obtained in Section 3. In Section 4, the discrete Schwarz method is described. The maximum pointwise error bounds are obtained in Section 5. Numerical experiments are presented in Section 6 and finally, conclusions are included in Section 7.

Notations: Through out the paper we use C, with or without subscript to denote a generic positive constant independent of the iteration k and the discretization parameter N.

Let $\boldsymbol{y} : D \to \mathbb{R}$, $D \subseteq \mathbb{R}$. The appropriate norm for studying the convergence of the numerical solution to the exact solution of a SPP is $||y||_D = \sup_{x \in D} |y(x)|$.

For a vector $\boldsymbol{y} = (y_1, y_2)^T$, we define $||\boldsymbol{y}|| = \max_{j=1,2} |y_j|$.

For a vector valued function $\boldsymbol{z} = (z_1, z_2)^T$, define $||\boldsymbol{z}||_\Omega = \max\{||z_1||_\Omega, ||z_2||_\Omega\}$.

Given any two vector valued functions, $\boldsymbol{z}$ and $\boldsymbol{y}$, $\boldsymbol{z} \geq \boldsymbol{y}$ if $z_j \geq y_j$ for all $j = 1, 2$. For a vector of mesh functions $\mathbf{Z}(x_i) = (Z_1(x_i), Z_2(x_i))^T$, define

$$||\boldsymbol{Z}||_{\Omega^N} = \max_{j=1,2} \left(\max_{x_i \in \Omega^N} |Z_j(x_i)| \right).$$

2. Continuous Schwarz Method

In this section, a continuous Schwarz method is described. This process generates a sequence of iterates $\{\boldsymbol{y}^{[k]}\}$, which converges as $k \to \infty$ to the exact solution $\boldsymbol{y}$. Further we prove the maximum principle for (7)-(8). Using this principle, a stability result is stated. Finally, the bounds on the derivatives of the regular and singular components of $\boldsymbol{y}$ is presented.

First, we split the domain into two overlapping subdomains as $\Omega_c = (0, 1 - \tau)$ and $\Omega_r = (1 - 2\tau, 1)$, where the subdomain parameter τ is an appropriate constant, defined in Section 4. The iterative process is defined as follows:

$$\boldsymbol{y}^{[0]}(x) \equiv 0, \ 0 < x < 1, \quad \boldsymbol{y}^{[0]}(0) = \boldsymbol{y}(0), \quad y_2^{[0]}(1) = y_2(1).$$

For $k \geq 1$, the iterates $\boldsymbol{y}^{[k]}(x)$ are defined by

$$\boldsymbol{y}^{[k]}(x) = \begin{cases} \boldsymbol{y}_c^{[k]}(x) & \text{for } x \in \bar{\Omega}_c, \\ \boldsymbol{y}_r^{[k]}(x) & \text{for } x \in \bar{\Omega}_r \backslash \bar{\Omega}_c, \end{cases}$$

where $\boldsymbol{y}_p^{[k]}, p = \{c, r\}$ are the solutions of the problems

$$\begin{aligned} \mathbf{L}\boldsymbol{y}_r^{[k]}(x) &= \mathbf{f} \text{ in } \Omega_r, \quad \boldsymbol{y}_r^{[k]}(1 - 2\tau) = \boldsymbol{y}^{[k-1]}(1 - 2\tau), \ y_{2,r}^{[k]}(1) = y_2(1) \text{ and} \\ \mathbf{L}\boldsymbol{y}_c^{[k]}(x) &= \mathbf{f} \text{ in } \Omega_c, \ \boldsymbol{y}_c^{[k]}(0) = \boldsymbol{y}(0), \quad y_{2,c}^{[k]}(1 - \tau) = y_{2,r}^{[k]}(1 - \tau). \end{aligned}$$

Letting $\Omega_p = (d, e)$, $\Omega_p^0 = (d, e]$, $\bar{\Omega}_p = [d, e]$, $p = \{c, r\}$, note that the BVP (7)-(8) satisfies the following maximum principle on each $\bar{\Omega}_p$.

Theorem 2.1. (Maximum Principle). *Consider the BVP (7)-(8). Let* $y_1(d) \geq 0$, $y_2(d) \geq 0$, $y_2(e) \geq 0$, $L_1\boldsymbol{y}(x) \geq 0$, *for* $x \in \Omega_p^0$, *and* $L_2\boldsymbol{y}(x) \geq 0$, *for* $x \in \Omega_p$. *Then,* $\boldsymbol{y}(x) \geq 0$, $\forall x \in \bar{\Omega}_p$.

An immediate consequence of this is the following stability result.

Lemma 2.2. (Stability Result). *If* $\boldsymbol{y}(x)$ *is the solution of the BVP (7)-(8) then*

$$||\boldsymbol{y}(x)|| \leq C \max\{|y_1(d)|, \ |y_2(d)|, \ |y_2(e)|, \ \max_{x \in \Omega_p^0} |L_1\boldsymbol{y}(x)|, \ \max_{x \in \Omega_p} |L_2\boldsymbol{y}(x)|\},$$

$\forall x \in \Omega_p$.

2.1. Asymptotic Expansion Approximation. Using a standard perturbation method [10], we can construct an asymptotic expansion for the solution of (7)-(8) as follows. In fact we have $\boldsymbol{y}_{as}(x,\varepsilon) = \boldsymbol{u}_0(x)+\boldsymbol{v}_0(x)+\varepsilon(\boldsymbol{u}_1(x)+\boldsymbol{v}_1(x))+O(\varepsilon^2)$. The zero order asymptotic approximation as $\boldsymbol{y}_{as}(x) = \boldsymbol{u}_0(x) + \boldsymbol{v}_0(x)$, where $\boldsymbol{u}_0(x) = (u_{0_1}(x), u_{0_2}(x))^T$ is the solution of the reduced problem of the BVP (7)-(8) given by

$$\begin{cases} u'_{0_1}(x) - u_{0_2}(x) = 0, \\ a(x)u'_{0_2}(x) + b(x)u_{0_2}(x) + c(x)u_{0_1}(x) = f(x), \\ u_{0_1}(0) = q_1, \quad u_{0_2}(0) = q_2, \end{cases} \tag{11}$$

$\boldsymbol{v}_0(x) = (v_{0_1}(x), v_{0_2}(x))^T$ is a layer correction term given by

$$\begin{cases} v_{0_1}(x) = (\varepsilon/a(0))(q_3 - u_{0_2}(1))e^{-a(0)(1-x)/\varepsilon}, \\ v_{0_2}(x) = (q_3 - u_{0_2}(1))e^{-a(0)(1-x)/\varepsilon}, \end{cases} \tag{12}$$

and $\boldsymbol{v}_0(x)$ satisfies

$$\begin{cases} v'_{0_1}(x) - v_{0_2}(x) = 0, \\ -\varepsilon v''_{0_2}(x) + a(0)v'_{0_2}(x) = 0, \\ v_{0_1}(0) = (\varepsilon/a(0))v_{0_2}(0), \\ v_{0_2}(0) = v_{0_2}(1)e^{-a(0)/\varepsilon}, \quad v_{0_2}(1) = q_3 - u_{0_2}(1). \end{cases} \tag{13}$$

3. Estimates of Derivatives

In this section, estimates of the derivatives of the solution are derived.

Lemma 3.1. *Let $\boldsymbol{y}(x)$ be the solution of the BVP (7)-(8). Then $y_1(x)$ and $y_2(x)$ satisfy*

$$|y_1^{(l)}(x)| \le C(1 + \varepsilon^{-(l-1)}\exp(-\alpha'(1-x)/\varepsilon)),$$
$$|y_2^{(l)}(x)| \le C(1 + \varepsilon^{-(l)}\exp(-\alpha'(1-x)/\varepsilon)), \quad \textit{for} \quad 0 \le l \le 4, \ \forall x \in \bar{\Omega},$$
$$\textit{where } \bar{\Omega} = (\bar{\Omega}_r \backslash \bar{\Omega}_c) \cup \bar{\Omega}_c.$$

Proof. Using the procedure adopted in [17] one can prove the lemma. □

In this section, estimates of the derivatives of the components of the solution of the BVP (7)-(8) are derived. In Section 5 we establish the convergence of the discrete Schwarz method described in Section 4. For this we need sharper bounds on the derivatives of the exact solution $\boldsymbol{y}$ of (7)-(8). Now, decompose the solution $\boldsymbol{y}(x)$ of (7)-(8) into smooth and singular components as

$$\boldsymbol{y}(x) = \boldsymbol{v}(x) + \boldsymbol{w}(x), \tag{14}$$

where

$$\boldsymbol{v}(x) = \boldsymbol{u}_0(x), \quad \boldsymbol{w}(x) = \boldsymbol{v}_0(x), \tag{15}$$

where $\boldsymbol{u}_0(x)$ is given by (11), $\boldsymbol{v}_0(x)$ satisfies (12)-(13).
The following lemma gives estimate of the derivatives of these components.

Lemma 3.2. *The smooth and singular components of the solution* $\boldsymbol{y}(x)$ *of the BVP (7)-(8) satisfy*

$$|\boldsymbol{v}_1^{(l)}(x)| \leq C, \quad |\boldsymbol{v}_2^{(l)}(x)| \leq C,$$

$$\text{and } |\boldsymbol{w}_1^{(l)}(x)| \leq C\varepsilon^{-(l-1)}e^{-\alpha'(1-x)/\varepsilon}, \quad |\boldsymbol{w}_2^{(l)}(x)| \leq C\varepsilon^{-(l)}e^{-\alpha'(1-x)/\varepsilon},$$

for $0 \leq l \leq 4$, $\forall x \in \bar{\Omega} = (\bar{\Omega}_r \backslash \bar{\Omega}_c) \cup \bar{\Omega}_c$, $\boldsymbol{v}(x)$ *and* $\boldsymbol{w}(x)$ *are given by (11)-(13).*

Proof. Following the method of proof adopted in [7] and using Lemma 2.2, we can get the desired estimates. □

4. Discrete Schwarz Method

The continuous overlapping Schwarz method described in Section 2 is discretized by introducing uniform meshes on each subdomain. The domain $\Omega = (0,1)$ is divided into two overlapping subdomains as $\Omega_c = (0, 1-\tau)$ and $\Omega_r = (1-2\tau, 1)$. The subdomain parameter τ is chosen to be the Shishkin transition point $\tau = \min\left\{\frac{1}{3}, \frac{4\varepsilon}{\alpha}\ln N\right\}$ as in [7]. In each subdomain, $\Omega_p = (d, e)$, $p = \{c, r\}$, construct a uniform mesh $\bar{\Omega}_p^N = \{d = x_0 < x_1 < x_2 < \cdots < x_N = e\}$ with $h_p = x_i - x_{i-1} = (e-d)/N$.

In the proposed scheme we use the combination of classical finite difference scheme and central finite difference scheme on a uniform mesh on the subdomain Ω_r and the midpoint difference scheme on a uniform mesh on the subdomain Ω_c. Then in each subdomain Ω_p^N, $p = \{c, r\}$, the corresponding discretization is,

$$\mathbf{L}^N\mathbf{Y}_c(x_i) \tag{16}$$

$$= \begin{cases} L_1^N\mathbf{Y}_c(x_i) = D^-Y_{1,c}(x_i) - \hat{Y}_{2,c}(x_i) = 0, \quad i = 1, \cdots, N, \\ L_2^N\mathbf{Y}_c(x_i) = -\varepsilon\delta^2 Y_{2,c}(x_i) + a_{i-1/2}D^-Y_{2,c}(x_i) + c_{i-1/2}\hat{Y}_{1,c}(x_i) + \\ b_{i-1/2}\hat{Y}_{2,c}(x_i) = f_{i-1/2}, \quad i = 1, \cdots, N-1, \end{cases} \tag{17}$$

$$\mathbf{L}^N\mathbf{Y}_r(x_i) = \begin{cases} L_1^N\mathbf{Y}_r = D^-Y_{1,r}(x_i) - Y_{2,r}(x_i) = 0, \quad i = 1, \cdots, N, \\ L_2^N\mathbf{Y}_r = -\varepsilon\delta^2 Y_{2,r}(x_i) + a_i D^0 Y_{2,r}(x_i) + b_i Y_{2,r}(x_i) + \\ c_i Y_{1,r}(x_i) = f_i, \quad i = 1, \cdots, N-1. \end{cases} \tag{18}$$

Where,

$$\delta^2 Y_{j,p}(x_i) = \frac{1}{h_p^2}\left(Y_{j,p}(x_{i+1}) - 2Y_{j,p}(x_i) + Y_{j,p}(x_{i-1})\right),$$

$$D^0 Y_{j,r}(x_i) = \frac{Y_{j,r}(x_{i+1}) - Y_{j,r}(x_{i-1})}{2h_r}, \quad D^- Y_{j,p}(x_i) = \frac{Y_{j,p}(x_i) - Y_{j,p}(x_{i-1})}{h_p},$$

$\hat{Y}_{j,c}(x_i) \equiv (Y_{j,c}(x_i) + Y_{j,c}(x_{i-1}))/2$, $a_{i-1/2} \equiv a((x_{i-1} + x_i)/2)$, and $a_i \equiv a(x_i)$;

similarly for $b_{i-1/2}$, $c_{i-1/2}$, $f_{i-1/2}$, b_i, c_i and f_i, $j=1,2$.
The discrete problem is $\mathbf{L}^N\mathbf{Y}_p(x_i)=\mathbf{f}(x_i)$, where

$$\mathbf{f}(x_i)=\begin{cases}\mathbf{f}_{i-\frac{1}{2}}, & x_i\in\bar{\Omega}_c^N,\\ \mathbf{f}_i, & x_i\in\bar{\Omega}_r^N.\end{cases}$$

Then the algorithm for discrete Schwarz method is defined as follows.
Step 1: We choose the initial mesh function

$$\mathbf{Y}^{[0]}(x_i)\equiv 0,\quad 0<x_i<1,\quad \mathbf{Y}^{[0]}(0)=\boldsymbol{y}(0),\quad Y_2^{[0]}(1)=y_2(1).$$

Step 2: We compute the mesh functions $\mathbf{Y}_p{}^{[k]}$, $p=\{r,c\}$ which are the solutions of the following discrete problems

$$\begin{aligned}\mathbf{L}^N\mathbf{Y}_r^{[k]}(x_i) &= \mathbf{f}_i,\ x_i\in\Omega_r^N,\quad \mathbf{Y}_r^{[k]}(1-2\tau)=\bar{\mathbf{Y}}^{[k-1]}(1-2\tau),\quad Y_{2,r}^{[k]}(1)=y_2(1),\\ \mathbf{L}^N\mathbf{Y}_c^{[k]}(x_i) &= \mathbf{f}_{i-\frac{1}{2}},\ x_i\in\Omega_c^N,\quad \mathbf{Y}_c^{[k]}(0)=\boldsymbol{y}(0),\quad Y_{2,c}^{[k]}(1-\tau)=\bar{Y}_{2,r}^{[k]}(1-\tau),\end{aligned}$$

where $\bar{\mathbf{Y}}^{[k]}$ denotes the piecewise linear interpolant of $\mathbf{Y}^{[k]}$ on the mesh $\bar{\Omega}^N:=(\bar{\Omega}_r^N\backslash\bar{\Omega}_c)\cup\bar{\Omega}_c^N$.
Step 3: We compute the mesh function $\mathbf{Y}^{[k]}$ by combining together the solutions on the subdomains

$$\mathbf{Y}^{[k]}(x_i)=\begin{cases}\mathbf{Y}_c^{[k]}(x_i), & \text{for } x_i\in\bar{\Omega}_c^N,\\ \mathbf{Y}_r^{[k]}(x_i), & \text{for } x_i\in\bar{\Omega}_r^N\backslash\bar{\Omega}_c.\end{cases}$$

Step 4: If the stopping criterion

$$||\mathbf{Y}^{[k+1]}-\mathbf{Y}^{[k]}||_{\bar{\Omega}^N}\le tol$$

is reached, then stop; otherwise go to **Step 2**. Here *tol* is the user prescribed accuracy.
The following are analogous results for the discrete problem.

Lemma 4.1. ***(Discrete maximum principle)*** *Assume that* $\boldsymbol{Y}(x_0)\ge\boldsymbol{0}$ *and* $Y_2(x_N)\ge 0$ *then* $\boldsymbol{L}^N\boldsymbol{Y}(x_i)\ge\boldsymbol{0}$, *for* $x_i\in\Omega_p^N$ *implies that* $\boldsymbol{Y}(x_i)\ge\boldsymbol{0}$, $\forall x_i\in\bar{\Omega}_p^N$.

Proof. Please refer to [8, 13]. □

An immediate consequence of this lemma is the following stability result.

Lemma 4.2. *If* $Y_j(x_i)$ *is any mesh function then for all* $x_i\in\bar{\Omega}_p^N$,

$$|Y_j(x_i)|\le C\max\{|Y_1(x_0)|,\,|Y_2(x_0)|,\,|Y_2(x_N)|,\,||L_1^N\,\boldsymbol{Y}||_{\Omega_p^{0}{}^N},\,||L_2^N\,\boldsymbol{Y}||_{\Omega_p^N}\},j=1,2.$$

Proof. Please refer to [8, 13]. □

5. Error Estimates

In this section, we estimate the error in discrete Schwarz iterates and prove that two iterations are required to attain second order convergence. Following the method of analysis adapted in [14] and [15] we derive error estimates. The analysis proceeds as follows.

Lemma 5.1. *Let* $\boldsymbol{y}$ *be the solution of (7)-(8) and let* $\mathbf{Y}^{[k]}$ *be the* k^{th} *iterate of the discrete Schwarz method described as in Section 4. Then, there are constants* C *such that*

$$|| \boldsymbol{Y}^{[k]} - \boldsymbol{y}||_{\bar{\Omega}^N} \leq C2^{-k} + CN^{-2}\ln^3 N,$$

where C *is independent of* k *and* N.

Proof. At the first iteration $(\mathbf{Y}^{[0]} - \boldsymbol{y})(0) = \mathbf{0}$ and $(\mathbf{Y}^{[0]} - \boldsymbol{y})(1) = \mathbf{0}$.
Since $\mathbf{Y}^{[0]}(x_i) = \mathbf{0}$ for $x_i \in \Omega^N := \{x_1 < x_2 < x_3 < \cdots < x_{N-1}\}$, we can use Lemma 2.2 to show that

$$||\mathbf{Y}^{[0]} - \boldsymbol{y}||_{\Omega^N} = ||\boldsymbol{y}||_{\Omega^N} \leq C.$$

Clearly, there are constants C such that $||\mathbf{Y}^{[0]} - \boldsymbol{y}||_{\bar{\Omega}^N} \leq C2^0 + CN^{-2}\ln^3 N$. Thus, the result holds for $k = 0$ and the proof is now completed by induction. Assume that, for an arbitrary integer $k \geq 0$, there exists C such that

$$||\mathbf{Y}^{[k]} - \boldsymbol{y}||_{\bar{\Omega}^N} \leq C2^{-k} + CN^{-2}\ln^3 N.$$

Case (i): Error bound estimation on $\bar{\Omega}_r^N$.
In the proposed scheme we use the combination of midpoint difference scheme and central finite difference scheme on $\bar{\Omega}_r^N$. One can deduce the following truncation error estimate as in [8] on $x_i \in \bar{\Omega}_r^N$ as

$$|(\mathbf{L}^N - \mathbf{L})\boldsymbol{y}||_{\Omega_r^N} \leq \begin{pmatrix} Ch_r^2||y_1^{(3)}||_{\Omega_r^N} \\ C\varepsilon h_r^2||y_2^{(4)}||_{\Omega_r^N} + Ch_r^2||y_2^{(3)}||_{\Omega_r^N} \end{pmatrix}. \tag{19}$$

In order to find a bound on $||\mathbf{L}^N(\mathbf{Y}_r^{[k+1]} - \boldsymbol{y})||_{\Omega_r^N}$ we must decompose $\boldsymbol{y}$ as in (14). Consider

$$\begin{aligned} ||\mathbf{L}^N(\mathbf{Y}_r^{[k+1]} - \boldsymbol{y})||_{\Omega_r^N} &= ||\mathbf{f} - \mathbf{L}^N\boldsymbol{y}||_{\Omega_r^N} \\ &= ||(\mathbf{L}^N - \mathbf{L})\boldsymbol{y}||_{\Omega_r^N} \\ &\leq ||(\mathbf{L}^N - \mathbf{L})\boldsymbol{v}||_{\Omega_r^N} + ||(\mathbf{L}^N - \mathbf{L})\boldsymbol{w}||_{\Omega_r^N}. \end{aligned} \tag{20}$$

For the first term on the right-hand side of (20), we use the local truncation error estimate (19), $h_r \leq CN^{-1}$, $\varepsilon \leq CN^{-1}$ and Lemma 3.2 to get

$$||(\mathbf{L}^N - \mathbf{L})\boldsymbol{v}||_{\Omega_r^N} \leq \begin{pmatrix} Ch_r^2||v_1^{(3)}||_{\Omega_r^N} \\ C\varepsilon h_r^2||v_2^{(4)}||_{\Omega_r^N} + Ch_r^2||v_2^{(3)}||_{\Omega_r^N} \end{pmatrix}$$

$$\leq \begin{pmatrix} CN^{-2} \\ CN^{-3} + CN^{-2} \end{pmatrix}$$
$$\leq CN^{-2}.$$

For the second term on the right-hand side of (20), when $\tau = \dfrac{4\varepsilon}{\alpha}\ln N,$ using the local truncation error estimate (19), and $h_r \leq C\varepsilon N^{-1}\ln N,$ we have

$$\begin{aligned} ||(\mathbf{L}^N - \mathbf{L})\boldsymbol{w}||_{\Omega_r^N} &\leq \begin{pmatrix} Ch_r^2||w_1^{(3)}||_{\Omega_r^N} \\ C\varepsilon h_r^2||w_2^{(4)}||_{\Omega_r^N} + Ch_r^2||w_2^{(3)}||_{\Omega_r^N} \end{pmatrix} \\ &\leq \begin{pmatrix} Ch_r^2\varepsilon^{-2} \\ Ch_r^2\varepsilon^{-3} \end{pmatrix} \\ &\leq C\varepsilon^{-1}N^{-2}\ln^2 N. \end{aligned}$$

Using the above estimates in (20), we have

$$||\mathbf{L}^N(\mathbf{Y}_r^{[k+1]} - \boldsymbol{y})||_{\Omega_r^N} \leq CN^{-2}\ln^3 N + C\varepsilon^{-1}N^{-2}\ln^2 N, \quad (21)$$

for some C. The end point of the subdomain Ω_r^N is $1-2\tau,$ which in general is not in $\Omega^N = \{x_1 < x_2 < x_3 < \cdots < x_{N-1}\},$ so we use a piecewise linear interpolant of the previous iterate to determine $\mathbf{Y}_r^{[k+1]}(1-2\tau).$
Now, using our inductive argument we have

$$\begin{aligned} ||(\mathbf{Y}_r^{[k+1]} - \boldsymbol{y})(1-2\tau)|| &= ||(\bar{\mathbf{Y}}^{[k]} - \boldsymbol{y})(1-2\tau)|| \\ &= ||(\mathbf{Y}^{[k]} - \boldsymbol{y})(1-2\tau)|| \\ &\leq ||(\mathbf{Y}^{[k]} - \bar{\boldsymbol{y}})(1-2\tau)|| \\ &+ ||(\bar{\boldsymbol{y}} - \boldsymbol{y})(1-2\tau)|| \end{aligned} \quad (22)$$

where $\bar{\boldsymbol{y}}$ is the piecewise linear interpolant of $\boldsymbol{y}$ using grid points of $\bar{\Omega}_c^N.$
For the second term on the right-hand side of (22), using solution decomposition $\boldsymbol{y}$ as in (14) we get

$$||(\bar{\boldsymbol{y}} - \boldsymbol{y})(1-2\tau)|| \leq ||(\bar{\boldsymbol{v}} - \boldsymbol{v})(1-2\tau)|| + ||(\bar{\boldsymbol{w}} - \boldsymbol{w})(1-2\tau)||. \quad (23)$$

Note that $(1-2\tau)$ lies in $\bar{\Omega}_c.$ For any $\mathbf{z} \in C^2(\bar{\Omega}_c),$ standard argument of piecewise linear interpolant $\bar{\mathbf{z}}$ gives

$$||(\mathbf{z} - \bar{\mathbf{z}})(1-2\tau)|| \leq Ch_c^2||\mathbf{z}^{(2)}||_{\bar{\Omega}_c} \quad \text{and} \quad ||(\mathbf{z} - \bar{\mathbf{z}})(1-2\tau)|| \leq C||\mathbf{z}||_{\bar{\Omega}_c}. \quad (24)$$

For the first term on the right-hand side of (23), we use the first bound of (24), $h_c \leq CN^{-1}$ and Lemma 3.2 to get

$$\begin{aligned} ||(\bar{\boldsymbol{v}} - \boldsymbol{v})(1-2\tau)|| &\leq Ch_c^2||\boldsymbol{v}^{(2)}||_{\bar{\Omega}_c} \\ &\leq CN^{-2}. \end{aligned}$$

For the second term on the right-hand side of (23), when $\tau = \dfrac{4\varepsilon}{\alpha}\ln N,$ note that the layer function $\boldsymbol{w}$ is monotonically increasing in the region $(1/3, 1-\tau) \subset \bar{\Omega}_c.$

Hence using the second bound of (24), we have

$$\begin{aligned} ||(\bar{\boldsymbol{w}}-\boldsymbol{w})(1-2\tau)|| &\leq C||\boldsymbol{w}||_{\bar{\Omega}_c} \\ &\leq Ce^{-\alpha\tau/\varepsilon} \leq CN^{-2}. \end{aligned}$$

$$\therefore\ ||(\bar{\boldsymbol{y}}-\boldsymbol{y})(1-2\tau)|| \leq CN^{-2}. \tag{25}$$

Now, using (25) in (22), we have

$$\begin{aligned} ||(\mathbf{Y}_r^{[k+1]}-\boldsymbol{y})(1-2\tau|| &\leq C2^{-k}+CN^{-2}\ln^3 N+CN^{-2} \\ &\leq C2^{-k}+CN^{-2}\ln^3 N. \end{aligned}$$

Consider the mesh function

$$\begin{aligned} \Psi^{\pm}(x_i) &= \mathbf{C}\left(\frac{1+x_i}{4}\right)2^{-k}+\mathbf{C}(1+x_i)N^{-2}\ln^3 N \\ &\quad +\mathbf{C}(x_i-(1-2\tau))\varepsilon^{-1}N^{-2}\ln^2 N \pm (\mathbf{Y}_r^{[k+1]}-\boldsymbol{y})(x_i), \end{aligned}$$

where $\mathbf{C}$ is positive constants to be chosen suitably, so that the following are satisfied.

Note that, $\Psi^{\pm}(1-2\tau)$

$$\begin{aligned} &\geq \mathbf{C}\left(\frac{1+(1-2\tau)}{4}\right)2^{-k}+\mathbf{C}(1+(1-2\tau))N^{-2}\ln^3 N \\ &\quad -\mathbf{C}2^{-k}-\mathbf{C}N^{-2}\ln^3 N, \\ &\geq \mathbf{C}\left(\frac{1}{4}\right)2^{-k}+\mathbf{C}N^{-2}\ln^3 N-\mathbf{C}2^{-k}-\mathbf{C}N^{-2}\ln^3 N>0. \end{aligned}$$

$$\Psi^{\pm}(1) = \mathbf{C}\left(\frac{1}{2}\right)2^{-k}+2\mathbf{C}N^{-2}\ln^3 N+2\mathbf{C}\tau\varepsilon^{-1}N^{-2}\ln^2 N \pm 0>0, \quad \text{and}$$

$$\begin{aligned} L^N\Psi^{\pm}(x_i) &\geq \alpha\left(\left(\frac{\mathbf{C}}{4}\right)2^{-k}+\mathbf{C}N^{-2}\ln^3 N+\mathbf{C}\varepsilon^{-1}N^{-2}\ln^2 N\right) \\ &\quad -\mathbf{C}N^{-2}\ln^3 N-\mathbf{C}\varepsilon^{-1}N^{-2}\ln^2 N>0. \end{aligned}$$

Using the discrete maximum principle for the operator L^N on $\bar{\Omega}_r^N$ we get,

$$\begin{aligned} ||(\mathbf{Y}_r^{[k+1]}-\boldsymbol{y})||_{\bar{\Omega}_r^N} &\leq C\left(\frac{1+x_i}{4}\right)2^{-k}+C(1+x_i)N^{-2}\ln^3 N \\ &\quad +C(x_i-(1-2\tau))\varepsilon^{-1}N^{-2}\ln^2 N. \end{aligned}$$

Consequently,

$$\begin{aligned} ||(\mathbf{Y}_r^{[k+1]}-\boldsymbol{y})||_{\bar{\Omega}_r^N\setminus\bar{\Omega}_c} &\leq C\left(\frac{1}{2}\right)2^{-k}+2CN^{-2}\ln^3 N+2C\tau\varepsilon^{-1}N^{-2}\ln^2 N, \\ &\leq C2^{-(k+1)}++CN^{-2}\ln^3 N+C\tau\varepsilon^{-1}N^{-2}\ln^2 N. \end{aligned}$$

But since $\tau = \dfrac{4\varepsilon}{\alpha}\ln N,$ this gives

$$||(\mathbf{Y}_r^{[k+1]} - \boldsymbol{y})||_{\bar{\Omega}_r^N\setminus\bar{\Omega}_c} \quad \leq \quad C2^{-(k+1)} + CN^{-2}\ln^3 N. \tag{26}$$

Case (ii): Error bound estimation on $\bar{\Omega}_c^N$.
We use solution decomposition as in (14) at each point $x_i \in \bar{\Omega}_c^N$, the difference $(\mathbf{Y}_c^{[k+1]} - \boldsymbol{y})$ can be written in the form

$$(\mathbf{Y}_c^{[k+1]} - \boldsymbol{y})(x_i) = (\mathbf{V}_c^{[k+1]} - \boldsymbol{v})(x_i) + (\mathbf{W}_c^{[k+1]} - \boldsymbol{w})(x_i). \tag{27}$$

Suppose that $(1-\tau)$ lies in $\bar{\Omega}_r$. For any $\mathbf{z} \in C^2(\bar{\Omega}_r)$, standard argument of piecewise linear interpolant $\bar{\mathbf{z}}$ gives

$$||(\mathbf{z} - \bar{\mathbf{z}})(1-\tau)|| \leq Ch_r^2||\mathbf{z}^{(2)}||_{\bar{\Omega}_r}. \tag{28}$$

In the proposed scheme we use the midpoint difference scheme on $\bar{\Omega}_c^N$. One can deduce the following truncation error estimate as in [8] on $x_i \in \bar{\Omega}_c^N$ as

$$||(\mathbf{L}^N - \mathbf{L})\boldsymbol{y}||_{\Omega_c^N} \quad \leq \quad \begin{pmatrix} Ch_c^2||y_1^{(3)}||_{\Omega_c^N} + Ch_c^2||y_2^{(2)}||_{\Omega_c^N} \\ C\varepsilon h_c||y_2^{(3)}||_{\Omega_c^N} + Ch_c^2(||y_2^{(3)}||_{\Omega_c^N} + ||y_1^{(2)}||_{\Omega_c^N}) \end{pmatrix}.$$

Subcase (i): For the first term on the right-hand side of (27), using the above local truncation error estimate, $h_c \leq CN^{-1}$, $\varepsilon \leq CN^{-1}$ and Lemma 3.2, we get

$$\begin{aligned} ||\mathbf{L}^N(\mathbf{V}_c^{[k+1]} - \boldsymbol{v})||_{\Omega_c^N} &= ||\mathbf{f} - \mathbf{L}^N\boldsymbol{v}||_{\Omega_c^N} \\ &= ||(\mathbf{L}^N - \mathbf{L})\boldsymbol{v}||_{\Omega_c^N} \\ &\leq \begin{pmatrix} Ch_c^2||v_1^{(3)}||_{\Omega_c^N} + Ch_c^2||v_2^{(2)}||_{\Omega_c^N} \\ C\varepsilon h_c||v_2^{(3)}||_{\Omega_c^N} + Ch_c^2(||v_2^{(3)}||_{\Omega_c^N} + ||v_1^{(2)}||_{\Omega_c^N}) \end{pmatrix} \\ &\leq \begin{pmatrix} CN^{-2} \\ CN^{-2} \end{pmatrix} \leq CN^{-2}. \end{aligned}$$

Now, using our inductive argument, the bound of (28), $h_r \leq CN^{-1}$, $\varepsilon \leq CN^{-1}$ and Lemma 3.2, we get

$$\begin{aligned} ||(\mathbf{V}_c^{[k+1]} - \boldsymbol{v})(1-\tau)|| &= ||(\bar{\mathbf{V}}_r^{[k+1]} - \boldsymbol{v})(1-\tau)|| \\ &= ||(\bar{\mathbf{V}} - \boldsymbol{v})(1-\tau)|| \\ &\leq Ch_r^2||\boldsymbol{v}^{(2)}||_{\bar{\Omega}_r} \\ &\leq CN^{-2}, \end{aligned}$$

where we have use the fact that $(1-\tau)$ is the mesh point of $\bar{\Omega}_r^N$.
Consider the mesh function

$$\Phi^{\pm}(x_i) \quad = \quad \mathbf{C}\left(\frac{x_i}{2(1-\tau)}\right)2^{-k} + (1+x_i)\mathbf{C}N^{-2} \pm (\mathbf{V}_c^{[k+1]} - \boldsymbol{v})(x_i),$$

where C is positive constants to be choosen suitably, so that the following expressions are satisfied.

$$\text{Note that,} \quad \Phi^{\pm}(0) = \mathbf{C}N^{-2} \pm 0 > 0,$$

$$\begin{aligned} \Phi^{\pm}(1-\tau) &\geq \mathbf{C}\left(\frac{1}{2}\right)2^{-k} + \mathbf{C}(2-\tau)N^{-2} - \mathbf{C}N^{-2}, \\ &\geq \left(\frac{\mathbf{C}}{2}\right)2^{-k} + \mathbf{C}N^{-2} - \mathbf{C}N^{-2} > 0 \quad \text{and} \\ L^N\Phi^{\pm}(x_i) &\geq \alpha\left(\left(\frac{\mathbf{C}}{2}\right)2^{-k} + \mathbf{C}N^{-2}\right) - \mathbf{C}N^{-2} > 0. \end{aligned}$$

We use the discrete maximum principle for the operator L^N on $\bar{\Omega}_c^N$ to get,

$$\begin{aligned} ||\mathbf{V}_c^{[k+1]} - \boldsymbol{v}||_{\bar{\Omega}_c^N} &\leq C\left(\frac{1}{2}\right)2^{-k} + C(2-\tau)N^{-2} \\ &\leq C2^{-[k+1]} + CN^{-2}. \end{aligned}$$

Subcase(ii): For the second term on the right-hand side of (27). When $\tau = \frac{4\varepsilon}{\alpha}\ln N$, using the arguments discussed as in ([9], Lemma 6), for $x_i \in \Omega_c^N$, we have $||\mathbf{W}_c^{[k+1]} - \boldsymbol{w}||_{\Omega_c^N} \leq CN^{-2}$.
Now, using error bound for the regular and layer parts we get

$$||(\mathbf{Y}_c^{[k+1]} - \boldsymbol{y})||_{\Omega_c^N} \leq C2^{-[k+1]} + CN^{-2}\ln^3 N. \tag{29}$$

On combining error bounds (26) and (29), we have

$$||\mathbf{Y}^{[k+1]} - \boldsymbol{y}||_{\bar{\Omega}^N} \leq C2^{-[k+1]} + CN^{-2}\ln^3 N.$$

□

Now we will show that the discrete Schwarz iterates converge at a higher rate than that proved in Lemma 5.1.

Lemma 5.2. *Let $\mathbf{Y}^{[k]}$ be the k^{th} iterate of the discrete Schwarz method described as in Section 5. Then there exists some C such that*

$$||\,\mathbf{Y}^{[k+1]} - \,\mathbf{Y}^{[k]}||_{\bar{\Omega}^N} \leq C\nu^k \quad \textit{where} \quad \nu = \left(1 + \frac{\tau\alpha}{2\varepsilon N}\right)^{-N/2} < 1$$

and C is independent of k and N. Furthermore if $\tau = \frac{4\varepsilon}{\alpha}\ln N$ then $\nu \leq 2N^{-1}$.

Proof. At the first iteration $||\mathbf{Y}^{[0]}||_{\Omega^N} = \mathbf{0}$.
Then clearly

$$||\mathbf{Y}^{[1]} - \mathbf{Y}^{[0]}||_{\Omega^N} = ||\mathbf{Y}^{[1]}||_{\Omega^N}.$$

$\mathbf{Y}_r^{[1]}$ satisfies

$$\mathbf{L}^N\mathbf{Y}_r^{[1]} = \mathbf{f}_i \quad \text{for} \quad x_i \in \Omega_r^N,$$

$$\mathbf{Y}_r^{[1]}(1-2\tau) = \bar{\mathbf{Y}}^{[0]}(1-2\tau), \quad Y_{2,r}^{[1]}(1) = y_2(1).$$

Therefore, we use Lemma 4.2 to obtain $||\mathbf{Y}_r^{[1]}||_{\bar{\Omega}_r^N} \leq C$.

$$\text{Consequently, } ||\mathbf{Y}_r^{[1]}||_{\bar{\Omega}_r^N \setminus \Omega_c} \leq C.$$

Also $\mathbf{Y}_c^{[1]}$ satisfies

$$\mathbf{L}^N \mathbf{Y}_c^{[1]} = \mathbf{f}_{i-1/2} \quad \text{for} \quad x_i \in \Omega_c^N,$$
$$\mathbf{Y}_c^{[1]}(0) = \boldsymbol{y}(0), \quad Y_{2,c}^{[1]}(1-\tau) = \bar{Y}_{2,r}^{[1]}(1-\tau).$$

Therefore, we can apply Lemma 4.2 to get $||\mathbf{Y}_c^{[1]}||_{\bar{\Omega}_c^N} \leq C$.
Combine all these estimates to obtain

$$||\mathbf{Y}^{[1]} - \mathbf{Y}^{[0]}||_{\bar{\Omega}^N} \leq C\nu^0.$$

Thus, the result holds for $k = 0$ and the proof is now completed by induction argument.
Assume that, for an arbitrary integer $k \geq 0$,

$$||\mathbf{Y}^{[k+1]} - \mathbf{Y}^{[k]}||_{\bar{\Omega}^N} \leq C\nu^k \quad \text{where} \quad \nu = \left(1 + \frac{\alpha\tau}{2\varepsilon N}\right)^{-N/2}.$$

Note that

$$\mathbf{L}^N(\mathbf{Y}_c^{[k+1]} - \mathbf{Y}_c^{[k]})(x_i) = \mathbf{0}, \quad \text{for} \quad x_i \in \Omega_c^N, \quad (\mathbf{Y}_c^{[k+1]} - \mathbf{Y}_c^{[k+1]})(0) = \mathbf{0},$$

and $|(Y_{2_c}^{[k+1]} - Y_{2_c}^{[k]})(1-\tau)| \leq C\nu^k$.
Let $\boldsymbol{E}_c^{[k+1]}(x_i) = \begin{pmatrix} E_{1,c}(x_i) \\ E_{2,c}(x_i) \end{pmatrix}$ be the solution of

$$\begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & -\varepsilon \end{pmatrix} \delta^2 \boldsymbol{E}_c^{[k+1]}(x_i) + \alpha D^- \boldsymbol{E}_c^{[k+1]}(x_i) + \beta \hat{\boldsymbol{E}}_c^{[k+1]}(x_i) = 0 \text{ for } x_i \in \Omega_c^N, \\ \boldsymbol{E}_c^{[k+1]}(0) = 0, \quad E_{2_c}^{[k+1]}(1-\tau) = C\nu^k. \end{cases} \tag{30}$$

Using the maximum principle argument we note that $\boldsymbol{E}_c^{[k+1]}(0) \geq 0$, $E_{2_c}^{[k+1]}(1-\tau) \geq 0$, $\boldsymbol{E}_c^{[k+1]}(x_i) \geq 0$ for $x_i \in \bar{\Omega}_c^N$, and thus one can easily deduce that $L^N \boldsymbol{E}_c^{[k+1]}(x_i) \geq 0$, for $x_i \in \Omega_c^N$. Hence

$$\begin{aligned} & L^N(\boldsymbol{E}_c^{[k+1]} - (\mathbf{Y}_c^{[k+1]} - \mathbf{Y}_c^{[k]}))(x_i) \\ = \; & L^N(\boldsymbol{E}_c^{[k+1]})(x_i) - L^N(\mathbf{Y}_c^{[k+1]} - \mathbf{Y}_c^{[k]})(x_i), \\ \geq \; & 0, \text{ as } L^N(\mathbf{Y}_c^{[k+1]} - \mathbf{Y}_c^{[k]})(x_i) = \mathbf{0} \text{ for } x_i \in \Omega_c^N, \end{aligned}$$

$\boldsymbol{E}_c^{[k+1]}(0) - (\mathbf{Y}_c^{[k+1]} - \mathbf{Y}_c^{[k]})(0) \geq \mathbf{0}$, $\quad E_{2_c}^{[k+1]}(1-\tau) - (Y_{2_c}^{[k+1]} - Y_{2_c}^{[k]})(1-\tau) \geq \mathbf{0}$.
Then by using Lemma 4.1, we have

$$(\mathbf{Y}_c^{[k+1]} - \mathbf{Y}_c^{[k]})(x_i) \leq \boldsymbol{E}_c^{[k+1]}(x_i) \text{ for } x_i \in \Omega_c^N. \tag{31}$$

The exact solution to the difference problem (30) is

$$\boldsymbol{E}_c^{[k+1]}(x_i) = \mathbf{C}\nu^k \frac{m_1^i - m_2^i}{m_1^N - m_2^N}, \tag{32}$$

where

$$\begin{aligned} m_1 &= \left(1+\frac{\alpha h}{2\varepsilon}+\frac{\beta h^2}{4\varepsilon}\right)+\sqrt{\left(1+\frac{\alpha h}{2\varepsilon}+\frac{\beta h^2}{4\varepsilon}\right)^2+\left(-1-\frac{\alpha h}{\varepsilon}+\frac{\beta h^2}{2\varepsilon}\right)}, \\ &\geq 1+\frac{\alpha h}{2\varepsilon}=\left(1+\frac{\alpha(1-\tau)}{2\varepsilon N}\right)\geq\left(1+\frac{\alpha\tau}{2\varepsilon N}\right). \end{aligned}$$

$$m_2=\left(1+\frac{\alpha h}{2\varepsilon}+\frac{\beta h^2}{4\varepsilon}\right)-\sqrt{\left(1+\frac{\alpha h}{2\varepsilon}+\frac{\beta h^2}{4\varepsilon}\right)^2+\left(-1-\frac{\alpha h}{\varepsilon}+\frac{\beta h^2}{2\varepsilon}\right)}.$$

Now

$$\mathbf{L}^N(\mathbf{Y}_r^{[k+2]}-\mathbf{Y}_r^{[k+1]})(x_i)=\mathbf{0},\quad \forall\ x_i\in\bar{\Omega}_r^N,$$
$$(\mathbf{Y}_r^{[k+2]}-\mathbf{Y}_r^{[k+1]})(1)=\mathbf{0}.$$

Using our inductive hypotheses and (31)

$$\begin{aligned} ||(\ \mathbf{Y}_r^{[k+2]}-\mathbf{Y}_r^{[k+1]})(1-2\tau)|| &= ||(\bar{\mathbf{Y}}_c^{[k+1]}-\bar{\mathbf{Y}}_c^{[k]})(1-2\tau)|| \\ &= ||(\mathbf{Y}_c^{[k+1]}-\mathbf{Y}_c^{[k]})(1-2\tau)|| \\ &\leq \boldsymbol{E}_c^{[k+1]}(1-2\tau), \end{aligned}$$

where we have used the fact that $(1-2\tau)$ is the mesh point of $\bar{\Omega}_c^N$.
Using Lemma 4.2 we obtain

$$||\mathbf{Y}_r^{[k+2]}-\mathbf{Y}_r^{[k+1]}||_{\bar{\Omega}_r^N}\quad\leq\quad \boldsymbol{E}_c^{[k+1]}(1-2\tau).$$

Here we used

$$\begin{aligned} \boldsymbol{E}_c^{[k+1]}(1-2\tau) &= \mathbf{C}\nu^k\frac{m_1^{N/2}-m_2^{N/2}}{m_1^N-m_2^N} \\ &\leq \frac{\mathbf{C}\nu^k}{m_1^{N/2}} \\ &= \frac{\mathbf{C}\nu^k}{\left(1+\frac{\tau\alpha}{2\varepsilon N}\right)^{N/2}} \\ &= \mathbf{C}\nu^k\left(1+\frac{\tau\alpha}{2\varepsilon N}\right)^{-N/2} \\ &= \mathbf{C}\nu^{k+1}. \end{aligned}$$

$$\therefore\ ||\mathbf{Y}_r^{[k+2]}-\mathbf{Y}_r^{[k+1]}||_{\bar{\Omega}_r^N}\leq C\nu^{k+1}. \tag{33}$$

$$\text{Consequently, } ||\mathbf{Y}_r^{[k+2]}-\mathbf{Y}_r^{[k+1]}||_{\bar{\Omega}_r^N\setminus\bar{\Omega}_c}=C\nu^{k+1}. \tag{34}$$

Finally note that

$$\mathbf{L}^N(\mathbf{Y}_c^{[k+2]}-\mathbf{Y}_c^{[k+1]})(x_i)=\mathbf{0},\quad \text{for } x_i\in\Omega_c^N,\quad (\mathbf{Y}_c^{[k+2]}-\mathbf{Y}_c^{[k+1]})(0)=\mathbf{0},$$

Using our inductive hypotheses and (33), we have

$$||(\mathbf{Y}_c^{[k+2]}-\mathbf{Y}_c^{[k+1]})(1-\tau)|| \quad=\quad ||(\bar{\mathbf{Y}}_r^{[k+2]}-\bar{\mathbf{Y}}_r^{[k+1]})(1-\tau)||$$

$$
\begin{aligned}
&= ||(\mathbf{Y}_r^{[k+2]} - \mathbf{Y}_r^{[k+1]})(1-\tau)|| \\
&\leq C\nu^{k+1},
\end{aligned}
$$

where we have used the fact that $(1-\tau)$ is the mesh point of $\bar{\Omega}_r^N$. Therefore, we can apply Lemma 4.2 to get

$$
||(\mathbf{Y}_c^{[k+2]} - \mathbf{Y}_c^{[k+1]})||_{\bar{\Omega}_c^N} \leq C\nu^{k+1}. \tag{35}
$$

Combining the estimates (34) and (35) we obtain,

$$
||\mathbf{Y}^{[k+2]} - \mathbf{Y}^{[k+1]}||_{\bar{\Omega}^N} \leq C\nu^{k+1}.
$$

For $\tau = \dfrac{4\varepsilon}{\alpha}\ln N$ using the arguments given in Lemma 5.1 of [7] we obtain,

$$
\begin{aligned}
\nu &= \left(1 + \frac{\tau\alpha}{2\varepsilon N}\right)^{-N/2}, \\
&= \left(1 + \frac{2\ln N}{N}\right)^{-N/2} \leq 2N^{-1},\ N \geq 1.
\end{aligned}
$$

□

The following theorem contains the main result of this paper, combining Lemmas 5.1 and 5.2, we prove that, two iterations are sufficient to attain second order convergence.

Theorem 5.3. *Let* $\boldsymbol{y}$ *be the solution to (7)-(8) and* $\mathbf{Y}^{[k]}$ *be the* k^{th} *iterate of the discrete Schwarz method described in Section 4. If* $\tau = \dfrac{4\varepsilon}{\alpha}\ln N$ *and* $N > 2$, *then*

$$
||\,\mathbf{Y}^{[k]} - \boldsymbol{y}||_{\bar{\Omega}^N} \leq CN^{-k} + CN^{-2}\ln^3 N,
$$

where C *is independent of* k *and* N.

Proof. From Lemma 5.2 there exists $\mathbf{Y}$ such that

$$
\mathbf{Y} := \operatorname*{lt}_{k\to\infty} \mathbf{Y}^{[k]}.
$$

We know from Lemma 5.1 that there exists C such that

$$
||\mathbf{Y}^{[k]} - \boldsymbol{y}||_{\bar{\Omega}^N} \leq C2^{-k} + CN^{-2}\ln^3 N.
$$

This implies that

$$
||\mathbf{Y} - \boldsymbol{y}||_{\bar{\Omega}^N} \leq CN^{-2}\ln^3 N. \tag{36}
$$

We also know from Lemma 5.2 that there exists C such that

$$
||\mathbf{Y}^{[k+1]} - \mathbf{Y}^{[k]}||_{\bar{\Omega}^N} \leq CN^{-k}.
$$

Consequently, for $N \geq 2$, there exists C such that

$$
||\mathbf{Y}^{[k]} - \mathbf{Y}||_{\bar{\Omega}^N} \leq C\sum_{l=k}^{\infty} N^{-l},
$$

$$\begin{aligned} &= C\left[\frac{N^{-k}}{1-N^{-1}}\right], \\ &\leq CN^{-k}. \end{aligned} \tag{37}$$

Thus, using (36) and (37), we conclude that

$$\begin{aligned} ||\mathbf{Y}^{[k]} - \boldsymbol{y}||_{\bar{\Omega}^N} &= ||\mathbf{Y}^{[k]} - \mathbf{Y} + \mathbf{Y} - \boldsymbol{y}||_{\bar{\Omega}^N}, \\ &\leq ||\mathbf{Y}^{[k]} - \mathbf{Y}||_{\bar{\Omega}^N} + ||\mathbf{Y} - \boldsymbol{y}||_{\bar{\Omega}^N}, \\ &\leq CN^{-k} + CN^{-2}\ln^3 N. \end{aligned}$$

□

6. Numerical Illustrations

In this section, we consider one example to illustrate the theoretical results for the BVP (1)-(2). The stopping criterion for the iterative procedure is taken to be $||\mathbf{Y}^{[k+1]} - \mathbf{Y}^{[k]}||_{\bar{\Omega}^N} \leq 10^{-14}$. Let $Y_j^N,\ j = 1,2$ be a Schwarz numerical approximation for the exact solution y_j on the mesh Ω^N and N is the number of mesh points. We normally omit the superscript k on the final Schwarz iterate and write simply Y_j^N. For a finite set of values of $\varepsilon = \{2^{-1} \cdots 2^{-39}\}$, we compute the maximum point-wise two mesh difference errors for $j = 1, 2$,

$$||Y_j^N - y_j||_{\Omega^N} \approx D_{\varepsilon,j}^N = ||Y_j^N - \bar{Y}_j^{2N}||_{\Omega^N}, \quad D_j^N = \max_{\varepsilon} D_{\varepsilon,j}^N,$$

where $\bar{Y}_j^{2N}$ is the numerical solution obtained on a mesh with the same transition points, but $2N$ intervals in each subdomain. From these quantities the order of convergence [2] are computed from $p_j^N = log_2\left\{\frac{D_j^N}{D_j^{2N}}\right\}$. The computed maximum point-wise errors D_j^N $(j = 1, 2)$ and the computed order of convergence are tabulated (Table 1). The nodal errors are plotted as graphs (Figure 1). Table 1 lists $D_1^N,\ p_1^N,\ D_2^N,\ p_2^N$ and k (the number of iterations computed) for various values of N and ε. We can see that the errors are independent of the singular perturbation parameter ε and decrease as N increases. The computed rates of convergence are almost second order, with the usual $\ln N$ factor associated with these techniques.

Example 6.1. Consider the BVP

$$\begin{aligned} -\varepsilon y^{'''}(x) + (2x+1)y''(x) + (4x+9)y'(x) - 6y(x) = sinhx,\ x \in \Omega \\ y(0) = 0, \quad y'(0) = 0, \quad y'(1) = 0. \end{aligned}$$

The numerical results are presented in Table 1.

TABLE 1. *Values of D_1^N, p_1^N and D_2^N, p_2^N for the solution components Y_1 and Y_2 respectively for the Example 6.1*

	Number of mesh points N				
	64	128	256	512	1024
2^{-1}	1.1852e-006	3.0394e-007	7.6946e-008	1.9357e-008	4.8544e-009
2^{-3}	2.0192e-006	5.1153e-007	1.2872e-007	3.2284e-008	8.0840e-009
2^{-5}	2.7844e-006	6.2110e-007	1.4262e-007	3.5708e-008	8.9339e-009
2^{-7}	4.5897e-006	1.1288e-006	2.7680e-007	6.7717e-008	1.6544e-008
2^{-9}	5.1993e-006	1.2969e-006	3.2364e-007	8.0647e-008	2.0074e-008
2^{-11}	5.4359e-006	1.3515e-006	3.3799e-007	8.4546e-008	2.1133e-008
2^{-13}	5.5599e-006	1.3752e-006	3.4297e-007	8.5788e-008	2.1461e-008
2^{-15}	5.6163e-006	1.3891e-006	3.4545e-007	8.6268e-008	2.1575e-008
2^{-17}	5.6344e-006	1.3958e-006	3.4708e-007	8.6542e-008	2.1624e-008
2^{-19}	5.6392e-006	1.3980e-006	3.4788e-007	8.6737e-008	2.1656e-008
2^{-21}	5.6404e-006	1.3985e-006	3.4815e-007	8.6835e-008	2.1680e-008
2^{-23}	5.6407e-006	1.3987e-006	3.4822e-007	8.6868e-008	2.1692e-008
2^{-25}	5.6408e-006	1.3987e-006	3.4823e-007	8.6876e-008	2.1696e-008
2^{-27}	5.6408e-006	1.3987e-006	3.4824e-007	8.6878e-008	2.1697e-008
2^{-29}	5.6408e-006	1.3987e-006	3.4824e-007	8.6879e-008	2.1697e-008
2^{-31}	5.6408e-006	1.3987e-006	3.4824e-007	8.6879e-008	2.1697e-008
2^{-33}	5.6408e-006	1.3987e-006	3.4824e-007	8.6879e-008	2.1697e-008
2^{-35}	5.6408e-006	1.3987e-006	3.4824e-007	8.6879e-008	2.1697e-008
2^{-37}	5.6408e-006	1.3987e-006	3.4824e-007	8.6879e-008	2.1697e-008
2^{-39}	5.6408e-006	1.3987e-006	3.4824e-007	8.6879e-008	2.1697e-008
D_1^N	5.6408e-006	1.3987e-006	3.4824e-007	8.6879e-008	2.1697e-008
p_1^N	**2.0118e+000**	**2.0059e+000**	**2.0030e+000**	**2.0015e+000**	-
k	2	2	2	2	2
2^{-1}	4.6881e-006	1.1814e-006	2.9650e-007	7.4268e-008	1.8585e-008
2^{-3}	4.1003e-006	1.0242e-006	2.5580e-007	6.3909e-008	1.5971e-008
2^{-5}	2.4274e-006	5.6902e-007	1.3339e-007	3.3585e-008	8.4238e-009
2^{-7}	3.5317e-006	1.0404e-006	2.7093e-007	6.7786e-008	1.6714e-008
2^{-9}	2.0472e-006	1.0271e-006	3.0922e-007	8.2121e-008	2.0941e-008
2^{-11}	1.9974e-006	5.4564e-007	2.7292e-007	8.2420e-008	2.1973e-008
2^{-13}	7.2388e-006	5.0118e-007	1.4096e-007	7.0053e-008	2.1146e-008
2^{-15}	1.0961e-005	1.8221e-006	1.2506e-007	3.5829e-008	1.7724e-008
2^{-17}	1.2597e-005	2.7549e-006	4.5689e-007	3.1202e-008	9.0319e-009
2^{-19}	1.3113e-005	3.1640e-006	6.9050e-007	1.1438e-007	7.7900e-009
2^{-21}	1.3252e-005	3.2931e-006	7.9283e-007	1.7284e-007	2.8613e-008
2^{-23}	1.3287e-005	3.3278e-006	8.2512e-007	1.9844e-007	4.3238e-008
2^{-25}	1.3296e-005	3.3366e-006	8.3379e-007	2.0651e-007	4.9638e-008
2^{-27}	1.3299e-005	3.3388e-006	8.3600e-007	2.0868e-007	5.1657e-008
2^{-29}	1.3299e-005	3.3394e-006	8.3656e-007	2.0923e-007	5.2199e-008
2^{-31}	1.3299e-005	3.3395e-006	8.3670e-007	2.0937e-007	5.2337e-008
2^{-33}	1.3299e-005	3.3396e-006	8.3673e-007	2.0941e-007	5.2372e-008
2^{-35}	1.3299e-005	3.3396e-006	8.3674e-007	2.0942e-007	5.2381e-008
2^{-37}	1.3299e-005	3.3396e-006	8.3674e-007	2.0942e-007	5.2383e-008
2^{-39}	1.3299e-005	3.3396e-006	8.3674e-007	2.0942e-007	5.2383e-008
D_2^N	1.3299e-005	3.3396e-006	8.3674e-007	2.0942e-007	5.2383e-008
p_2^N	**1.9936e+000**	**1.9968e+000**	**1.9984e+000**	**1.9992e+000**	-
k	2	2	2	2	2

7. Conclusion

A singularly perturbed third order ODEs of convection-diffusion problem is considered. It is shown that a designed discrete Schwarz method produces numerical approximations which converge in the maximum norm to the exact solution . This convergence is shown to be of almost second order. Note that from

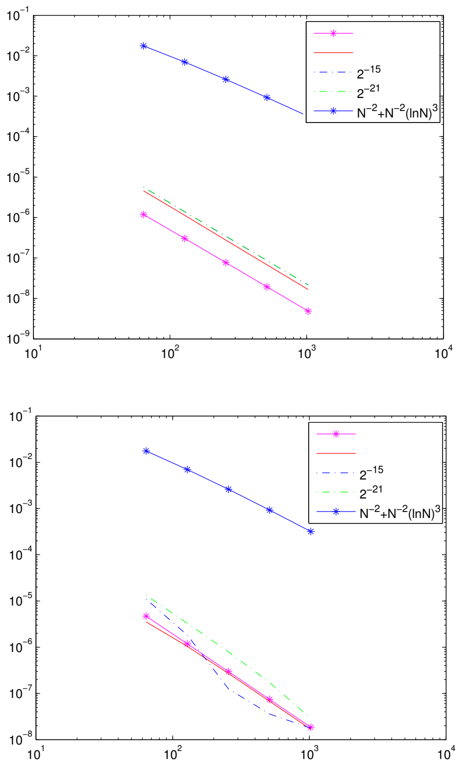

FIGURE 1. Nodal error for the component Y_1 and Y_2 of the Example 6.1

Theorem 5.3, for $k \geq 2$ the $N^{-2}\ln^3 N$ term dominates the error bound. Thus, two iterations are sufficient to attained the desired accuracy.

Numerical experiment validate the theoretical result. The graph plotted in the figure is convergent curves in the maximum norm at nodal points for the different values of ε for the example considered. This graph clearly indicate that the optimal error bound is of order $O(N^{-k}+N^{-2}\ln^3 N)$ as predicted. The main advantages of this method used with the proposed scheme are it reduced iteration counts very much, easily identified in which iteration the Schwarz iterate terminates and it produce almost second order convergence.

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