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GENERALIZATION OF MEROMORPHIC FUNCTIONS SHARING A NONZERO POLYNOMIAL WITH FINITE WEIGHT

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ABSTRACT. *The purpose of the paper is to study the meromorphic functions sharing a nonzero polynomial with finite weight. The results of the paper improve and generalize the recent results due to Pulak Sahoo and Sajahan Seikh* [9].

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1. Introduction and main results

In this paper, a meromorphic function will mean meromorphic in the whole complex plane. Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, $f - a$ and $g - a$ have the same set of zeros with the same multiplicities, we say that f and g share the value a CM(counting multiplicities), and if we do not consider the multiplicities then f and g are said to share the value a IM(ignoring multiplicities).

We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [4, 13]. A meromorphic function a is said to be a small function of f provided that $T(r, a) = S(r, f)$, that is $T(r, a) = o(T(r, f))$ as $r \to \infty$, outside of a possible exceptional set of finite linear measure.

In 1959, W.K.Hayman (see [4], Corollary of Theorem 9) proved the following theorem.
Theorem A. Let f be a transcendental meromorphic function and $n(\geq 3)$ is an integer. Then $f^n f' = 1$ has infinitely many solutions.

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Corresponding to Theorem A, C.C.Yang and H.X.Hua [13] proved the following result.

Theorem B. Let f and g be two non-constant meromorphic functions, $n \geq 11$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$ or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.

In 2002, Fang and Qiu [2] proved the following theorem.

Theorem C. Let f and g be two non-constant meromorphic functions, and $n \in N$ such that $n \geq 11$. If $f^n f' - z$ and $g^n g' - z$ share 0 CM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three nonzero complex numbers satisfying $4(c_1 c_2)^{n+1} c^2 = -1$ or $f = tg$ for a complex number t such that $t^{n+1} = 1$.

In 2010, X.M.Li and Gao [8] proved the following result.

Theorem D. Let f and g be two transcendental meromorphic functions, let $n \geq 11$ be a positive integer, and let $P \not\equiv 0$ be a polynomial with its degree $\gamma_p \leq 11$. If $f^n f' - P$ and $g^n g' - P$ share 0 CM, then either $f = tg$ for a complex number t satisfying $t^{n+1} = 1$, or $f(z) = c_1 e^{CQ}$, $g(z) = c_2 e^{-CQ}$, where c_1, c_2 and c are three nonzero complex numbers satisfying $(c_1 c_2)^{n+1} c^2 = -1$, Q is a polynomial satisfying $Q = \int_0^{z_0} P(\eta)\, d\eta$.

We now explain the notation of weighted sharing of values, introduced by I.Lahiri [5, 6].

Definition 1. [5, 6] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a, g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Recently, Pulak Sahoo and S. Seikh[9] proved the following theorems.

Theorem E. Let f and g be two transcendental meromorphic functions, let n, k be two positive integers such that $n \geq 3k + 9$, and let $P \not\equiv 0$ be a polynomial with its degree $\gamma_p \leq n - 1$. Let $(f^n)^{(k)} - P$ and $(g^n)^{(k)} - P$ share (0, 2). Then (i) if $k = 1$, either $f \equiv tg$ for a complex number t satisfying $t^n = 1$ or $f = c_1 e^{CQ}$ and $g = c_2 e^{-CQ}$ where c_1, c_2 and c are three non-zero complex number satisfying

$(c_1c_2)^n c^2 = -1$, Q is a polynomial satisfying $Q = \int_0^z P(\eta)\,d\eta$.
(ii) if $k \geq 2$, either $(f^n)^{(k)}(g^n)^{(k)} = P^2$ or $f \equiv tg$ for a complex number t satisfying $t^n = 1$.

Theorem F. Let f and g be two transcendental meromorphic functions, let n, m, k be three positive integers, and let $P \not\equiv 0$ be a polynomial. If $(f^n(f-1)^m)^{(k)} - P$ and $(g^n(g-1)^m)^{(k)} - P$ share (0, 2) then each of the following hold:
(i) When $m = 1, n \geq 3k+12$ and $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n}$, then either $(f^n(f-1)^m)^{(k)}(g^n(g-1)^m)^{(k)} = P^2$ or $f = g$.
(ii) When $m \geq 2$ and $n \geq 3k+m+11$, then either $(f^n(f-1)^m)^{(k)}(g^n(g-1)^m)^{(k)} = P^2$ or $f = g$ or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n(\omega_1 - 1)^m - \omega_2^n(\omega_2 - 1)^m$.
The possibility $(f^n(f-1)^m)^{(k)}(g^n(g-1)^m)^{(k)} = P^2$ does not arise for $k = 1$.

In this paper we will prove one theorem which will improve and generalizes Theorems E and F.

Theorem 1. Let f and g be two transcendental meromorphic functions, let $p(z)$ be a non-zero polynomial with $deg(p) \leq n-1$, $n(\geq 1), k(\geq 1)$ and $m(\geq 0)$ be three integers such that $n > 3k+m+8$. Let $(f^nP(f))^{(k)} - p$ and $(g^nP(g))^{(k)} - p$ share (0, 2) and f and g share ∞ IM then one of the following three cases hold:
(i) $f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = GCD(n+m, ..., n+m-i, ..., n), a_{m-i} \neq 0$ for some $i = 1, 2, ..., m$.
(ii) f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + ... + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + ... + a_0)$.
(iii) $P(z)$ reduces to a nonzero monomial namely, namely $P(z) = a_iz^i \not\equiv 0$ for some $i \in \{0, 1, 2, ..., m\}$; if $p(z)$ is not a constant, then $f = c_1e^{cQ(z)}$, $g = c_2e^{-cQ(z)}$, where $Q(z) = \int_0^z p(z)\,dz$, $c_1, c_2, c \in \mathbb{C}$ such that $a_i^2(c_1c_2)^{n+i}[(n+i)c]^2 = -1$, if $p(z) = b(\neq 0)$, then $f = c_3e^{cz}$, $g = c_4e^{-cz}$, where $c_3, c_4, c \in \mathbb{C}$ such that $(-1)^k a_i^2(c_3c_4)^{n+i}[(n+i)c]^{2k} = b^2$.

2. Lemmas

Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We denote by H the function as follows:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{1}$$

Lemma 1.[4] Suppose that f is a non-constant meromorphic function, $k \geq 2$ is an integer. If

$$N(r, \infty; f) + N(r, 0; f) + N(r, 0; f^{(k)}) = S(r, f'/f),$$

then $f = e^{az+b}$, where $a \neq 0, b$ are constants.

Lemma 2.[3] Let $f(z)$ be a non-constant entire function and let $k \geq 2$ be a positive integer. If $f(z)f^{(k)}(z) \neq 0$, then $f(z) = e^{az+b}$, where $a \neq 0, b$ are constants.

Lemma 3.[3] Let f be a non-constant meromorphic function and let k be a positive integer. Suppose that $f^{(k)} \not\equiv 0$, then

$$N(r,0;f^{(k)}) \leq N(r,0;f) + k\overline{N}(r,\infty;f) + S(r,f).$$

Lemma 4.[14] Let $f_j (j = 1,2,3)$ be a meromorphic and f_1 be non-constant. Suppose that

$$\sum_{j=1}^{3} f_j \equiv 1$$

and

$$\sum_{j=1}^{3} N(r,0;f_j) + 2\sum_{j=1}^{3} \overline{N}(r,\infty;f_j) < (\lambda + o(1))T(r),$$

as $r \to +\infty$, $r \in I$, $\lambda < 1$ and $T(r) = max_{1\leq j\leq 3}T(r,f_j)$. Then $f_2 \equiv 1$ or $f_3 \equiv 1$.

Lemma 5.[12] Let f be a nonconstant meromorphic function and let $a_n(z)(\not\equiv 0), a_{n-1}(z), ...a_0(z)$ be meromorphic functions such that $T(r,a_i(z)) = S(r,f)$ for $i = 0,1,2,...n$. Then
$T(r, a_nf^n + a_{n-1}f^{n-1} + ... + a_1f + a_0) = nT(r,f) + S(r,f)$.

Lemma 6.[16] Let f be a non-constant meromorphic function, and $p,k \in N$. Then

$$N_p(r,0;f^{(k)}) \leq T(r,f^k) - T(r,f) + N_{p+k}(r,0;f) + S(r,f), \tag{2}$$

$$N_p(r,0;f^{(k)}) \leq k\overline{N}(r,\infty;f) + N_{p+k}(r,0;f) + S(r,f). \tag{3}$$

Lemma 7.[7] If $N(r,0;f^{(k)} \mid f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity, then

$$N(r,0;f^{(k)} \mid f \neq 0) \leq k\overline{N}(r,\infty;f) + N(r,0;f \mid< k) + k\overline{N}(r,0;f \mid\geq k) + S(r,f).$$

Lemma 8. ([15], Lemma 6) If $H \equiv 0$, then F,G share 1 CM. If further F,G share ∞ IM then F,G share ∞ CM.

Lemma 9.[17] Let f, g be non-constant meromorphic functions, let n, k be two positive integers with $n > k+2$, and let $P(w) = a_mw^m+a_{m-1}w^{m-1}+...+a_1w+a_0$ be a non zero polynomial. Let $\alpha(\not\equiv 0,\infty)$ be a small function with respect to f with finitely many zeros and poles. If $[f^nP(f)]^{(k)}[g^nP(g)]^{(k)} \equiv \alpha^2$, f and g share $\infty\, IM$, then $P(w)$ is reduced to a nonzero monomial, namely $P(w) = a_iw^i \not\equiv 0$ for some $i \in \{0,1,...,m\}$.

Lemma 10. Let f, g be two transcendental meromorphic functions and let $p(z)$ be a non-zero polynomial with $deg(p) \leq n-1$, where n and k be two positive integers such that $n > k$. Let $[f^n]^{(k)} - p$, $[g^n]^{(k)} - p$ share 0 CM and f,g share ∞ IM. Now when $[f^n]^{(k)}[g^n]^{(k)} \equiv p^2$,
(i) if $p(z)$ is not a constant, then $f(z) = c_1e^{cQ(z)}$, $g(z) = c_2e^{-cQ(z)}$ where

$Q(z) = \int_0^z p(t)\,\mathrm{d}t$, $c_1, c_2, c \in \mathbb{C}$ such that $(nc)^2(c_1c_2)^n = -1$,
(ii) if $p(z)$ is a non-zero constant b, then $f(z) = c_3e^{dz}$, $g(z) = c_4e^{-dz}$, where c_3, c_4 and d are constants such that $(-1)^k(c_3c_4)^n(nd)^{2k} = b^2$.

Proof: Suppose

$$(f^n)^k(g^n)^k \equiv p^2. \tag{4}$$

Since f and g share ∞ IM, (4) one can easily say that f and g are transcendental entire functions. We consider the following cases.

Case 1. Let $deg(p(z)) = l(\geq 1)$. Since $n > k$, it follows that $N(r, 0; f) = O(logr)$ and $N(r, 0; g) = O(logr)$. Let

$$F_1 = \frac{(f^n)^{(k)}}{p} \quad and \quad G_1 = \frac{(g^n)^{(k)}}{p}. \tag{5}$$

From (4) we get

$$F_1G_1 \equiv 1. \tag{6}$$

If $F_1 \equiv cG_1$, where c is a non-zero constant, then by (6), F_1 is a constant and so f is a polynomial, which contradicts our assumption. Hence $F_1 \not\equiv G_1$.
Let

$$\phi = \frac{[f^n]^{(k)} - p}{[g^n]^{(k)} - p}. \tag{7}$$

We deduce from (7) that

$$\phi \equiv e^{\beta}, \tag{8}$$

where β is an entire function.
Let $f_1 = F_1$, $f_2 = -e^{\beta}G_1$ and $f_3 = e^{\beta}$. Here f_1 is transcendental. Now from (8), we have $f_1 + f_2 + f_3 \equiv 1$.
Hence by Lemma 3, we get

$$\begin{aligned}\sum_{j=1}^{3} N(r, 0; f_j) + 2\sum_{j=1}^{3} \overline{N}(r, \infty; f_j) &\leq N(r, 0; F_1) + N(r, 0; e^{\beta}G_1) + O(logr)\\ &\leq (\lambda + o(1))T(r),\end{aligned}$$

as $r \to +\infty$, $r \in I, \lambda < 1$ and $T(r) = max_{1\leq j\leq 3}T(r, f_j)$. So by Lemma 4, we get either $e^{\beta}G_1 \equiv -1$ or $e^{\beta} \equiv 1$. But here the only possibility is that $e^{\beta}G_1 \equiv -1$, i.e, $[g^n]^{(k)} \equiv -e^{-\beta}p(z)$ and so from (4), we obtain

$$F_1 \equiv e^{\gamma_1}G_1,$$

$$i.e., [f^n]^{(k)} \equiv e^{\gamma_1}[g^n]^{(k)},$$

where γ_1 is a non-constant entire function. Now from (4) we get

$$(f^n)^{(k)} \equiv ce^{\frac{1}{2}\gamma_1}p(z) \;\; , \;\; (g^n)^{(k)} \equiv ce^{\frac{-1}{2}\gamma_1}p(z), \tag{9}$$

where $c = \pm 1$. Since $N(r, 0; f) = O(logr)$ and $N(r, 0; g) = O(logr)$, so we can take

$$f(z) = h_1(z)e^{\alpha(z)} \;\; , \;\; g(z) = h_2(z)e^{\beta(z)} \tag{10}$$

where h_1 and h_2 are non-zero polynomials and α, β are two non-constant entire functions.
We deduce from (4) and (10) that either both α and β are transcendental entire functions or both are polynomials.
We consider the following cases:
Subcase 1.1: Let $k \geq 2$.

First we suppose both α and β are transcendental entire functions.
Let $\alpha_1 = \alpha' + \frac{h_1'}{h_1}$ and $\beta_1 = \beta' + \frac{h_2'}{h_2}$. Clearly both α_1 and β_1 are transcendental functions.
Note that
$S(r, n\alpha_1) = S(r, \frac{(f^n)'}{f^n})$, $S(r, n\beta_1) = S(r, \frac{(g^n)'}{g^n})$. Moreover we see that
$N(r, 0; (f^n)^{(k)}) \leq N(r, 0; p^2) = O(logr)$ and $N(r, 0; (g^n)^{(k)}) \leq N(r, 0; p^2) = O(logr)$.
From these and using (10) we have

$$N(r, \infty; f^n) + N(r, 0; f^n) + N(r, 0; (f^n)^{(k)}) = S(r, n\alpha_1) = S(r, \frac{(f^n)'}{f^n}) \quad (11)$$

and

$$N(r, \infty; g^n) + N(r, 0; g^n) + N(r, 0; (g^n)^{(k)}) = S(r, n\beta_1) = S(r, \frac{(g^n)'}{g^n}). \quad (12)$$

Then from (11), (12) and Lemma 1 we must have

$$f(z) = e^{az+b} \ , \ g(z) = e^{cz+d}, \quad (13)$$

where $a \neq 0, b, c \neq 0$ and d are constants. But these types of f and g do not satisfy relation (4).

Next we suppose α and β are both polynomials. Also from (4) we get $\alpha + \beta \equiv c$ i.e., $\alpha' \equiv -\beta'$. Therefore $deg(\alpha) = deg(\beta)$. We deduce from (10) that

$$(f^n)^k \equiv Ah_1^{n-k}[h_1^k(\alpha')^k + P_{k-1}(\alpha', h_1')]e^{n\alpha} \equiv A_1 p e^{n\alpha} \quad (14)$$

and

$$(g^n)^k \equiv Bh_2^{n-k}[h_2^k(\beta')^k + Q_{k-1}(\beta', h_2')]e^{n\beta} \equiv B_1 p e^{n\beta} \quad (15)$$

where A, B, A_1, B_1 are non-zero constants, $P_{k-1}(\alpha', h_1')$ and $Q_{k-1}(\beta', h_2')$ are differential polynomials in α', h_1' and β', h_2' respectively. By virtue of polynomial p, from (14) and (15) we conclude that both h_1 and h_2 are nonzero constants. So we can rewrite f and g as follows:

$$f = e^{\gamma_2}, g = e^{\delta_2} \quad (16)$$

where $\gamma_2 + \delta_2 \equiv C$ and $deg(\gamma_2) = deg(\delta_2)$. If $deg(\gamma_2) = deg(\delta_2) = 1$, then we again get a contradiction from (4). Next we suppose $deg(\gamma_2) = deg(\delta_2) \geq 2$.

We deduce from (16) that

$$\begin{aligned}
(f^n)' &= n\gamma_2 e^{n\gamma_2}\\
(f^n)'' &= [n^2(\gamma_2')^2 + n\gamma_2'']e^{n\gamma_2}\\
(f^n)''' &= [n^3(\gamma_2')^3 + 3n^2\gamma_2'\gamma_2'' + n\gamma_2''']e^{n\gamma_2}\\
(f^n)^{(iv)} &= [n^4(\gamma_2')^4 + 6n^2(\gamma_2')^2\gamma_2'' + 3n^2(\gamma_2'')^2 + 4n^2\gamma_2'\gamma_2'' + n\gamma_2^{(iv)}]e^{n\gamma_2}\\
(f^n)^{(v)} &= [n^5(\gamma_2')^5 + 10n^4(\gamma_2')^3\gamma_2'' + 15n^3\gamma_2'(\gamma_2'')^2 + 10n^3(\gamma_2')^2\gamma_2'''\\
&\qquad + 10n^2\gamma_2''\gamma_2''' + 5n^2\gamma_2'\gamma_2^{(iv)} + n\gamma_2^{(v)}]e^{n\gamma_2}\\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
(f^n)^{(k)} &= [n^k(\gamma_2')^k + K(\gamma_2')^{k-2}\gamma_2'' + P_{k-2}(\gamma_2')]e^{n\gamma_2}.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
(g^n)^{(k)} &= [n^k(\delta_2')^k + K(\delta_2')^{k-2}\delta_2'' + P_{k-2}(\delta_2')]e^{n\delta_2}\\
&= [(-1)^{(k)}n^k(\gamma_2')^k - K(-1)^{k-2}(\gamma_2')^{k-2}\gamma_2'' + P_{k-2}(-\gamma_2')]e^{n\delta_2},
\end{aligned}$$

where K is a suitably positive integer and $P_{k-2}(\gamma_2')$ is a differential polynomial in γ_2'.
Since $deg(\gamma_2) \geq 2$, we observe that $deg((\gamma_2')^{(k)}) \geq kdeg(\gamma_2')$ and so $(\gamma_2')^{k-2}\gamma_2''$ is either a non-zero constant or $deg((\gamma_2')^{k-2}\gamma_2'') \geq (k-1)deg(\gamma_2') - 1$.
Also we see that

$$deg((\gamma_2')^k) > deg((\gamma_2')^{k-2}\gamma_2'') > deg(P_{k-2}(\gamma_2'))\ \ (or\ \ deg(P_{k-2}(-\gamma_2'))).$$

Since $[f^n]^{(k)}$ and $[g^n]^{(k)}$share 0 CM, the polynomials $n^k(\gamma_2')^k + K(\gamma_2')^{k-2}\gamma_2'' + P_{k-2}(\gamma_2')$ and $(-1)^k n^k(\gamma_2')^k - K(-1)^{k-2}(\gamma_2')^{k-2}\gamma_2'' + P_{k-2}(-\gamma_2')$ must be identical but this is impossible for $k \geq 2$. Actually the terms $n^k(\gamma_2')^k + K(\gamma_2')^{k-2}\gamma_2''$ and $(-1)^k n^k(\gamma_2')^k - K(-1)^{k-2}(\gamma_2')^{k-2}\gamma_2''$ cannot be identical for $k \geq 2$.

Subcase 1.2. Let $k = 1$. Now from (4) we get

$$f^{n-1}f'g^{n-1}g' \equiv p_1^2, \tag{17}$$

where $p_1^2 = \frac{1}{n^2}p^2$.
First we suppose that both α and β are transcendental entire functions.
Let $h = fg$. Clearly h is a transcendental entire function. Then from (17) we get

$$\left(\frac{g'}{g} - \frac{1}{2}\frac{h'}{h}\right)^2 \equiv \frac{1}{4}\left(\frac{h'}{h}\right)^2 - h^{-n}p_1^2. \tag{18}$$

Let

$$\alpha_2 = \frac{g'}{g} - \frac{1}{2}\frac{h'}{h}.$$

From (18) we get

$$\alpha_2^2 = \frac{1}{4}\left(\frac{h'}{h}\right)^2 - h^{-n}p_1^2. \tag{19}$$

First we suppose $\alpha_2 \equiv 0$. Then we get $h_1^{-n}p_1^2 \equiv \frac{1}{4}(\frac{h'}{h})^2$ and so $T(r,h) = S(r,h)$, which is impossible. Next we suppose that $\alpha_2 \not\equiv 0$. Differentiating (19) we get

$$2\alpha_2\alpha_2' \equiv \frac{1}{2}\frac{h'}{h}\left(\frac{h'}{h}\right)' + nh'h^{-n-1}p_1^2 - 2h^{-n}p_1p_1'.$$

Applying (19) we obtain

$$h^{-n}\left(-n\frac{h'}{h}p_1^2 + 2p_1p_1' - 2\frac{\alpha_2'}{\alpha_2}p_1^2\right) \equiv \frac{1}{2}\frac{h'}{h}\left(\left(\frac{h'}{h}\right)' - \frac{h'}{h}\frac{\alpha_2'}{\alpha_2}\right). \tag{20}$$

First we suppose that $-n\frac{h'}{h}p_1^2 + 2p_1p_1' - 2\frac{\alpha_2'}{\alpha_2}p_1^2 \equiv 0$.
Then there exist a non-zero constant c such that $\alpha_2^2 \equiv ch^{-n}p_1^2$ and so from (19) we get

$$(c+1)h^{-n}p_1^2 \equiv \frac{1}{4}\left(\frac{h'}{h}\right)^2.$$

If $c = -1$, then h will be a constant. If $c \neq -1$, then we have $T(r,h) = S(r,h)$, which is impossible. Next we suppose that $-n\frac{h'}{h}p_1^2 + 2p_1p_1' - 2\frac{\alpha_2'}{\alpha_2}p_1^2 \not\equiv 0$.
Then by (20) we have

$$\begin{aligned} nT(r,h) = n\,m(r,h) &\leq m\left(r, h^n\frac{1}{2}\frac{h'}{h}\left(\left(\frac{h'}{h}\right)' - \frac{h'}{h}\frac{\alpha_2'}{\alpha_2}\right)\right) \\ &\quad + m\left(r, \frac{1}{\frac{1}{2}\frac{h'}{h}((\frac{h'}{h})' - \frac{h'}{h}\frac{\alpha_2'}{\alpha_2}))}\right) + O(1) \\ &\leq T\left(r, \frac{1}{2}\frac{h'}{h}\left(\left(\frac{h'}{h}\right)' - \frac{h'}{h}\frac{\alpha_2'}{\alpha_2}\right)\right) \\ &\quad + m\left(r, n\frac{h'}{h}p_1^2 - 2p_1p_1' + 2\frac{\alpha_2'}{\alpha_2}p_1^2\right) \\ &\leq \overline{N}(r,0;\alpha_2) + S(r,h) + S(r,\alpha_2) \end{aligned} \tag{21}$$

From (19) we get $T(r,\alpha_2) < \frac{1}{2}nT(r,h) + S(r,h)$.
Now from (21) we get $\frac{1}{2}nT(r,h) \leq S(r,h)$, which is impossible.
Thus α and β are both polynomials. Also from (4) we can conclude that $\alpha(z) + \beta(z) \equiv C$ for a constant C and so $\alpha'(z) + \beta'(z) \equiv 0$. We deduce from (4) that

$$[f^n]' \equiv n[h_1^n\alpha' + h_1^{n-1}h_1']e^{n\alpha} \equiv p(z)e^{n\alpha} \tag{22}$$

and

$$[g^n]' = n[h_2^n\beta' + h_2^{n-1}h_2']e^{n\beta} \equiv p(z)e^{n\beta}. \tag{23}$$

Since $deg(p) \leq n-1$ from (22) and (23) we conclude that both h_1 and h_2 are nonzero constants. So we can rewrite f and g as follows:

$$f = e^{\gamma_2} \ , \ g = e^{\delta_2}. \tag{24}$$

Now from (4) we get

$$n^2 \gamma_2' \delta_2' e^{n(\gamma_2+\delta_2)} \equiv p^2. \tag{25}$$

Also from (25) we can conclude that $\gamma_2(z) + \delta_2(z) \equiv C$ for a constant C and so $\gamma_2'(z) + \delta_2'(z) \equiv 0$.
Thus from (25) we get $n^2 e^{nC} \gamma_2' \delta_2' \equiv p^2(z)$. By computation we get

$$\gamma_2' = cp(z), \delta_2' = -cp(z). \tag{26}$$

Hence

$$\gamma_2 = cQ(z) + b_1 \ , \ \delta_2 = -cQ(z) + b_2, \tag{27}$$

where $Q(z) = \int_0^z p(z)\,\mathrm{d}z$ and b_1, b_2 are constants. Finally we take f and g as $f(z) = c_1 e^{cQ(z)}$, $g(z) = c_2 e^{-cQ(z)}$, where c_1, c_2 and c are constants such that $(nc)^2 (c_1 c_2)^n = -1$.

Case 2. Let $p(z)$ be a nonzero constant b. In this case we see that f and g have no zeros and so we can take f and g as follows:

$$f = e^{\alpha}, \ g = e^{\beta}, \tag{28}$$

where $\alpha(z), \beta(z)$ are two non-constant entire functions.
We now consider the following two subcases:

Subcase 2.1. Let $k \geq 2$.
We see that $N(r, 0; [f^n]^k) = 0$. From this and using (28) we have

$$f^n(z)[f^n(z)]^{(k)} \neq 0. \tag{29}$$

Similarly we have

$$g^n(z)[g^n(z)]^{(k)} \neq 0. \tag{30}$$

Then from (29), (30) and Lemma 2 we must have

$$f = e^{az+b} \ , \ g = e^{cz+d}, \tag{31}$$

where $a \neq 0, b, c \neq 0$ and d are constants.

Subcase 2.2. Let $k = 1$. Considering Subcase 1.2 one can easily get

$$f = e^{az+b} \ , \ g = e^{cz+d}, \tag{32}$$

where $a \neq 0, b, c \neq 0$ and d are constants. Finally we can take f and g as $f = c_3 e^{dz}, g = c_4 e^{-dz}$, where c_3, c_4 and d are non-zero constants such that $(-1)^k (c_3 c_4)^n (nd)^{2k} = b^2$. This completes the proof.

Lemma 11. Let f and g be two transcendental meromorphic functions, let $p(z)$ be a nonzero polynomial with $deg(p) \leq n-1$, let n and k be two positive integers with $n > k+2$. Let $P(w)$ be defined as in Lemma 9 and $(f^n P(f))^{(k)}$, $(g^n P(g))^{(k)}$ share p CM and also f and g share ∞ IM. Suppose that $[f^n P(f)]^{(k)} [g^n P(g)]^{(k)} \equiv$

p^2, then $P(z)$ reduces to a nonzero monomial namely, namely $P(z) = a_i z^i \not\equiv 0$ for some $i \in \{0, 1, ..., m\}$; if $p(z)$ is not a constant, then $f = c_1 e^{cQ(z)}$, $g = c_2 e^{-cQ(z)}$, where $Q(z) = \int_0^z p(z)\,\mathrm{d}z$, $c_1, c_2, c \in \mathbb{C}$ such that $a_i^2 (c_1 c_2)^{n+i}[(n+i)c]^2 = -1$, if $p(z) = b(\neq 0)$, then $f = c_3 e^{cz}$, $g = c_4 e^{-cz}$, where $c_3, c_4, c \in \mathbb{C}$ such that $(-1)^k a_i^2 (c_3 c_4)^{n+i}[(n+i)c]^{2k} = b^2$.
Proof: The proof of lemma follows from Lemmas 9 and 10.

3. Proof of the Theorem

Proof of Theorem 1.

Let $F = \frac{[f^n P(f)]^{(k)}}{p}$ and $G = \frac{[g^n P(g)]^{(k)}}{p}$. It follows that F and G share (1, 2) except for the zeros of p.

Case 1. Let $H \not\equiv 0$. From (1) it can be easily calculated that the possible poles of H occur at (i) multiple zeros of F and G, (ii) those 1-points of F and G whose multiplicities are different,(iii) poles of F and G, (iv) zeros of $F'(G')$ which are not the zeros of $F(F-1)(G(G-1))$.
Since H has only simple poles we get

$$\begin{aligned} N(r,\infty;H) \leq{}& \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + \overline{N}_*(r,1;F,G) + \overline{N}(r,0;F \mid\geq 2) \\ &+ \overline{N}(r,0;G \mid\geq 2) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') \end{aligned} \tag{33}$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G')$ is similarly defined.
Here we see that

$$N(r,1;F \mid= 1) \leq N(r,0;H) \leq N(r,\infty;H) + S(r,F) + S(r,G). \tag{34}$$

Note that $\overline{N}_*(r,1;F,G) = 0$ and $\overline{N}_*(r,\infty;F,G) \leq \overline{N}(r,\infty;f)$.
Now in view of Lemma 7 we get

$$\begin{aligned} &\overline{N}_0(r,0;G') + \overline{N}(r,1;F \mid\geq 2) + \overline{N}_*(r,1;F,G) \\ &\leq \overline{N}_0(r,0;G') + \overline{N}(r,1;F \mid\geq 2) + \overline{N}(r,1;F \mid\geq 3) \\ &= \overline{N}_0(r,0;G') + \overline{N}(r,1;G \mid\geq 2) + \overline{N}(r,1;G \mid\geq 3) \\ &\leq \overline{N}_0(r,0;G') + N(r,1;G) - \overline{N}(r,1;G) \\ &\leq N(r,0;G' \mid G \neq 0) \leq \overline{N}(r,0;G) + \overline{N}(r,\infty;g) + S(r,g) \end{aligned} \tag{35}$$

Hence using (33),(34),(35), Lemmas 5 and 6 we get from second fundamental theorem

$$\begin{aligned} (n+m)T(r,f) &\leq T(r,F) + N_{k+2}(r,0;f^n P(f)) - N_2(r,0;F) + S(r,f) \\ &\leq \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,1;F) + N_{k+2}(r,0;f^n P(f)) \\ &\quad - N_2(r,0;F) - N_0(r,0;F') \end{aligned}$$

$$\begin{aligned}(n+m)T(r,f) &\leq \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+N(r,1;F\mid=1)+\overline{N}(r,1;F\mid\geq 2)\\ &+N_{k+2}(r,0;f^nP(f))-N_2(r,0;F)-N_0(r,0;F')+S(r,f)\\ &\leq 2\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_2(r,0;F)+N_2(r,0;G)\\ &+N_{k+2}(r,0;f^nP(f))-N_2(r,0;F)+S(r,f)+S(r,g)\\ &\leq 2\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_{k+2}(r,0;f^nP(f))+k\overline{N}(r,\infty;g)\\ &+N_{k+2}(r,0;g^nP(g))+S(r,f)+S(r,g)\\ &\leq (k+m+4)\overline{N}(r,\infty;f)+(2k+m+4)T(r,g)+S(r,f)+S(r,g)\\ &\leq (k+m+4)T(r,f)+(2k+m+4)T(r,g)+S(r,f)+S(r,g)\\ &\leq (3k+2m+8)T(r)+S(r).\end{aligned} \tag{36}$$

In a similar way we can obtain

$$(n+m)T(r,g) \leq (3k+8+2m)T(r)+S(r), \tag{37}$$

where $T(r)=max\left\{T(r,f),T(r,g)\right\}$.
Combining (36) and (37) we see that

$$(n-3k-8-m)T(r) \leq S(r). \tag{38}$$

Since $n>3k+m+8$, (38) leads to a contradiction.

Case 2. Let $H\equiv 0$. Then by Lemma 11 (see [[10], p.166]) We have either

$$[f^nP(f)]^{(k)}[g^nP(g)]^{(k)} \equiv p^2, \tag{39}$$

or

$$f^nP(f) \equiv g^nP(g). \tag{40}$$

From (40) we get

$$f^n(a_mf^m+a_{m-1}f^{m-1}+...+a_0) \equiv g^n(a_mg^m+a_{m-1}g^{m-1}+...+a_0). \tag{41}$$

Let $h=f/g$. If h is a constant, then substituting $f=gh$ into (41) we deduce that

$$a_mg^{n+m}(h^{n+m}-1)+a_{m-1}g^{n+m-1}(h^{n+m-1}-1)+...+a_0g^n(h^n-1)\equiv 0,$$

which implies $h^d=1$, where $d=GCD(n+m,...,n+m-i,...n), a_{m-i}\neq 0$ for some $i=0,1,...,m$. Thus $f\equiv tg$ for a constant t such that $t^d=1$, where $d=GCD(n+m,...,n+m-i,...n)$, $a_{m-i}\neq 0$ for some $i=0,1,...m$. If h is not a constant, then we know by (41) that f and g satisfying the algebraic equation $R(f,g)=0$, where $R(\omega_1,\omega_2)=\omega_1^n(a_m\omega_1^m+a_{m-1}\omega_1^{m-1}+...a_0)-\omega_2^n(a_m\omega_2^m+a_{m-1}\omega_2^{m-1}+...+a_0)$. Remaining part of the theorem follows from (39) and Lemma 11. This completes the proof of the theorem.

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