

**IDENTITIES OF SYMMETRY FOR GENERALIZED
CARLITZ'S q -TANGENT POLYNOMIALS ASSOCIATED WITH
 p -ADIC INTEGRAL ON \mathbb{Z}_p^\dagger**

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ABSTRACT. In this paper, we discover symmetric properties for generalized Carlitz's q -tangent polynomials.

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1. Introduction

The area of the Bernoulli, Euler, Bernoulli, Genocchi, and tangent polynomials have been worked by many authors. Those polynomials possess many interesting properties and are of great importance in pure mathematics, for example, number theory, mathematical analysis and in the calculus of finite differences. Those polynomials also have various applications in other branches of science(see [1-14]). The Carlitz type tangent numbers and polynomials possess many interesting properties and arising in many areas of mathematics and physics. Recently, many mathematicians have studied in the area of the q -extension of tangent numbers and polynomials. In this paper, our aim in this paper is to discover special symmetric properties for generalized Carlitz's q -tangent polynomials.

Throughout this paper we use the following notations. By \mathbb{Z}_p we denote the ring of p -adic rational integers, \mathbb{Q}_p denotes the field of p -adic rational numbers, \mathbb{C}_p denotes the completion of algebraic closure of \mathbb{Q}_p , \mathbb{N} denotes the set of natural numbers, \mathbb{Z} denotes the ring of rational integers, \mathbb{Q} denotes the field of rational numbers, \mathbb{C} denotes the set of complex numbers, and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let ν_p be the normalized exponential valuation of \mathbb{C}_p with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When

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one talks of q -extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p -adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q - 1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. Throughout this paper we use the notation:

$$[x]_q = \frac{1 - q^x}{1 - q}, \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q} \quad (\text{cf. [4-14]}).$$

Hence, $\lim_{q \rightarrow 1} [x] = x$ for any x with $|x|_p \leq 1$ in the present p -adic case. Let

$$g \in UD(\mathbb{Z}_p) = \{g|g : \mathbb{Z}_p \rightarrow \mathbb{C}_p \text{ is uniformly differentiable function}\}.$$

Let a fixed positive integer d with $(p, d) = 1$, set

$$\begin{aligned} X &= X_d = \varprojlim_N (\mathbb{Z}/dp^N\mathbb{Z}), \quad X_1 = X, \\ X^* &= \bigcup_{\substack{0 < a < dp \\ (a,p)=1}} a + dp\mathbb{Z}_p, \\ a + dp^N\mathbb{Z}_p &= \{x \in X \mid x \equiv a \pmod{dp^N}\}, \end{aligned}$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^N$. For $g \in UD(\mathbb{Z}_p)$, the p -adic invariant integral on \mathbb{Z}_p is defined by Kim to be

$$I_{-1}(g) = \int_X g(x) d\mu_{-1}(x) = \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} g(x)(-1)^x, \quad \text{see [4].} \quad (1.1)$$

First, we introduce the Carlitz's type q -tangent numbers $T_{n,q}$ and polynomials $T_{n,q}(x)$ and investigate their properties(see [6]).

For $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$, the Carlitz's q -tangent polynomials $T_{n,q}(x)$ are defined by

$$T_{n,q}(x) = \int_{\mathbb{Z}_p} q^y [2y + x]_q^n d\mu_{-1}(y). \quad (1.2)$$

Since $[x + 2y]_q = [x]_q + q^x [2y]_q$, we easily see that

$$T_{n,q}(x) = \sum_{l=0}^n \binom{n}{l} [x]_q^{n-l} q^{xl} T_{l,q} = 2 \sum_{m=0}^{\infty} (-1)^m q^m [x + 2m]_q^n,$$

with the usual convention of replacing $(T_q)^n$ by $T_{n,q}$. When $x = 0$, $T_{n,q}(0) = T_{n,\chi,q}$ is called the n -th generalized Carlitz's q -tangent numbers.

By using p -adic integral, we obtain

$$\begin{aligned} T_{n,q}(x) &= 2 \left(\frac{1}{1-q}\right)^n \sum_{l=0}^n \binom{n}{l} (-1)^l q^{xl} \frac{1}{1+q^{2l+1}}, \\ T_{n,q} &= 2 \left(\frac{1}{1-q}\right)^n \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{1}{1+q^{2l+1}}. \end{aligned}$$

2. Symmetric identities for generalized Carlitz's q -tangent numbers and polynomials

Our primary goal of this section is to obtain symmetric identities for generalized Carlitz's q -tangent numbers $T_{n,\chi,q}$ and polynomials $T_{n,\chi,q}(x)$. For $q \in \mathbb{C}_p$ with $|q - 1|_p < 1$, generalized Carlitz's q -tangent polynomials $T_{n,\chi,q}(x)$ are defined by

$$T_{n,\chi,q}(x) = \int_X \chi(y)q^y [x + 2y]_q^n d\mu_{-1}(y). \quad (2.1)$$

When $x = 0$, $T_{n,\chi,q}(0) = T_{n,\chi,q}$ is called the n -th generalized Carlitz's q -tangent numbers.

Let w_1 and w_2 be odd numbers. Then we have

$$\begin{aligned} & \int_X \chi(y)q^{w_1 y} e^{\left[w_2 x + \frac{2w_2}{w_1} j + 2y \right]_{q^{w_1}} [w_1]_{q^t}} d\mu_{-1}(y) \\ &= \lim_{N \rightarrow \infty} \sum_{y=0}^{dw_2 p^N - 1} \chi(y) e^{[w_1 w_2 x + 2w_2 j + 2w_1 y]_{q^t}} (-1)^y \\ &= \lim_{N \rightarrow \infty} \sum_{i=0}^{dw_2 - 1} \sum_{y=0}^{p^N - 1} \chi(i) q^{w_1(i + w_2 dy)} e^{[w_1 w_2 x + 2w_2 j + 2w_1(i + w_2 dy)]_{q^t}} (-1)^{i + w_2 dy} \end{aligned} \quad (2.2)$$

From (2.2), we can derive the following equation (2.3):

$$\begin{aligned} & \sum_{j=0}^{dw_1 - 1} \chi(j) (-1)^j q^{w_2 j} \int_X \chi(y) q^{w_1 y} e^{\left[w_2 x + \frac{2w_2}{w_1} j + 2y \right]_{q^{w_1}} [w_1]_{q^t}} d\mu_{-1}(y) \\ &= \lim_{N \rightarrow \infty} \sum_{j=0}^{dw_1 - 1} \sum_{i=0}^{dw_2 - 1} \sum_{y=0}^{p^N - 1} \chi(i) \chi(j) (-1)^{i+j} q^{w_2 j} q^{w_1 i} q^{dw_1 w_2 y} \\ & \quad \times e^{[w_1 w_2 x + 2w_2 j + 2w_1 i + 2dw_1 w_2 y]_{q^t}} (-1)^y \end{aligned} \quad (2.3)$$

By the same method as (2.3), we have

$$\begin{aligned} & \sum_{j=0}^{dw_2 - 1} \chi(j) (-1)^j q^{w_1 j} \int_X \chi(y) q^{w_2 y} e^{\left[w_1 x + \frac{2w_1}{w_2} j + 2y \right]_{q^{w_2}} [w_2]_{q^t}} d\mu_{-1}(y) \\ &= \lim_{N \rightarrow \infty} \sum_{j=0}^{dw_2 - 1} \sum_{i=0}^{dw_1 - 1} \sum_{y=0}^{p^N - 1} \chi(j) \chi(i) (-1)^{i+j} q^{w_1 j} q^{w_2 i} q^{dw_1 w_2 y} \\ & \quad \times e^{[w_1 w_2 x + 2w_1 j + 2w_2 i + 2w_1 w_2 dy]_{q^t}} (-1)^y \end{aligned} \quad (2.4)$$

Therefore, by (2.3) and (2.4), we have the following theorem.

Theorem 2.1. For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod{2}$, $w_2 \equiv 1 \pmod{2}$, we have

$$\begin{aligned} & \sum_{j=0}^{dw_1-1} \chi(j)(-1)^j q^{w_2 j} \int_X \chi(y) q^{w_1 y} e^{\left[w_2 x + \frac{2w_2}{w_1} j + 2y \right]_{q^{w_1}} [w_1]_q t} d\mu_{-1}(y) \\ &= \sum_{j=0}^{dw_2-1} \chi(j)(-1)^j q^{w_1 j} \int_X \chi(y) q^{w_2 y} e^{\left[w_1 x + \frac{2w_1}{w_2} j + 2y \right]_{q^{w_2}} [w_2]_q t} d\mu_{-1}(y). \end{aligned} \quad (2.5)$$

By substituting Taylor series of e^{xt} into (2.5) and after calculations, we obtain the following corollary.

Corollary 2.2. For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod{2}$, $w_2 \equiv 1 \pmod{2}$, we have

$$\begin{aligned} & [w_1]_q^n \sum_{j=0}^{dw_1-1} \chi(j)(-1)^j q^{w_2 j} \int_X \chi(y) q^{w_1 y} \left[w_2 x + \frac{2w_2}{w_1} j + 2y \right]_{q^{w_1}}^n d\mu_{-1}(y) \\ &= [w_2]_q^n \sum_{j=0}^{dw_2-1} \chi(j)(-1)^j q^{w_1 j} \int_X \chi(y) q^{w_2 y} \left[w_1 x + \frac{2w_1}{w_2} j + 2y \right]_{q^{w_2}}^n d\mu_{-1}(y). \end{aligned} \quad (2.6)$$

By (2.1) and Corollary 2.2, we have the following theorem.

Theorem 2.3. For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod{2}$, $w_2 \equiv 1 \pmod{2}$, we have

$$\begin{aligned} & [w_1]_q^n \sum_{j=0}^{dw_1-1} \chi(j)(-1)^j q^{w_2 j} T_{n, \chi, q^{w_1}} \left(w_2 x + \frac{2w_2}{w_1} j \right) \\ &= [w_2]_q^n \sum_{j=0}^{dw_2-1} \chi(j)(-1)^j q^{w_1 j} T_{n, \chi, q^{w_2}} \left(w_1 x + \frac{2w_1}{w_2} j \right). \end{aligned}$$

By (2.6), we can derive the following equation (2.7):

$$\begin{aligned} & \int_X \chi(y) q^{w_1 y} \left[w_2 x + \frac{2w_2}{w_1} j + 2y \right]_{q^{w_1}}^n d\mu_{-1}(y) \\ &= \sum_{i=0}^n \binom{n}{i} \left(\frac{[w_2]_q}{[w_1]_q} \right)^i [2j]_{q^{w_2}}^i q^{w_2(n-i)j} \int_X \chi(y) q^{w_1 y} [w_2 x + 2y]_{q^{w_1}}^{n-i} d\mu_{-1}(y) \\ &= \sum_{i=0}^n \binom{n}{i} \left(\frac{[w_2]_q}{[w_1]_q} \right)^i [2j]_{q^{w_2}}^i q^{w_2(n-i)j} T_{n-i, \chi, q^{w_1}}(w_2 x). \end{aligned} \quad (2.7)$$

By (2.7), and Theorem 2.3, we have

$$\begin{aligned}
 & [w_1]_q^n \sum_{j=0}^{dw_1-1} \chi(j)(-1)^j q^{w_2 j} \int_X \chi(y) q^{w_1 y} \left[w_2 x + \frac{2w_2}{w_1} j + 2y \right]_{q^{w_1}}^n d\mu_{-1}(y) \\
 &= \sum_{j=0}^{dw_1-1} \chi(j)(-1)^j q^{w_2 j} \sum_{i=0}^n \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i} [2j]_{q^{w_2}}^i q^{w_2(n-i)j} T_{n-i, \chi, q^{w_1}}(w_2 x) \\
 &= \sum_{i=0}^n \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i} T_{n-i, \chi, q^{w_1}}(w_2 x) \sum_{j=0}^{dw_1-1} \chi(j)(-1)^j q^{w_2(n-i+1)j} [2j]_{q^{w_2}}^i \\
 &= \sum_{i=0}^n \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i} T_{n-i, \chi, q^{w_1}}(w_2 x) \mathcal{S}_{n,i}(dw_1, q^{w_2} | \chi),
 \end{aligned} \tag{2.8}$$

where

$$\mathcal{S}_{n,i}(w_1, q | \chi) = \sum_{j=0}^{w_1-1} \chi(j)(-1)^j q^{(n-i+1)j} [2j]_{q^i}$$

is called as the sums of powers. By the same method as (2.8), we get

$$\begin{aligned}
 & [w_2]_q^n \sum_{j=0}^{dw_2-1} \chi(j)(-1)^j q^{w_1 j} \int_X \chi(y) q^{w_2 y} \left[w_1 x + \frac{2w_1}{w_2} j + 2y \right]_{q^{w_2}}^n d\mu_{-1}(y) \\
 &= \sum_{i=0}^n \binom{n}{i} [w_1]_q^i [w_2]_q^{n-i} T_{n-i, \chi, q^{w_2}}(w_1 x) \mathcal{S}_{n,i}(dw_2, q^{w_1} | \chi).
 \end{aligned} \tag{2.9}$$

By (2.8) and (2.9), we have the following theorem.

Theorem 2.4. For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod{2}$, $w_2 \equiv 1 \pmod{2}$, we have

$$\begin{aligned}
 & \sum_{i=0}^n \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i} \mathcal{S}_{n,i}(dw_1, q^{w_2} | \chi) T_{n-i, \chi, q^{w_1}}(w_2 x) \\
 &= \sum_{i=0}^n \binom{n}{i} [w_1]_q^i [w_2]_q^{n-i} \mathcal{S}_{n,i}(dw_2, q^{w_1} | \chi) T_{n-i, \chi, q^{w_2}}(w_1 x).
 \end{aligned}$$

By setting $x = 0$ in Theorem 2.4, we have the following corollary.

Corollary 2.5. For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod{2}$, $w_2 \equiv 1 \pmod{2}$, we have

$$\begin{aligned}
 & \sum_{i=0}^n \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i} \mathcal{S}_{n,i}(dw_1, q^{w_2} | \chi) T_{n-i, \chi, q^{w_1}} \\
 &= \sum_{i=0}^n \binom{n}{i} [w_1]_q^i [w_2]_q^{n-i} \mathcal{S}_{n,i}(dw_2, q^{w_1} | \chi) T_{n-i, \chi, q^{w_2}}.
 \end{aligned}$$

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