J. Appl. Math. & Informatics Vol. **36**(2018), No. 1 - 2, pp. 107 - 114
https://doi.org/10.14317/jami.2018.107

NEW FRACTIONAL INTEGRAL INEQUALITIES OF TYPE OSTROWSKI THROUGH GENERALIZED CONVEX FUNCTION

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Abstract. We establish some new ostrowski type inequalities for MT-convex function including first order derivative via Niemann-Trouvaille fractional integral. It is interesting to mention that our results provide new estimates on these types of integral inequalities for MT-convex functions.

AMS Mathematics Subject Classification : 26D15, 26A33, 26A51.
Key words and phrases : Ostrowski type inequality, MT-convex function, Holder's integral inequality, Niemann-Trouvaille fractional integrals.

1. Introduction and Preliminaries

The ostrowski inequality is very important and well-known in the literature. This inequality is stated as: Suppose $f : I \subset [0, \infty) \to R$ be a differentiable function on I^0 (interior of I), where $a, b \in I$ with $a < b$ such that $f' \in L[a, b]$. If $|f'(x)| \leq M$, then the following inequality holds:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u) du \right| \leq \frac{M}{b-a} \left[\frac{(x-a)^2 + (b-x)^2}{2} \right],$$

for all $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller one.
Recently, convex function plays a major role in the development of many well known inequalities. So many authors have generalized the classical version of famous inequalities such as Hermite-Hadamard inequality, Simpson's inequality, Ostrowski inequality etc. for different classes of convex functions. For more details, readers are refereed to [1-10],[20-27].
First we recall some definitions and preliminary facts of convex function and

Received August 24, 2017. Revised November 30, 2017. Accepted December 10, 2017.
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fractional calculus theory which will be used in the sequel.
A function $f : I \to R(\ \varphi \neq I \subseteq R)$ is said to be convex on the interval I of real numbers, if

$$f\left(\lambda x + (1-\lambda)y\right) \leq \lambda f\left(x\right) + \left(1-\lambda\right) f\left(y\right),$$

where $a, b \in I$ and $\lambda \in [0,1]$,

Definition 1.1. A function $f : I \subseteq R \to R$ is said to be in the class of $MT\left(I\right)$, if it is nonnegative and satisfies the inequality:

$$f\left(\lambda a + (1-\lambda)\, b\right) d\lambda \leq \frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}} f\left(a\right) + \frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}} f\left(b\right),$$

for all $x, y \in I$ and $\lambda \in [0,1]$.

Remark 1.1. In above inequality, if we take $\lambda = 1/2$, the inequality reduces to Jensen convex.

Theorem 1.2. *Let $f \in MT\left(I\right)$, where $a, b \in I$ with $a < b$ and $f \in L_1\left[a,b\right]$, then the following holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f\left(x\right) dx,$$

and

$$\frac{2}{b-a}\int_a^b \tau\left(x\right) f\left(x\right) dx \leq \frac{f\left(a\right)+f\left(b\right)}{2},$$

where $\tau\left(x\right) = \frac{\sqrt{(b-x)(x-a)}}{b-a}$, $x \in [a,b]$.

Fraction calculus [11, 12, 13, 14, 15, 16, 17, 18, 19] was introduced at the end of the nineteenth century by Niemann and Trouvaille, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences, biological sciences, economics and engineering.

Definition 1.3. Let $f \in L_1\left[a,b\right]$. The Niemann-Trouvaille fractional integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $\alpha \geq 0$ are defined by

$$J_{a+}^{\alpha} f\left(x\right) = \frac{1}{\Gamma\left(\alpha\right)}\int_a^x \left(x-t\right)^{\alpha-1} f\left(t\right) dt, \ \ \left(a < x\right),$$

and

$$J_{b-}^{\alpha} f\left(x\right) = \frac{1}{\Gamma\left(\alpha\right)}\int_x^b \left(t-x\right)^{\alpha-1} f\left(t\right) dt, \ \ \left(b > x\right),$$

respectively. Here $\Gamma\left(\alpha\right) = \int_0^{\infty} e^{-u} u^{\alpha-1} du$ and $J_{a+}^{0} f\left(x\right) = J_{b-}^{0} f\left(x\right) = f\left(x\right)$.
In case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Lemma 1.4. *If $f : [a,b] \to \mathrm{R}$ be a differentiable mapping on (a,b) with $a < b$ and $f' \in L_1[a,b]$, then we have*

$$\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{b-a} f(x) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J^{\alpha}_{\alpha+} f(b) + J^{\alpha}_{\alpha-} f(a) \right]$$
$$\leq \frac{(x-a)^{\alpha+1}}{b-a} \int_0^1 \lambda^{\alpha} f'(\lambda x + (1-\lambda) a) \, d\lambda - \frac{(b-x)^{\alpha+1}}{b-a} \int_0^1 \lambda^{\alpha} f'(\lambda x + (1-\lambda) b) \, d\lambda,$$

for all $x \in [a,b]$ with $\alpha > 0$,

2. Main results

We are in position to derive our main results.

Theorem 2.1. *Let $f : [a,b] \to \mathrm{R}$ be a differentiable mapping such that $f' \in L_1[a,b]$. If $|f'|$ is MT-convex on $[a,b]$ and $|f'(x)| \leq M$, then we have*

$$\left| \frac{(x-a)^{\alpha}+(x-b)^{\alpha}}{b-a} f(x) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J^{\alpha}_{\alpha+} f(b) + J^{\alpha}_{\alpha-} f(a) \right] \right|$$
$$\leq 2\beta\left(\alpha + \tfrac{1}{2}, \tfrac{1}{2}\right) \left(\frac{M\left[(x-a)^{\alpha+1}+(x-b)^{\alpha+1}\right]}{b-a} \right),$$

for all $x \in [a,b]$ and $\alpha > 0$. Where

$$\beta(x,y) = \int_0^1 \lambda^{x-1} (1-\lambda)^{y-1} \, d\lambda, \ x > 0, \quad y > 0$$

represents the beta function.

Proof. Using Lemma 1.4, Holder's inequality, and $MT-$convexity of $|f'|$, we get

$$\left| \frac{(x-a)^{\alpha}+(x-b)^{\alpha}}{b-a} f(x) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}} \left[J^{\alpha}_{\alpha+} f(b) + J^{\alpha}_{\alpha-} f(a) \right] \right|$$
$$= \left| \frac{(x-a)^{\alpha+1}}{b-a} \int_0^1 \lambda^{\alpha} f'(\lambda x + (1-\lambda) a) \, d\lambda - \frac{(x-b)^{\alpha+1}}{b-a} \int_0^1 \lambda^{\alpha} f'(\lambda x + (1-\lambda) b) \, d\lambda \right|$$
$$\leq \frac{(x-a)^{\alpha+1}}{b-a} \int_0^1 \lambda^{\alpha} |f'(\lambda x + (1-\lambda) a)| \, d\lambda + \frac{(x-b)^{\alpha+1}}{b-a} \int_0^1 \lambda^{\alpha} |f'(\lambda x + (1-\lambda) b)| \, d\lambda$$
$$\leq \frac{(x-a)^{\alpha+1}}{b-a} \int_0^1 \lambda^{\alpha} \left[\frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}} |f'(x)| + \frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}} |f'(a)| \right] d\lambda$$
$$+ \frac{(x-b)^{\alpha+1}}{b-a} \int_0^1 \lambda^{\alpha} \left[\frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}} |f'(x)| + \frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}} |f'(b)| \right] d\lambda$$
$$\leq \frac{(x-a)^{\alpha+1}}{b-a} \left(\beta\left(\alpha + \tfrac{3}{2}, \tfrac{1}{2}\right) |f'(x)| + \beta\left(\alpha + \tfrac{1}{2}, \tfrac{3}{2}\right) |f'(a)| \right)$$
$$+ \frac{(x-b)^{\alpha+1}}{b-a} \left(\beta\left(\alpha + \tfrac{3}{2}, \tfrac{1}{2}\right) |f'(x)| + \beta\left(\alpha + \tfrac{1}{2}, \tfrac{3}{2}\right) |f'(b)| \right)$$
$$\leq 2\beta\left(\alpha + \tfrac{1}{2}, \tfrac{1}{2}\right) \left(\frac{M\left[(x-a)^{\alpha+1}+(x-b)^{\alpha+1}\right]}{b-a} \right).$$

This completes the proof. □

Corollary 2.2. *Let $f:[a,b]\rightarrow \mathrm{R}$ be a differentiable mapping such that $f'\in L_1[a,b]$. If $|f'|$ is $MT-$convex on $[a,b]$ and $|f'(x)|\leq M$, then we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq\frac{\pi M\left[(x-a)^2+(b-x)^2\right]}{b-a},$$

for all $x\in[a,b]$.

Remark 2.1. (1). If we choose $x=\frac{a+b}{2}$, in Corollary 2.2, then we have midpoint inequality

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq\pi M(b-a).$$

(2). If we choose $x=a$ in Corollary 2.2, then we have

$$\left|f(a)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq\pi M(b-a).$$

(3). If we choose $x=b$ in Corollary 2.2, then we have

$$\left|f(b)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq\pi M(b-a).$$

Theorem 2.3. *Let $f:[a,b]\rightarrow \mathrm{R}$ be a differentiable mapping such that $f'\in L_1[a,b]$. If $|f'|^q$ is $MT-$convex on $[a,b]$, $p,q>1$ and $|f'(x)|\leq M$, then we have the following inequality for fractional integral:*

$$\begin{aligned}&\left|\tfrac{(x-a)^\alpha+(b-x)^\alpha}{b-a}f(x)-\tfrac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\alpha^+}f(b)+J^\alpha_{\alpha^-}f(a)\right]\right|\\&\leq M\left(\tfrac{\pi}{4}\right)^{\frac{1}{q}}\left[\tfrac{1}{p\alpha+1}\right]^{1/p}\left[\tfrac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\right],\end{aligned}$$

for all $x\in[a,b]$, $\alpha>0$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 1.4, Holder's inequality, and $MT-$convexity of $|f'|^q$, we get

$$\begin{aligned}&\left|\tfrac{(x-a)^\alpha+(b-x)^\alpha}{b-a}f(x)-\tfrac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\alpha^+}f(b)+J^\alpha_{\alpha^-}f(a)\right]\right|\\&=\left|\tfrac{(x-a)^{\alpha+1}}{b-a}\int_0^1\lambda^\alpha f'(\lambda x+(1-\lambda)a)\,d\lambda+\tfrac{(x-b)^{\alpha+1}}{b-a}\int_0^1\lambda^\alpha f'(\lambda x+(1-\lambda)b)\,d\lambda\right|\\&\leq\tfrac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1\lambda^{p\alpha}\right)^{\frac{1}{p}}\left(\int_0^1|f'(\lambda x+(1-\lambda)a)|^q\,d\lambda\right)^{\frac{1}{q}}\\&+\tfrac{(x-b)^{\alpha+1}}{b-a}\left(\int_0^1\lambda^{p\alpha}\right)^{\frac{1}{p}}\left(\int_0^1|f'(\lambda x+(1-\lambda)b)|^q\,d\lambda\right)^{\frac{1}{q}}\end{aligned}$$

$$\leq \frac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1 \lambda^{p\alpha}\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}\left|f'(x)\right|^q+\frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}\left|f'(a)\right|^q\right)d\lambda\right)^{\frac{1}{q}}$$
$$+\frac{(x-b)^{\alpha+1}}{b-a}\left(\int_0^1 \lambda^{p\alpha}\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}\left|f'(x)\right|^q+\frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}\left|f'(b)\right|^q\right)d\lambda\right)^{\frac{1}{q}}$$
$$\leq M\left(\frac{\pi}{4}\right)^{\frac{1}{q}}\left[\frac{1}{p\alpha+1}\right]^{1/p}\left[\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\right].$$

This completes the proof . □

Corollary 2.4. *Let $f:[a,b]\to \mathrm{R}$ be a differentiable mapping such that $f'\in L_1[a,b]$. If $|f'|^q$ is $MT-$convex on $[a,b]$, $p,q>1$ and $|f'(x)|\leq M$, then we have the following inequality for fractional integral:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq M\left[\frac{1}{p+1}\right]^{1/p}\left(\frac{\pi}{4}\right)^{\frac{1}{q}}\left(\frac{M\left[(x-a)^2+(b-x)^2\right]}{b-a}\right),$$

for all $x\in[a,b]$ and $\frac{1}{p}+\frac{1}{q}=1$.

Remark 2.2. (1). If we choose $x=\frac{a+b}{2}$ in Corollary 2.4, then we have midpoint inequality

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq\frac{M(b-a)}{2}\left[\frac{1}{p+1}\right]^{1/p}\left(\frac{\pi}{4}\right)^{\frac{1}{q}}.$$

(2). If we choose $x=a$ in Corollary 2.4, then we have

$$\left|f(a)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq M(b-a)\left[\frac{1}{p+1}\right]^{1/p}\left(\frac{\pi}{4}\right)^{\frac{1}{q}}.$$

(3). If we choose $x=b$ in Corollary 2.4, then we have

$$\left|f(b)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq M(b-a)\left[\frac{1}{p+1}\right]^{1/p}\left(\frac{\pi}{4}\right)^{\frac{1}{q}}.$$

Theorem 2.5. *Let $f:[a,b]\to \mathrm{R}$ be a differentiable mapping such that $f'\in L_1[a,b]$. If $|f'|^q$ is $MT-$convex on $[a,b]$, $p,q>1$ and $|f'(x)|\leq M$, then we have the following inequality for fractional integral:*

$$\left|\frac{(x-a)^\alpha+(b-x)^\alpha}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{\alpha^+}^\alpha f(b)+J_{\alpha^-}^\alpha f(a)\right]\right|$$
$$\leq M\left(\frac{\pi}{4}\right)^{\frac{1}{q}}\left[\frac{1}{p\alpha+1}\right]^{1/p}\left[\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\right].$$

for all $x\in[a,b]$, $\alpha>0$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 1.4, Power mean inequality and $MT-$convexity of $|f'|^q$, we get

$$\left|\frac{(x-a)^{\alpha}+(x-b)^{\alpha}}{b-a}f(x)-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\alpha+}^{\alpha}f(b)+J_{\alpha-}^{\alpha}f(a)\right]\right|$$
$$=\left|\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1\lambda^{\alpha}f'(\lambda x+(1-\lambda)a)\,d\lambda+\frac{(x-b)^{\alpha+1}}{b-a}\int_0^1\lambda^{\alpha}f'(\lambda x+(1-\lambda)b)\,d\lambda\right|$$
$$\leq\frac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1\lambda^{\alpha}d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1\lambda^{\alpha}\left|f'(\lambda x+(1-\lambda)a)\right|^q d\lambda\right)^{\frac{1}{q}}$$
$$+\frac{(x-b)^{\alpha+1}}{b-a}\left(\int_0^1\lambda^{\alpha}d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1\lambda^{\alpha}\left|f'(\lambda x+(1-\lambda)b)\right|^q d\lambda\right)^{\frac{1}{q}}$$
$$\leq\frac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1\lambda^{\alpha}d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1\lambda^{\alpha}\left(\frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}\left|f'(x)\right|^q+\frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}\left|f'(a)\right|^q\right)d\lambda\right)^{\frac{1}{q}}$$
$$+\frac{(x-b)^{\alpha+1}}{b-a}\left(\int_0^1\lambda^{\alpha}d\lambda\right)^{1-\frac{1}{q}}\left(\int_0^1\lambda^{\alpha}\left(\frac{\sqrt{\lambda}}{2\sqrt{1-\lambda}}\left|f'(x)\right|^q+\frac{\sqrt{1-\lambda}}{2\sqrt{\lambda}}\left|f'(b)\right|^q\right)d\lambda\right)^{\frac{1}{q}}$$
$$\leq M\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(2\beta\left(\alpha+\frac{1}{2},\frac{1}{2}\right)\right)^{\frac{1}{q}}\left[\frac{(x-a)^{\alpha+1}+(x-b)^{\alpha+1}}{b-a}\right].$$

This completes the proof . □

Corollary 2.6. *Let $f:[a,b]\to\mathrm{R}$ be a differentiable mapping such that $f'\in L_1[a,b]$. If $|f'|^q$ is $MT-$convex on $[a,b]$, $p,q>1$ and $|f'(x)|\leq M$, then we have:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq M^q\left(\frac{1}{2}\right)^{1-\frac{1}{q}}(\pi)^{\frac{1}{q}}\left(\frac{\left[(x-a)^2+(b-x)^2\right]}{b-a}\right),$$

for all $x\in[a,b]$ and $\frac{1}{p}+\frac{1}{q}=1$.

Remark 2.3. (1). If we choose $x=\frac{a+b}{2}$ in Corollary 2.6, then we have midpoint inequality

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq M^q\left(\frac{1}{2}\right)^{1-\frac{1}{q}}(\pi)^{\frac{1}{q}}\left(\frac{b-a}{2}\right).$$

(2). If we choose $x=a$ in Corollary 2.6, then we have

$$\left|f(a)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq M^q(b-a)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}(\pi)^{\frac{1}{q}}.$$

(3). If we choose $x=b$ in Corollary 2.6, then we have

$$\left|f(b)-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq M^q(b-a)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}(\pi)^{\frac{1}{q}}.$$

Acknowledgement. The authors are thankful to the anonymous reviewers for their valuable comments towards the improvement of the paper.

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