

CLASSIFICATIONS OF METRIC FUNCTIONS ON THE PLANE[†]

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ABSTRACT. There are many metric functions in the plane. In this paper we are to classify the metric functions on the plane using two ways such as using sum of distances between some points when start point and end point is fixed, and using area of transferred triangle consisted of distances between 3 points.

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1. Introduction

Recently there are many results about more practical metric functions than Euclidean metric in the plane[1,2,3,4,5,6,7,8]. It is much helpful to understand geometric properties and useful way in the plane. Euclid metric is easy to understand and useful but is not proper to set as the distance of route in city. To improve the metric, Manhattan metric[6] introduced for taxi-moving route and its generalization, α -metric[2,3] are useful to grasp more practical route in city. Nowadays, there are many researches that adopt new metrics to reflect more reality and study properties of metric given spaces. In this paper we are to classify the metric functions by two ways: The first method of classification using sum of distances between some points when start point and end point is fixed. The other method using area of transferred triangle consisted of distances between three points.

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2. Classifications based on summation of distances between several points

In this chapter we are going to classify the metric functions using summation of distances between several points.

Definition 2.1. Let d and \bar{d} be two metrics on the same set X . If there exist positive constant α, β such that $\alpha d(x, y) \leq \bar{d}(x, y) \leq \beta d(x, y)$ for all $x, y \in X$, then we say d is equivalent to \bar{d} [6].

Definition 2.2. If functions $f(x)$ and $g(x)$ satisfy $0 < \lim_{x \rightarrow \infty} \left| \frac{f(x)}{g(x)} \right| < \infty$, then we say that $f(x) = \Theta(g(x))$. Also, if it satisfy $\lim_{x \rightarrow \infty} \left| \frac{f(x)}{g(x)} \right| < \infty$, then we write $f(x) = O(g(x))$.

Let $A_0(x_0, y_0), A_1(x_1, y_1), \dots, A_n(x_n, y_n)$ be points on a plane which satisfy $x_0 = y_0 = 0, x_i \leq x_{i+1}, y_i \leq y_{i+1} (i = 0, 1, \dots, n-1), x_n = y_n = a$. And for function $d : X \times X \rightarrow [0, \infty)$ on a set X , we can consider

$$m \leq d(A_0, A_1) + d(A_1, A_2) + \dots + d(A_{n-1}, A_n) \leq M$$

Let $d_{\min}(n)$ be the maximum value of m and $d_{\max}(n)$ be the minimum value of M .

We see that $d_{\min}(n), d_{\max}(n)$ are function of n and d . Now, define a α -classification $\alpha_d(n)$ by

$$\alpha_d(n) = \frac{d_{\max}(n)}{d_{\min}(n)}, \quad \alpha_d(n) = O(\beta_d(n))$$

Example 2.3. Let us consider for the Euclidean metric function d . It is trivial that $d_{\min}(n) = \sum_i d(A_i, A_{i+1}) = d(A_0, A_n) = \sqrt{2}a$. Let B_1, \dots, B_n be the points which the summation of the distances is maximum. Since

$$\sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2} \leq |x_i - x_{i+1}| + |y_i - y_{i+1}|,$$

$$d_{\max}(n) = \sum_i d(A_i, A_{i+1}) \leq \sum_i (|x_i - x_{i+1}| + |y_i - y_{i+1}|) = 2a$$

So, α -classification for Euclidean metric function is $\alpha_d(n) = \sqrt{2}, \beta_d(n) = 1$.

Example 2.4. Let us apply the classification on a metric function $d(x, y) = \left| \log\left(\frac{y}{x}\right) \right|$. It is trivial that $d_{\min}(n) = d(A_0, A_n) = \left| \log\left(\frac{x_n}{x_0}\right) \right|$. $d_{\max}(n) = \sum_i d(A_i, A_{i+1}) = \sum_i \log\left(\frac{x_{i+1}}{x_i}\right) = \log\left(\frac{x_n}{x_0}\right)$. So, the α -classification is $\alpha_d(n) = 1, \beta_d(n) = 1$. For $\beta_d(n)$, we have

Theorem 2.5. If $\beta_d(n) = n^c$ (c is a constant), then $c \geq 0$

Proof. Assume that $c < 0$. Since $\lim_{n \rightarrow \infty} \beta_d(n) = \lim_{n \rightarrow \infty} n^c = 0$, we see that $\lim_{n \rightarrow \infty} \alpha_d(n) = 0$. Now according to the definition of $\alpha_d(n)$, the distance between

two discrete points X, Y , $d(X, Y)$ satisfies $d(X, Y) \leq d(A_0, X) + d(X, Y) + d(Y, A_n) \leq \lim_{n \rightarrow \infty} d_{\max}(n) = 0$. This contradicts the definition of the metric function that $d(X, Y) \leq 0$ for two discrete points X, Y . \square

Theorem 2.6. *If $\beta_d(n) = n^c$ (c is a constant), then $c \leq 1$*

Proof. Assume that $c > 1$. Since $\lim_{n \rightarrow \infty} \frac{\beta_d(n)}{n} = \lim_{n \rightarrow \infty} n^{c-1} = \infty$, we see that $\lim_{n \rightarrow \infty} \frac{\alpha_d(n)}{n} = \lim_{n \rightarrow \infty} \frac{d_{\max}(n)}{nd_{\min}(n)} = \lim_{n \rightarrow \infty} \frac{\sum_i d(A_i, A_{i+1})}{nd_{\min}(n)} = \infty$. Here we know that there is a maximum value among $d(A_0, A_1), \dots, d(A_{n-1}, A_n)$ as they are all real numbers. Let $d(A_j, A_{j+1})$ be the maximum value thus having $\frac{d(A_j, A_{j+1})}{n} \geq \sum_i d(A_i, A_{i+1})$. There exists a positive number $\delta > 0$ for all positive number $\epsilon > 0$. Since $\frac{d(A_j, A_{j+1})}{d_{\min}(n)} > \frac{\sum_i d(A_i, A_{i+1})}{nd_{\min}(n)} > \frac{\epsilon}{d_{\min}(n)}$ for all positive number r which satisfies $r > \delta$, we get $d(A_j, A_{j+1}) > \epsilon$. This states that we can select two points so that the distance between those two points would be bigger than random number. This contradicts the fact that $\beta_d(n) = n^c$ (c is a constant) because $\beta_d(n)$ would fixed as ∞ . \square

Remark 2.7. We have shown that if we can select two points so that the distance between those two points would be bigger than random number, it shows that $d_{\max}(n) \rightarrow \infty$ and $\beta_d(n) \rightarrow \infty$. We see that α -classification would fall in to two categories. First, $\beta_d(n)$ would be expressed as a function of n and the second, $\beta_d(n) \rightarrow \infty$. If $\beta_d(n) \rightarrow \infty$, then $d_{\min}(n)$ is also limitless which infers that two points exists so that the distance between those two points would be bigger than random number. Hence we see that if $\beta_d(n)$ would be expressed as a function of n , then the distance between any two points is bounded. For example, for a fixed point A , if $d(x, A)$ is continuous then we get $\beta_d(n)$ for all points $\forall x \in \mathbf{R}^2$. Hence forth we will strong $\beta_d(n)$ for $\beta_d(n)$ can be expressed by a function of n .

Theorem 2.8. *For every $c \in [0, 1]$, there exist a distance function d which satisfies $\beta_d(n) = n^c$ (c is a constant).*

Proof. It is known that for the positive number $k \geq 1$, it is known that $e = d_E(X, Y)^{\frac{1}{k}}$ is also a distance function. Now lets prove that $\beta_d(n) = n^{1-\frac{1}{k}}$. It satisfies that $\alpha_e(n) = \frac{e_{\max}(n)}{e_{\min}(n)} = \frac{e_{\max}(n)}{e(A_0, A_n)} = \frac{e_{\max}(n)}{(\sqrt{2}a)^{\frac{1}{k}}}$. Let k be the summation of euclidean distance between n number of points. For a fixed constant k , let we consider the maximum for the summation of e -distance between n number of points.

For $d_E(A_0, A_1) + d_E(A_1, A_2) + \dots + d_E(A_{n-1}, A_n) = k$, if we substitute $d_E(A_i, A_{i+1}) = x_{i+1}$, we have $\sum_{s=1}^n x_s = k$. The Cauchy-Schwarz inequality

$(\sum_{s=1}^n x_s^{\frac{1}{k}})^k \leq (\sum_{s=1}^n x_s)(\sum_{s=1}^n 1)^{k-1} = k \times n^{k-1}$. So implies

$$e_{\max}(n) = \sum_{s=1}^n e(A_{s-1}, A_s) = \sum_{s=1}^n x_s^{\frac{1}{k}} \leq k \times n^{1-\frac{1}{k}}$$

is valid. Hence we have $\alpha_e(n) = \frac{e_{\max}(n)}{(\sqrt{2a})^{\frac{1}{k}}} = C \times n^{1-\frac{1}{k}}$ and proves the Theorem. \square

Theorem 2.9. For a metric function $d(X_1, X_2) = \{|x_1 - x_2|^k + |y_1 - y_2|^k\}^{\frac{1}{k}}$, it satisfies $\beta_d(n) = 1$.

Proof. It is known that d is a metric function.[6] Since $(|x_1 - x_2|^k + |y_1 - y_2|^k)^{\frac{1}{k}} \leq |x_i - x_{i+1}| + |y_i - y_{i+1}|$ for $k > 1$. $\sum_i d(A_i, A_{i+1}) = \sum_i (|x_1 - x_2|^k + |y_1 - y_2|^k)^{\frac{1}{k}} \leq |x_n - x_0| + |y_n - y_0| = 2a$. So the α -classification of the metric function d would be $\alpha_d(n) = \frac{d_{\max}(n)}{d_{\min}(n)} = \frac{2a}{2^{\frac{1}{k}}a} = 2^{1-\frac{1}{k}}$ and $\beta_d(n) = 1$. \square

Remark 2.10. We see that for a metric function d classified as $\beta_d(n) = 1$, for every two points upon a plane x, y , there should exist a point z which satisfies $d(x, z) + d(y, z) = d(x, y)$. According to Minkowski's inequality we obtain

$$(|x_1 - x_2|^k + |y_1 - y_2|^k)^{\frac{1}{k}} + (|x_2 - x_3|^k + |y_2 - y_3|^k)^{\frac{1}{k}} \geq (|x_1 - x_3|^k + |y_1 - y_3|^k)^{\frac{1}{k}}$$

That is, $d(X, Y) + d(Y, Z) = d(X, Z)$, if three points X, Y, Z on \mathbf{R}^2 should lie on a straight line.

Theorem 2.11. If d and d' is equivalent in $[0, a]^2$, then $\alpha_d(n) = \Theta(\alpha_{d'}(n))$.

Proof. Because of equivalency between two metric functions, there are p, q , which satisfy $p \cdot d(x, y) \geq d'(x, y) \geq q \cdot d(x, y)$ for all x, y in $[0, a]^2$. Since $d_{\max}(n)$ is defined as sum of n distances, $p \cdot d_{\max}(n) \geq d'_{\max}(n)$, $d'_{\max}(n) \geq q \cdot d_{\max}(n)$. Hence we get

$$\begin{aligned} \frac{\alpha_d(n)}{\alpha_{d'}(n)} &= \frac{d'(O, A) \cdot d_{\max}(n)}{d(O, A) \cdot d'_{\max}(n)} \\ \frac{d'(O, A)}{d(O, A) \cdot q} &\leq \frac{\alpha_d(n)}{\alpha_{d'}(n)} \leq \frac{d'(O, A)}{d(O, A) \cdot p} \end{aligned}$$

Thus $\alpha_d(n) = \theta(\alpha_{d'}(n))$. \square

Remark 2.12. A metric function $d_{x|x|}(P_1, P_2) = |x_1|x_1| - x_2|x_2|| + |y_1|y_1| - y_2|y_2||$ and Manhattan metric $d_T(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$ are not equivalent but $\alpha_{d_{x|x|}}(n) = \Theta(\alpha_{d_T}(n))$. Hence we see that the converse of theorem 4.1.6 is not true.

Theorem 2.13. For a metric function d and convex continuous function $f : R_0^+ \rightarrow R_0^+$, if $f(0) = 0$, then

$$\alpha_{f(d)}(n) = O\left(\frac{f(d_m(n)) \cdot \alpha_d(n)}{d_m(n)}\right),$$

where $d_m(n)$ is smallest $d(A_k, A_{k+1})$ in $(f \circ d)_{\max}(n)$.

Proof. By Jensen inequality, $f(0 + \lambda x) \geq \lambda f(x)$ for $0 \leq \lambda \leq 1$. So we get

$$\frac{f(\lambda x)}{\lambda} \geq f(x), \quad \frac{f(\lambda x)}{\lambda x} \geq \frac{f(x)}{x}$$

and $\frac{f(x)}{x}$ is an descent function. Thus,

$$(f \circ d)_{\min}(n) = f(d_{\min}(n))$$

$$\begin{aligned} (f \circ d)_{\max}(n) &= \sum_{k=0}^{n-1} f(d(A_k, A_{k+1})) \geq \frac{f(d_m(n))}{d_m(n)} \cdot \sum_{k=0}^{n-1} d(A_k, A_{k+1}) \\ &\geq \frac{f(d_m(n))}{d_m(n)} \cdot (f \circ d)_{\max}(n) \end{aligned}$$

$$\alpha_{f(d)}(n) = \frac{(f \circ d)_{\max}(n)}{(f \circ d)_{\min}(n)} \leq \frac{\frac{f(d_m(n))}{d_m(n)} \cdot d_{\max}(n)}{f(d_{\min}(n))} = \frac{f(d_m(n)) \cdot d_{\max}(n)}{d_m(n) \cdot f(d_{\min}(n))}$$

so, $\alpha_{f(d)}(n) = O\left(\frac{f(d_m(n)) \cdot \alpha_d(n)}{d_m(n)}\right)$. □

3. Classification using the area of a triangle having distance functions as length of edge

If d is given metric on R^2 and three points X, Y, Z , then we can consider triangle having $d(X, Y), d(Y, Z), d(Z, X)$ as length of the edges. (Let us consider the case when the three points are on one line too.) Let us say the triangle is $\Delta X'Y'Z'$.

$$d_E(X', Y') = d(X, Y), d_E(Y', Z') = d(Y, Z), d_E(Z', X') = d(Z, X)$$

Let S the area of ΔXYZ be fixed and S'_{\min} is the minimum and S'_{\max} the maximum of the area of $\Delta X'Y'Z'$. Now we can classify distance functions as following.

- Type1: $S'_{\min} = 0, S'_{\max} = \infty$
- Type2: $S'_{\min} = 0, S'_{\max} < \infty, (S'_{\max}, \text{ can be } 0)$
- Type3: $S'_{\min} > 0, S'_{\max} = \infty$
- Type4: $S'_{\min} > 0, S'_{\max} < \infty$

Example 3.1. $d(X, Y) = c > 0 (X \neq Y)$ and $d(X, X) = 0$: It is trivial that this function is a distance function. If X, Y, Z are three different points, $\Delta X'Y'Z'$ is a equilateral triangle with length of an edge c . So,

$$S'_{\min} = S'_{\max} = \frac{\sqrt{3}}{4} c^2$$

Example 3.2. $d(X, Y) = d_E(X, Y) : \Delta X'Y'Z' \equiv \Delta XYZ$. So, $S'_{\min} = S'_{\max} = S$

Example 3.3. $d(X, Y) = d_E(X, Y)^c$ ($0 < c < 1$): By Heron's Formula we get

$$S = \frac{\sqrt{(x+y+z) \prod_{cyc} (x+y-z)}}{4}, S' = \frac{\sqrt{(x^c+y^c+z^c) \prod_{cyc} (x^c+y^c-z^c)}}{4},$$

where $z = d_E(X, Y)$, $y = d_E(Z, X)$, $x = d_E(Y, Z)$. Now consider S'_{max} . For fixed S , if we have $z = x + y - \epsilon$ ($\epsilon \ll x = y$) and, ϵ is very small then we could get x, y very big. In this case

$$x^c + y^c - z^c > x^c + y^c - (x+y)^c = (2-2^c)x^c,$$

and $x^c + y^c + z^c > x^c$, $x^c - y^c + z^c > x^c$, $-x^c + y^c + z^c > x^c$. Hence

$$S'_{max} > \frac{\sqrt{2-2^c}}{4} x^{2c}.$$

If $x \rightarrow \infty$, then we see that $S'_{max} = \infty$

Now let us calculate S'_{min} . We will show the existence of the lower bound of S'_{min} . Let $f(z) = \frac{x^c+y^c-z^c}{(x+y-c)^c}$, then

$$\begin{aligned} f'(z) &= -c(x+y-z)^{c-1} \frac{(x^c+y^c-z^c) - (x+y-z)z^{c-1}}{(x+y-z)^{2c}} \\ &= -c \frac{x^c+y^c-z^{c-1}(x+y)}{(x+y-z)^{1+c}}. \end{aligned}$$

So, we get $f(z)$ is minimum when $z = \left\{ \frac{x^c+y^c}{x+y} \right\}^{\frac{1}{c-1}}$. In this case,

$$f\left(\left(\frac{x^c+y^c}{x+y}\right)^{\frac{1}{c-1}}\right) = \frac{x^c+y^c - \left(\frac{x^c+y^c}{x+y}\right)\left(\frac{x^c+y^c}{x+y}\right)^{\frac{1}{c-1}}}{\left(x+y - \left(\frac{x^c+y^c}{x+y}\right)^{\frac{1}{c-1}}\right)^c} = \frac{x^c+y^c}{(x+y)\left(x+y - \left(\frac{x^c+y^c}{x+y}\right)^{\frac{1}{c-1}}\right)^{c-1}}$$

and

$$\frac{x^c+y^c-z^c}{(x+y-z)^c} \geq \frac{\frac{x^c+y^c}{x+y}}{\left(x+y - \left(\frac{x^c+y^c}{x+y}\right)^{\frac{1}{c-1}}\right)^{c-1}} = \frac{\frac{1+t^c}{1+t}}{\left(1+t - \left(\frac{1+t^c}{1+t}\right)^{\frac{1}{c-1}}\right)^{c-1}} \left(t = \frac{y}{x}\right)$$

So, we get a lower bound given in a function of c . If we denote it by $g(c)$, then

$$\begin{aligned} S'_{min} &\geq \frac{\sqrt{(x+y+z)^c g(c)^3 \prod_{cyc} (x+y-z)^3}}{4} = \frac{\sqrt{g(c)^3}}{4} (4S)^3 \\ &\quad (\because x^c + y^c + z^c \geq (x+y+z)^c) \end{aligned}$$

So, we get a lower bound of S'_{min} .

Example 3.4. For an injective function $f : R^2 \rightarrow R$ and $d(X, Y) = |f(X) - f(Y)|$ is metric function [6]. For three different point X, Y, Z on R^2 , we can assume that $f(X) < f(Y) < f(Z)$. Then

$$d(X, Z) = f(Z) - f(X) = f(Z) - f(Y) + f(Y) - f(X) = d(X, Y) + d(Y, Z).$$

So, we get the S' always 0. Hence $S'_{min} = S'_{max} = 0$.

Example 3.5. $d(X, Y) = d_T(X, Y)$: Let the coordinates of X, Y, Z be $(a_1, b_1), (a_2, b_2), (a_3, b_3)$ respectively. If $a_1 \leq a_2 \leq a_3, b_1 \leq b_2 \leq b_3$ then,

$$\begin{aligned} d(X, Z) &= a_3 - a_1 + b_3 - b_1 \\ &= a_3 - a_2 + b_3 - b_2 + a_2 - a_1 + b_2 - b_1 = d(X, Y) + d(Y, Z), \end{aligned}$$

and we get $S' = 0$, so, $S'_{\min} = 0$. Consider for the case of $S' > 0$. Then without loss of generality, we can say

$$a_1 \leq a_2 \leq a_3, b_1 \leq b_3 \leq b_2.$$

Then, if we let $a_2 - a_1 = a, a_3 - a_2 = b, b_3 - b_1 = c, b_2 - b_3 = d$, then

$$d(X, Y) = a + b + c, d(Y, Z) = b + d, d(Z, X) = a + c + d.$$

So, by Heron's formula $S' = \sqrt{bd(a+c)(a+c+b+d)}$, $S = \frac{bc+bd+ad}{2}$. If we take a, b very big and c, d very small with S fixed, then we can get S' very big. So, $S'_{\max} = \infty$

Lemma 3.6. All real number r can be expressed in the form of

$$r = \sum_{i=-\infty}^{\infty} a_i 10^i (a_i \in Z, -1 \leq a_i \leq 8)$$

uniquely. In this case, this is pseudo-decimal expression. Here we consider $1. - 1 - 1 - 1 - 1 - 1 \dots = 0.88888\dots$ as $8888\dots$

Proof. For all real number r , prove the existence of sequence a_i that satisfies

$$r = \sum_{i=-\infty}^{\infty} a_i 10^i (a_i \in Z, -1 \leq a_i \leq 8).$$

First, consider a big natural number N that $|r| < 10^{N-1}$. Then,

$$10^N > \frac{10^N}{9} + r > \frac{10^N}{9} - 10^{N-1} > 0.$$

So, if we get a decimal expression of $\frac{10^N}{9} + r$,

$$\frac{10^N}{9} + r = \sum_{k=-\infty}^{N-1} x_k 10^k (x_k \in Z, 0 \leq x_k \leq 9).$$

Then, we get

$$r = \sum_{k=-\infty}^{N-1} x_k 10^k - \sum_{k=-\infty}^{N-1} 10^k = \sum_{k=-\infty}^{N-1} (x_k - 1) 10^k.$$

So, if we let

$$a_i = 0 (i \geq N), a_i = x_i - 1 (i < N),$$

then, $r = \sum_{i=-\infty}^{\infty} a_i 10^i$ ($a_i \in Z, -1 \leq a_i \leq 8$). So, if there is a real number r that is expressed in two different ways by

$$r = \sum_{i=-\infty}^{\infty} a_i 10^i = \sum_{i=-\infty}^{\infty} b_i 10^i \quad (a_i, b_i \in Z, -1 \leq a_i, b_i \leq 8).$$

Then, if we let $r_i = a_i - b_i$ then, $0 = \sum_{i=-\infty}^{\infty} r_i 10^i$. Since $-9 \leq r_i \leq 9$, for two sets

$$A = \{i | 1 \leq r_i \leq 9\}, B = \{i | 1 \leq -r_i \leq 9\},$$

$t = \sum_{i \in A} r_i 10^i = \sum_{j \in B} (-r_j) 10^j$. So $A \cap B = \emptyset$ and by the uniqueness of the decimal expression of t , we get A and B are empty. So, we get the uniqueness of pseudo-decimal expression. \square

Theorem 3.7. *When using pseudo-decimal expression, if we define $f : R^2 \rightarrow R$ by*

$$f\left(\sum_{k=-\infty}^{\infty} a_k 10^k, \sum_{k=-\infty}^{\infty} b_k 10^k\right) = \sum_{k=-\infty}^{\infty} (a_k 10^{2k} + b_k 10^{2k+1}),$$

then f is a bijective function.

Proof. For arbitrary $x, y \in R$, we know that pseudo-decimal expression is uniquely exist. By the Theorem 3.6. \square

Theorem 3.8. *Since for all real number r , the pseudo decimal expression of*

$$r = \sum_{k=-\infty}^{\infty} (a_k 10^{2k} + b_k 10^{2k+1})$$

is unique, the corresponding

$$x = \sum_{k=-\infty}^{\infty} a_k 10^k, y = \sum_{k=-\infty}^{\infty} b_k 10^k$$

is also unique too. So, f is a bijective function.

Theorem 3.9. *For $\emptyset \neq S_0 \subseteq R^+$, there exist distance function $d : R^2 \rightarrow R$ such that*

$$S_0 = \{d(X, Y) | X \neq Y\} \dots (*)$$

Proof. By Theorem 3.7, there exist a bijection $f : R^2 \rightarrow R$. Now, let us prove the existence of surjective function $g : R \rightarrow S_0$. First, s_0 is a fixed element of S_0 that satisfies the following conditions.

- (1) When S_0 is bounded below, let s_0 be the greatest lower bound of S_0
- (2) When S_0 is not bounded below, let s_0 be an arbitrary element of S_0 .

Then for $s \in S_0$, $g(s) = s$ and for all $t \notin S_0$, $g(t) = s_0$. Then, we get that g is a surjective function.

If we let $h = f \circ g : R^2 \rightarrow S_0$ then, h is a surjective function, and if

$$A_s = \{x|h(x) = s, x \in R^2\} \neq \emptyset,$$

then, we get

$$\bigcup_{s \in S_0} A_s = R^2, \forall_{s_1 \neq s_2 \in S_0} : A_{s_1} \cap A_{s_2} = \emptyset.$$

Construce $d : R^2 \rightarrow R$ satisfying

- ① $d(x, x) = 0$
- ② $d(x, y) = d(y, x) = \max(s_1, s_2) (x \in A_{s_1}, y \in A_{s_2}, x \neq y)$

For $x \in A_{s_1}, y \in A_{s_2}, z \in A_{s_3}$, if we assume $s_1 \leq s_2 \leq s_3$, then

$$\begin{aligned} d(x, y) &= \max(s_1, s_2) = s_2, \\ d(y, z) &= \max(s_2, s_3) = s_3, \\ d(z, x) &= \max(s_3, s_1) = s_3. \end{aligned}$$

So, for all x, y, z , $d(x, y), d(y, z), d(z, x)$ are the edges of a triangle and this means d is a distance function. Now let's prove that d satisfies (*).

- (1) When S_0 is bounded below, for all $t < s_0, t \notin S_0$. So, $g(t) = s_0$ and that means there are infinitely many x such that $g(x) = s_0$. So, $|A_{s_0}| = \infty$ and for all $x \neq y \in A_{s_0}$, we get $d(x, y) = s_0$, and for different $s_0 \neq s \in S_0$, $x \in A_{s_0}, y \in A_s$, we get $d(x, y) = \max(s, s_0) = s$. Hence $S_0 = \{d(X, Y)|X \neq Y\}$.
- (2) When S_0 is not bounded below, for all $s \in S_0$ there exist $s > s' \in S_0$. So, for $x \in A_s, y \in A_{s'}$

$$d(x, y) = \max(s, s') = s$$

So, we get

$$S_0 = \{d(X, Y)|X \neq Y\}.$$

Hence we complete the proof. □

Remark 3.10. If S_0 is a form of $[a, b]$, then we can divide R^2 into

$$A_a = \{(x, y)|x \leq a\}, A_c = \{(x, y)|x = c\} (a < c < b), A_b = \{(x, y)|x \geq b\},$$

and we can construct a distance function d .

Theorem 3.11. For all real number $\alpha < \beta$, there exist Type4 distance function d that satisfies following condition

$$S'_{min} = \alpha, S'_{max} = \beta.$$

Proof. First, there exist $a > b$ that satisfy

$$\beta = \frac{\sqrt{3}}{4}a^2, \alpha = \frac{b}{4}\sqrt{4a^2 - b^2}.$$

Then, if we think of distance function d that $S_0 = \{a, b\}$ in Theorem 4.8, then actually the possible S' are α and β . So, $S'_{\min} = \alpha, S'_{\max} = \beta$. \square

Definition 3.12. For two distance function d, d' , we say d is half equivalent to d' if and only if there exist real number α that for all x, y , $\frac{d(x,y)}{d'(x,y)} \leq \alpha$.

Theorem 3.13. If Type4 distance function d is an isometric to d_E , d is half equivalent to d_E .

Proof. Let say $d(x, y) = d_E(f(x), f(y))$. First, if $f(x), f(y), f(z)$ lies on a line, $S' = 0$ and we see that if $S > 0$ then, it contradicts to (S'_{\min}, S'_{\max}) is bounded. So, $S = 0$ and that means x, y, z , has to lie on a line too. So, we get that a line on $f(R^2)$ has to be on a line on R^2 too.

Now, let's show that f^{-1} is an affine transformation. Since f is injective and the range of f is R^2 , $f^{-1}: R^2 \rightarrow R^2$ preserve the straight line. So, for points $x, y \in R^2$, the trails of point $z \in R^2$ that the area of Δxyz is S are two lines. Also, for $f(x), f(y) \in R^2$ the trails of point $w \in R^2$ that the area of $\Delta f(x)f(y)w$ is more than S'_{\min} and less than S'_{\max} is a form of two bands on both sides of $\overline{f(x)f(y)}$. If $S(\Delta xyz_1) = S(\Delta xyz_2) = S$, then

$$S'_{\min} \leq S(\Delta f(x)f(y)f(z_1)), S(\Delta f(x)f(y)f(z_2)) \leq S'_{\max},$$

Hence $f(z_1), f(z_2)$ has to be in the form of band mentioned above. In this case, the $z \in R^2$ such that $f(z) \in \overleftarrow{f(z_1)f(z_2)}$ becomes to be $z \in \overleftarrow{z_1z_2}$. Then $S(\Delta xyz) = S$, and so

$$S'_{\min} \leq S(\Delta f(x)f(y)f(z)) \leq S'_{\max}.$$

But, since f is a bijective function, there exist z that $f(z)$ becomes the intersection point of $\overleftarrow{f(z_1)f(z_2)}$ and $\overleftarrow{f(x)f(y)}$ and that is a contradiction because

$$S(\Delta f(x)f(y)f(z)) = 0$$

So, $\overleftarrow{f(z_1)f(z_2)}$ is parallel to $\overleftarrow{f(x)f(y)}$. That is, we proved that two parallel lines transformed by f has to be parallel. Hence, f^{-1} becomes an affine transformation. If we think of the plane as a vector space then there exist matrix A and vector B that $f^{-1}(x) = Ax + B$. Also, since f^{-1} is a injective function, f has a form of

$$f(x) = A^{-1}x - BA^{-1}$$

Now, if

$$f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} a & c \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

then, $f((x, y)) = (ax + by + \alpha, cx + dy + \beta)$. So, for two points $A(x, y), B(z, w)$. Hence we get

$$d_E(A, B) = \sqrt{(x - z)^2 + (y - w)^2},$$

$$d(A, B) = \sqrt{\{a(x - z) + b(y - w)\}^2 + \{c(x - z) + d(y - w)\}^2}$$

$$\leq \sqrt{(a^2 + c^2 + |ab + cd|)(x - z)^2 + (b^2 + d^2 + |ab + cd|)(y - w)^2}.$$

So, without loss of generality, if we let $b^2 + d^2 \geq a^2 + c^2$, then we have

$$\frac{d(A, B)}{d_E(A, B)} \leq \sqrt{b^2 + d^2 + |ab + cd|}.$$

□

Definition 3.14. Let S_{E_1} be the set of $d : R^2 \rightarrow R$ such that there exists a function $f : R^+ \cup \{0\} \rightarrow R^+ \cup \{0\}$ satisfies

- (1) $d(x, y) = f(d_E(x, y))$
- (2) $f(x) = 0 \Leftrightarrow x = 0$
- (3) $f(x) + f(y) \geq f(z)$ for all $y + z \geq x \geq |y - z|$

Theorem 3.15. For the function f in definition 3.13, d is a distance function.

Proof.

- 1) It is trivial that $d(x, y) \geq 0$.
- 2) If $d(x, y) = 0$, $f(d_E(x, y)) = 0$ so $d_E(x, y) = 0$ (\because (2)) then, $x = y$.
Also, if $x = y$, $d_E(x, y) = 0$ so, $d(x, y) = 0$. So, $d(x, y) = 0 \Leftrightarrow x = y$.
- 3) $d(x, y) = f(d_E(x, y)) = f(d_E(y, x)) = d(y, x)$.
- 4) $d(x, y) + d(y, z) = f(d_E(x, y)) + f(d_E(y, z)) \geq f(d_E(x, z)) = d(x, z)$.

□

Definition 3.16. The function f is call to pseudo polynomial if and only if there exist a finite set A that satisfy $f(x) = \sum_{a \in A} g(a)x^a (g(a) \neq 0)$ for some g .

Theorem 3.17. For a distance function $d \in S_{E_1}$, if f is a pseudo polynomial, then there exists real number c that $f(x) = cx$.

Proof. Let there is a Type4 distance function $d \in S_{E_1}$ that f is a pseudo polynomial. In this case, for all positive real number x , $f(x) \geq 0$.

If we let, $d_E(x, y) = c, d_E(y, z) = a, d_E(z, x) = b$, then by Heron's formula

$$S = \frac{1}{4} \sqrt{(a + b + c)(a + b - c)(a + c - b)(b + c - a)},$$

S'

$$= \frac{1}{4} \sqrt{(f(a) + f(b) + f(c))(f(a) + f(b) - f(c))(f(a) + f(c) - f(b))(f(b) + f(c) - f(a))}.$$

In this case S' satisfies $S'_{\min} > 0$ and $S'_{\max} < \infty$.

If $b = c$, then

$$S = \frac{a}{4}\sqrt{4b^2 - a^2}, S' = \frac{f(a)}{4}\sqrt{4f(b)^2 - f(a)^2} > \frac{f(a)}{4}(2f(b) - f(a)).$$

Then, by condition (3) of definition 4.13 we get that for all $x, y > 0, y \leq 2x$,

$$f(y) \leq 2f(x) \dots (**)$$

If the maximal element a_0 of A is $a_0 > 1$ and $g(a_0) > 0$, then

$$\lim_{x \rightarrow \infty} \frac{f(2x)}{f(x)} = 2^{a_0} > 2.$$

It contradicts to (**). Also, if $g(a_0) < 0$, then $\lim_{x \rightarrow \infty} f(x) = -\infty$, and it is a contradiction. So, we get $g(a_0) > 0$ and $a_0 \geq 1$. Now if $2b - a = \epsilon$ (S is fixed), then

$$S = \frac{2b - \epsilon}{4}\sqrt{\epsilon(4b - \epsilon)} < b\sqrt{b\epsilon},$$

and we get

$$b > \left(\frac{S}{\sqrt{\epsilon}}\right)^{\frac{2}{3}}$$

Since f is a pseudo polynomial and $g(a_0) > 0$, there exist some large N such that f is an increasing function in $x \in [N, \infty)$. (Because, $f'(x)$ is also a pseudo polynomial with positive leading coefficient, there exist some large N that for all $x > N$, $f'(x) > 0$.)

Now consider the case when there is an element of A that is smaller than 1.

If $a_0 < 1$, then

$$2f(b) - f(a) = 2f(b) - f(2b - \epsilon) > 2f(b) - f(2b),$$

and

$$\lim_{b \rightarrow \infty} (2f(b) - f(2b - \epsilon)) \geq \lim_{b \rightarrow \infty} (2f(b) - f(2b)) = \lim_{b \rightarrow \infty} g(a_0)(2 - 2^{a_0})b^{a_0} = \infty,$$

which contradicts to d is Type4.

Also, when $a_0 = 1$, if we let

$$h(x) = f(x) - g(1)x,$$

by condition(*), if we let a_1 is the biggest element of A except a_0 , then $g(a_1) > 0$. So,

$$2f(b) - f(a) = 2f(b) - f(2b - \epsilon) > 2f(b) - f(2b),$$

and we get

$$\lim_{b \rightarrow \infty} (2f(b) - f(2b - \epsilon)) \geq \lim_{b \rightarrow \infty} (2h(b) - h(2b)) = \lim_{b \rightarrow \infty} g(a_1)(2 - 2^{a_0})b^{a_0} = \infty,$$

and it contradicts to d is Type4.

So, if $f(x)$ is a pseudo polynomial, then $a_0 = 1$ and the only element of A has to be 1. So, there exists real number c that $f(x) = cx$. \square

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