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A NOTE ON MULTIPLICATIVE (GENERALIZED)-DERIVATION IN SEMIPRIME RINGS

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ABSTRACT. In this article we study two Multiplicative (generalized)- derivations $\mathcal{G}$ and $\mathcal{H}$ that satisfying certain conditions in semiprime rings and tried to find out some information about the associated maps. Moreover, an example is given to demonstrate that the semiprimeness imposed on the hypothesis of the various results is essential.

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1. Introduction

Let $\mathfrak{R}$ be an associative ring. The center of $\mathfrak{R}$ is denoted by $Z(\mathfrak{R})$. An additive map δ from $\mathfrak{R} \to \mathfrak{R}$ is called a derivation of $\mathfrak{R}$ if $\delta(x_1x_2) = \delta(x_1)x_2 + x_1\delta(x_2)$ holds $\forall\ x_1, x_2 \in \mathfrak{R}$. Let $\mathfrak{F} : \mathfrak{R} \to \mathfrak{R}$ be a map associated with another map $\delta : R \to R$ so that $\mathfrak{F}(x_1x_2) = \mathfrak{F}(x_1)x_2 + x_1\delta(x_2)$ holds $\forall\ x_1, x_2 \in \mathfrak{R}$. If $\mathfrak{F}$ is additive and δ is a derivation of $\mathfrak{R}$, then $\mathfrak{F}$ is said to be a generalized derivation of $\mathfrak{R}$ that was introduced by Brešar [2]. In [7], Hvala gave the algebraic study of generalized derivations of prime rings. We note that if $\mathfrak{R}$ has the property that $\mathfrak{R}x_1 = (0)$ implies $x_1 = 0$ and $\psi : \mathfrak{R} \to \mathfrak{R}$ is any function, and $\chi : \mathfrak{R} \to \mathfrak{R}$ is any additive map such that $\chi(x_1x_2) = \psi(x_1)x_2 + x_1\psi(x_2)\ \forall\ x_1, x_2 \in \mathfrak{R}$, then χ is uniquely determined by ψ and moreover ψ must be a derivation by ([2, Remark 1]). Obviously, every derivation is a generalized derivation of $\mathfrak{R}$. Following [5], a multiplicative derivation of $\mathfrak{R}$ is a map $\mathcal{G} : \mathfrak{R} \to \mathfrak{R}$ which satisfies $\mathcal{G}(x_1x_2) = \mathcal{G}(x_1)x_2 + x_1\mathcal{G}(x_2)\ \forall\ x_1, x_2 \in \mathfrak{R}$. Of course these maps are not additive. We consider $\mathbb{R} = \mathbb{C}[0, 1]$, the ring of all continuous (real or complex

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valued) functions and define a map $\mathfrak{g} : \mathfrak{R} \to \mathfrak{R}$ as follows:

$$\mathcal{G}(\mathfrak{g})(x_1) = \begin{cases} \mathfrak{g}(x_1)\log|\mathfrak{f}(x_1)|, & when \mathfrak{f}(x_1) \neq 0 \\ 0, & otherwise. \end{cases}$$

Then, it is easy to verify that $\mathcal{G}$ satisfies $\mathcal{G}(\mathfrak{f}\mathfrak{g}) = \mathcal{G}(\mathfrak{f})\mathrm{g}+\mathfrak{f}\mathcal{G}(\mathfrak{g})\ \forall\ \mathfrak{f}, \mathfrak{g} \in \mathbb{C}[0, 1]$, but $\mathcal{G}$ is not additive. Daif and Tammam's [6] extended multiplicative generalized derivations as follows: a map $\mathcal{G} : \mathfrak{R} \to \mathfrak{R}$ is called a multiplicative generalized derivation if there exists a derivation $\mathfrak{g}$ such that $\mathcal{G}(x_1x_2) = \mathcal{G}(x_1)x_2 + x_1\mathfrak{g}(x_2)\ \forall\ x_1, x_2 \in \mathfrak{R}$. The notion of multiplicative (generalized)-derivation introduced by Dhara and Ali [3] as follows: a map $\mathcal{G} : \mathfrak{R} \to \mathfrak{R}$ (not necessarily additive) is said to be a multiplicative (generalized)-derivation if $\mathcal{G}(x_1x_2) = \mathcal{G}(x_1)x_2 + x_1\mathfrak{g}(x_2)$ holds $\forall\ x_1, x_2 \in \mathfrak{R}$, where $\mathfrak{g}$ is any map (not necessarily a derivation or an additive map). Hence, the concept of multiplicative (generalized)-derivation covers the concept of multiplicative derivation. Moreover, if $\mathfrak{g} = 0$ the multiplicative (generalized)-derivation covers the notion of multiplicative centralizers (not necessarily additive). One can find an example of multiplicative generalized derivation, which is neither a derivation nor generalized derivation.

Example 1.1. Consider t

$$\mathfrak{R} = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}.$$

Define $\mathcal{G}, \mathfrak{g} : R \to R$ as

$$\mathcal{G}\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a^2c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \mathfrak{g}\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a^2 & cb \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then it is straightforward to verify that $\mathcal{G}$ is not additive map in $\mathfrak{R}$, Hence $\mathcal{G}$ is a multiplicative (generalized)- derivation associated with the mapping $\mathfrak{g}$ on $\mathfrak{R}$, but $\mathcal{G}$ is not a generalized derivation of $\mathfrak{R}$.

Motivated by the results obtained by Tiwari et.al. [9] in the present paper we study semiprime ring admitting two multiplicative (generalized)-derivations $\mathcal{G}$, $\mathcal{H}$ associated with the mappings $\mathfrak{g}$, $\mathfrak{h}$ respectively and φ be a any mapping satisfying certain identities on a asubset of (i) $\mathcal{G}(x_1x_2) + \mathcal{H}(x_1)\mathcal{H}(x_2) \pm x_1x_2 \in Z(\mathfrak{R})$ (ii) $\mathcal{G}(x_1x_2)+\mathcal{H}(x_1)\mathcal{H}(x_2)\pm x_2x_1 \in Z(\mathfrak{R})$ (iii) $\mathcal{G}(x_1x_2)+\mathcal{H}(x_2)\mathcal{H}(x_1)\pm x_2x_1 \in Z(\mathfrak{R})$ (iv) $\mathcal{G}(x_1x_2) + \mathcal{H}(x_2)\mathcal{H}(x_1) \pm [x_1, \varphi(x_2)] \in Z(\mathfrak{R})$ (v) $\mathcal{G}(x_1x_2) + \mathcal{H}(x_2)\mathcal{H}(x_1) \pm [\varphi(x_1), x_2] \in Z(\mathfrak{R})\ \forall\ x_1, x_2 \in \mathfrak{I}$, where $\mathfrak{I}$ is a nonzero ideal.

2. Preliminaries

We shall use without explicit mention the following basic identities:

$[x_1x_2, x_3] = [x_1, x_3]x_2 + x_1[x_2, x_3]$ and $[x_1, x_3x_2] = [x_1, x_3]x_2 + x_3[x_1, x_2]$.

we begin with the following known lemmas:

Lemma 2.1. [1, Theorem 3] *Let $\mathfrak{R}$ be a semiprime ring and $\mathfrak{I}$ a nonzero left ideal of $\mathfrak{R}$. If $\mathfrak{R}$ admits a derivation $\mathfrak{g}$ which is nonzero on $\mathfrak{I}$ and centralizing on $\mathfrak{I}$, then $\mathfrak{R}$ contains a nonzero central ideal.*

Lemma 2.2. [4, Fact-4] *Let $\mathfrak{R}$ be a semiprime ring, $\mathfrak{g}$ a nonzero derivation of $\mathfrak{R}$ such that $x_1[[\mathfrak{g}(x_1), x_1], x_1] = 0\ \forall\ x_1 \in \mathfrak{R}$. Then $\mathfrak{g}$ maps $\mathfrak{R}$ into its center.*

3. Main results

Theorem 3.1. *Let $\mathfrak{R}$ be a semiprime ring, $\mathfrak{I}$ be a nonzero left ideal of $\mathfrak{R}$. If $\mathfrak{R}$ admits a multiplicative (generalized)-derivations $\mathcal{G}$ and $\mathcal{H}$ associated with the mappings $\mathfrak{g}$ and $\mathfrak{h}$ respectively on $\mathfrak{R}$ such that $\mathcal{G}(x_1x_2) + \mathcal{H}(x_1)\mathcal{H}(x_2) \pm x_1x_2 \in Z(\mathfrak{R})\ \forall\ x_1, x_2 \in \mathfrak{I}$, then $\mathfrak{I}[\mathfrak{h}(x_3), x_3] = (0)$ and $\mathfrak{I}[\mathfrak{g}(x_3), x_3] = (0)\ \forall\ x_3 \in \mathfrak{I}$.*

Proof. We have

$$\mathcal{G}(x_1x_2) + \mathcal{H}(x_1)\mathcal{H}(x_2) - x_1x_2 \in Z(\mathfrak{R})\ \forall\ x_1, x_2 \in \mathfrak{I}. \tag{1}$$

Replace x_2 by x_2x_3 in (1)and use (1), to get

$$[x_1x_2\mathfrak{g}(x_3) + \mathcal{H}(x_1)x_2\mathfrak{h}(x_3), x_3] = 0. \tag{2}$$

Replacing x_1 by x_1x_3 in (2), we have

$$[x_1x_3x_2\mathfrak{g}(x_3), x_3] + [\mathcal{H}(x_1)x_3x_2\mathfrak{h}(x_3), x_3] + [x_1\mathfrak{h}(x_3)x_2\mathfrak{h}(x_3), x_3] = 0 \tag{3}$$

Putting $x_2 = x_3x_2$ in (2) yields that

$$[x_1x_3x_2\mathfrak{g}(x_3), x_3] + [\mathcal{H}(x_1)x_3x_2\mathfrak{h}(x_3), x_3] = 0 \tag{4}$$

Comparing (4) and (3), we get

$$[x_1\mathfrak{h}(x_3)x_2\mathfrak{h}(x_3), x_3] = 0 \tag{5}$$

In (5), we replace x_1 with $\mathfrak{h}(x_3)x_1$ and from (5) we find that

$$[\mathfrak{h}(x_3), x_3]x_1\mathfrak{h}(x_3)x_2\mathfrak{h}(x_3) = 0$$

$\forall\ x_1, x_2, x_3 \in \mathfrak{I}$. This implies that $[\mathfrak{h}(x_3), x_3]x_1[\mathfrak{h}(x_3), x_3]x_2[\mathfrak{h}(x_3), x_3] = 0\ \forall\ x_1, x_2, x_3 \in \mathfrak{I}$. Thus, $(\mathfrak{I}[\mathfrak{h}(x_3), x_3])^3 = (0)\ \forall\ x_3 \in \mathfrak{I}$. Since $\mathfrak{R}$ is semiprime, it contains no nilpotent left ideal, implying $\mathfrak{I}[\mathfrak{h}(x_3), x_3] = (0)\ \forall\ x_3 \in \mathfrak{I}$, as desired.

Again from from equation (2) and the condition $\mathfrak{I}[\mathfrak{h}(x_3), x_3] = (0)$, we have

$$[x_1x_2\mathfrak{g}(x_3), x_3] + [\mathcal{H}(x_1), x_3]x_2\mathfrak{h}(x_3) + \mathcal{H}(x_1)[x_2, x_3]\mathfrak{h}(x_3) = 0 \tag{6}$$

Replacing x_2 by x_2x_3, we get

$$[x_1x_2x_3\mathfrak{g}(x_3), x_3] + [\mathcal{H}(x_1), x_3]x_2x_3\mathfrak{h}(x_3) + \mathcal{H}(x_1)[x_2, x_3]x_3\mathfrak{h}(x_3) = 0 \tag{7}$$

Right multiplying (6) by x_3 and subtraction from (7) and using the fact that $\mathfrak{I}[\mathfrak{h}(x_3), x_3] = (0)$,we find that

$$[x_1x_2, x_3][\mathfrak{g}(x_3), x_3] + x_1x_2[[\mathfrak{g}(x_3), x_3], x_3] = 0 \tag{8}$$

Now, replacing x_1 by tx_1 in (8) and using (8), we obtain $[t, x_3]x_1x_2[\mathfrak{g}(x_3), x_3] = 0$ $\forall\ x_1, x_2, x_3, t \in \mathfrak{I}$ and again replace t by $\mathfrak{g}(x_3)t$, to get

$$[\mathfrak{g}(x_3), x_3]tx_1x_2[\mathfrak{g}(x_3), x_3] = 0 \forall x_1, x_2, x_3, t \in \mathfrak{I}.$$

Replacing x_1 by $[\mathfrak{g}(x_3), x_3]$ we get

$$[\mathfrak{g}(x_3), x_3]t[\mathfrak{g}(x_3), x_3]x_2[\mathfrak{g}(x_3), x_3] = 0 \forall x_2, x_1, t \in \mathfrak{I}.$$

Thus, $(\mathfrak{I}[\mathfrak{g}(x_3), x_3])^3 = (0)\ \forall\ x_3 \in \mathfrak{I}$. Since $\mathfrak{R}$ is semiprime, it contains no nilpotent left ideal, we conclude that $\mathfrak{I}[\mathfrak{g}(x_3), x_3] = (0)\ \forall\ x_3 \in \mathfrak{I}$.

We may obtain the same conclusion by the same argument, when $\mathcal{G}(x_1x_2) + \mathcal{H}(x_1)\mathcal{H}(x_2) + x_1x_2 \in Z(\mathfrak{R})\ \forall\ x_1, x_2 \in \mathfrak{I}$. □

Using a similar approach as above we can prove the following:

Theorem 3.2. *Let $\mathfrak{R}$ be a semiprime ring, $\mathfrak{I}$ be a nonzero left ideal of $\mathfrak{R}$. If $\mathfrak{R}$ admits a multiplicative (generalized)-derivations $\mathcal{G}$ and $\mathcal{H}$ associated with the mappings $\mathfrak{g}$ and $\mathfrak{h}$ respectively on $\mathfrak{R}$ such that $\mathcal{G}(x_1x_2) - \mathcal{H}(x_1)\mathcal{H}(x_2) \pm x_1x_2 \in Z(\mathfrak{R})\ \forall\ x_1, x_2 \in \mathfrak{I}$, then $\mathfrak{I}[\mathfrak{h}(x_3), x_3] = (0)$ and $\mathfrak{I}[\mathfrak{g}(x_3), x_3] = (0)\ \forall\ x_3 \in \mathfrak{I}$.*

Corollary 3.3. *Let $\mathfrak{R}$ be a semiprime ring. If $\mathfrak{R}$ admits a multiplicative (generalized) - derivations $\mathcal{G}$ and $\mathcal{H}$ associated with the mappings $\mathfrak{g}$ and $\mathfrak{h}$ respectively on $\mathfrak{R}$ such that $\mathcal{G}(x_1x_2) - \mathcal{H}(x_1)\mathcal{H}(x_2) \pm x_1x_2 \in Z(\mathfrak{R})\ \forall\ x_1, x_2 \in \mathfrak{R}$, then $\mathfrak{h}$ is a commuting map on $\mathfrak{R}$ and $\mathfrak{g}$ is a commuting map on $\mathfrak{R}$.*

In view of Theorem 3.1 and Lemma 2.1, we immediately get the following corollary.

Corollary 3.4. *Let $\mathfrak{R}$ be a semiprime ring and $\mathfrak{R}$ admitting two multiplicative (generalized)- derivations $\mathcal{G}$ and $\mathcal{H}$ associated with the derivations $\mathfrak{g}$ and $\mathfrak{h}$ respectively on $\mathfrak{R}$. If $\mathcal{G}(x_1x_2) + \mathcal{H}(x_1)\mathcal{H}(x_2) \pm x_1x_2 \in Z(\mathfrak{R})\ \forall\ x_1, x_2 \in \mathfrak{R}$, then $\mathfrak{h} = 0$ and $\mathfrak{g} = 0$ or $\mathfrak{R}$ contains a nonzero central ideal.*

Theorem 3.5. *Let $\mathfrak{R}$ be a semiprime ring, $\mathfrak{I}$ be a nonzero left ideal of $\mathfrak{R}$. If $\mathfrak{R}$ admits a multiplicative (generalized)-derivations $\mathcal{G}$ and $\mathcal{H}$ associated with the mappings $\mathfrak{g}$ and $\mathfrak{h}$ respectively on $\mathfrak{R}$ such that $\mathcal{G}(x_1x_2) + \mathcal{H}(x_1)\mathcal{H}(x_2) \pm x_2x_1 \in Z(\mathfrak{R})\ \forall\ x_1, x_2 \in \mathfrak{I}$, then $\mathfrak{I}[\mathfrak{h}(x_3), x_3] = (0)\ \forall\ x_3 \in \mathfrak{I}$.*

Proof. We have

$$\mathcal{G}(x_1x_2) + \mathcal{H}(x_1)\mathcal{H}(x_2) + x_2x_1 \in Z(\mathfrak{R})\ \forall\ x_1, x_2 \in \mathfrak{I}. \tag{9}$$

Replacing x_2 with x_2x_3 in (9), we get

$$\begin{aligned}&\{\mathcal{G}(x_1x_2) + \mathcal{H}(x_1)\mathcal{H}(x_2) + x_2x_1\}x_3 + x_1x_2\mathfrak{g}(x_3)\\&+\mathcal{H}(x_1)x_2\mathfrak{h}(x_3) + x_2[x_3, x_1] \in Z(\mathfrak{R})\end{aligned} \tag{10}$$

$\forall\ x_1, x_2, x_3 \in \mathfrak{I}$. Commuting both sides with x_3, we obtain

$$[x_1x_2\mathfrak{g}(x_3), x_3] + [\mathcal{H}(x_1)x_2\mathfrak{h}(x_3), x_3] + [x_2[x_3, x_1], x_3] = 0. \tag{11}$$

Putting $x_1 = x_1x_3$ in the above relation we find that

$$\begin{array}{l}[x_1x_3x_2\mathfrak{g}(x_3), x_3] + [(\mathcal{H}(x_1)x_3 + x_1\mathfrak{h}(x_3))x_2\mathfrak{h}(x_3), x_3] \\ +[x_2[x_3, x_1x_3], x_3] = 0.\end{array} \tag{12}$$

Putting $x_2 = x_3x_2$ in (11), we get

$$[x_1x_3x_2\mathfrak{g}(x_3), x_3] + [\mathcal{H}(x_1)x_3x_2\mathfrak{h}(x_3), x_3] + x_3[x_2[x_3, x_1], x_3] = 0. \tag{13}$$

Subtracting (13) from (12), we have

$$[x_1\mathfrak{h}(x_3)x_2\mathfrak{h}(x_3), x_3] + [[x_2[x_3, x_1], x_3], x_3] = 0. \tag{14}$$

Putting $x_1 = x_1x_3$, the above relation gives that

$$[[x_2[x_3, x_1], x_3], x_3]x_3 + [x_1x_3\mathfrak{h}(x_3)x_2\mathfrak{h}(x_3), x_3] = 0. \tag{15}$$

Right multiplying (14) by x_3 and then subtracting it from (15), we get

$$[x_1[\mathfrak{h}(x_3)x_2\mathfrak{h}(x_3), x_3], x_3] = 0 \tag{16}$$

$\forall\ x_1, x_2, x_3 \in I$. Now we substitute $\mathfrak{h}(x_3)x_2\mathfrak{h}(x_3)x_1$ for x_1 in (16) and get

$$\begin{array}{rcl} 0 &=& [\mathfrak{h}(x_3)x_2\mathfrak{h}(x_3)x_1[\mathfrak{h}(x_3)x_2\mathfrak{h}(x_3), x_3], x_3] \\ &=& \mathfrak{h}(x_3)x_2\mathfrak{h}(x_3)[x_1[\mathfrak{h}(x_3)x_2\mathfrak{h}(x_3), x_3], x_3] \\ && +[\mathfrak{h}(x_3)x_2\mathfrak{h}(x_3), x_3]x_1[\mathfrak{h}(x_3)x_2\mathfrak{h}(x_3), x_3]. \end{array} \tag{17}$$

Using (16), it reduces to $[\mathfrak{h}(x_3)x_2\mathfrak{h}(x_3), x_3]x_1[\mathfrak{h}(x_3)x_2\mathfrak{h}(x_3), x_3] = 0\ \forall$ $x_1, x_2, x_3 \in \mathfrak{I}$. Since $\mathfrak{I}$ is a left ideal, it follows that

$$x_1[\mathfrak{h}(x_3)x_2\mathfrak{h}(x_3), x_3]\mathfrak{R}x_1[\mathfrak{h}(x_3)x_2\mathfrak{h}(x_3), x_3] = (0)$$

and hence $x_1[\mathfrak{h}(x_3)x_2\mathfrak{h}(x_3), x_3] = 0$ that is,

$$x_1(\mathfrak{h}(x_3)x_2\mathfrak{h}(x_3)x_3 - x_3\mathfrak{h}(x_3)x_2\mathfrak{h}(x_3)) = 0. \tag{18}$$

Now we put $x_2 = x_2\mathfrak{h}(x_3)x_1$, where $x_1 \in \mathfrak{I}$, and then obtain

$$x_1(\mathfrak{h}(x_3)x_2\mathfrak{h}(x_3)x_1\mathfrak{h}(x_3)x_3 - x_3\mathfrak{h}(x_3)x_2\mathfrak{h}(x_3)x_1\mathfrak{h}(x_3)) = 0.$$

By (18), this can be written as

$$x_1(\mathfrak{h}(x_3)x_2x_3\mathfrak{h}(x_3)x_1\mathfrak{h}(x_3) - \mathfrak{h}(x_3)x_2\mathfrak{h}(x_3)x_3x_1\mathfrak{h}(x_3)) = 0$$

that is, $x_1\mathfrak{h}(x_3)x_2[\mathfrak{h}(x_3), x_3]x_1\mathfrak{h}(x_3) = 0$. This implies

$$x_1[\mathfrak{h}(x_3), x_3]x_2[\mathfrak{h}(x_3), x_3]x_1[\mathfrak{h}(x_3), x_3] = 0$$

and hence we find that $(I[\mathfrak{g}(x_3), x_3])^3 = (0)\ \forall\ x_3 \in \mathfrak{I}$. Since a semiprime ring contains no nonzero nilpotent left ideals (see [17]), it follows that $\mathfrak{I}[\mathfrak{h}(x_3), x_3] = (0)$ $\forall\ x_3 \in \mathfrak{I}$, as desired.

By the similar technique, the same conclusion holds for $\mathcal{G}(x_1x_2)+\mathcal{H}(x_1)\mathcal{H}(x_2)-x_2x_1 \in Z(\mathfrak{R})\ \forall\ x_1, x_2 \in \mathfrak{I}$. □

Using the similar approach as used in the proof of Theorem 3.5 one can prove the following:

Theorem 3.6. *Let $\mathfrak{R}$ be a semiprime ring, $\mathfrak{I}$ be a nonzero left ideal of $\mathfrak{R}$. If $\mathfrak{R}$ admits a multiplicative (generalized)-derivations $\mathcal{G}$ and $\mathcal{H}$ associated with the mappings $\mathfrak{g}$ and $\mathfrak{h}$ respectively on $\mathfrak{R}$ such that $\mathcal{G}(x_1x_2) - \mathcal{H}(x_1)\mathcal{H}(x_2) \pm x_2x_1 \in Z(\mathfrak{R})$ $\forall$ $x_1, x_2 \in \mathfrak{I}$, then $\mathfrak{I}[\mathfrak{h}(x_3), x_3] = (0)$ $\forall$ $x_3 \in \mathfrak{I}$.*

Corollary 3.7. *Let $\mathfrak{R}$ be a semiprime ring and $\mathfrak{R}$ admitting two multiplicative (generalized)-derivations $\mathcal{G}$ and $\mathcal{H}$ associated with the mappings $\mathfrak{g}$ and $\mathfrak{h}$ respectively on $\mathfrak{R}$. If $\mathcal{G}(x_1x_2) + \mathcal{H}(x_1)\mathcal{H}(x_2) \pm x_2x_1 \in Z(\mathfrak{R})$ $\forall$ $x_1, x_2 \in \mathfrak{R}$, then $\mathfrak{h}$ is a commuting map on $\mathfrak{R}$.*

In view of Theorem 3.5 and Lemma 2.1, we immediately get the following corollary.

Corollary 3.8. *Let R be a semiprime ring and $\mathfrak{R}$ admitting two multiplicative (generalized)-derivations $\mathcal{G}$ and $\mathcal{H}$ associated with a derivation $\mathfrak{h} : \mathfrak{R} \to \mathfrak{R}$ and a mapping $\mathfrak{g} : \mathfrak{R} \to \mathfrak{R}$ respectively. If $\mathcal{G}(x_1x_2) + \mathcal{H}(x_1)\mathcal{H}(x_2) \pm x_2x_1 \in Z(\mathfrak{R})$ $\forall$ $x_1, x_2 \in \mathfrak{R}$, then $\mathfrak{h} = 0$ or $\mathfrak{R}$ contains a nonzero central ideal.*

Theorem 3.9. *Let $\mathfrak{R}$ be a semiprime ring, $\mathfrak{I}$ be a nonzero left ideal of $\mathfrak{R}$. If $\mathfrak{R}$ admits a multiplicative (generalized)-derivations $\mathcal{G}$ and $\mathcal{H}$ associated with the mappings $\mathfrak{g}$ and $\mathfrak{h}$ respectively on $\mathfrak{R}$ such that If $\mathcal{G}(x_1x_2) + \mathcal{H}(x_2)\mathcal{H}(x_1) \pm x_2x_1 \in Z(\mathfrak{R})$ $\forall$ $x_1, x_2 \in \mathfrak{I}$, then $x_1[\mathfrak{h}(x_1), x_1]_2 = (0)$ $\forall$ $x_1 \in \mathfrak{I}$.*

Proof. By our hypothesis

$$\mathcal{G}(x_1x_2) + \mathcal{H}(x_2)\mathcal{H}(x_1) + x_2x_1 \in Z(\mathfrak{R}) \ \forall \ x_1, x_2 \in \mathfrak{I}. \tag{19}$$

Replace x_1 by x_1x_3, to get

$$\mathcal{G}(x_1)x_3x_2 + x_1\mathfrak{g}(x_3x_2) + \mathcal{H}(x_2)(\mathcal{H}(x_1)x_3 + x_1\mathfrak{h}(x_3)) + x_2x_1x_3 \in Z(\mathfrak{R}) \tag{20}$$

that is

$$\begin{aligned}&\mathcal{G}(x_1)x_3x_2 + x_1\mathfrak{g}(x_3x_2) - \mathcal{G}(x_1x_2)x_3 + \mathcal{H}(x_2)x_1\mathfrak{h}(x_3)\\ &+(\mathcal{G}(x_1x_2) + \mathcal{H}(x_2)\mathcal{H}(x_1) + x_2x_1)x_3 \in Z(\mathfrak{R}).\end{aligned} \tag{21}$$

Since $\mathcal{G}(x_1x_2) + \mathcal{H}(x_2)\mathcal{H}(x_1) + x_2x_1 \in Z(\mathfrak{R})$, we obtain

$$[(\mathcal{G}(x_1x_2) + \mathcal{H}(x_2)\mathcal{H}(x_1) + x_2x_1)x_3, x_3] = 0.$$

Thus we find that

$$[\mathcal{G}(x_1)[x_3, x_2], x_3] + [x_1\mathfrak{g}(x_3x_2)x_1\mathfrak{g}(x_2)x_3, x_3] + [\mathcal{H}(x_2)x_1\mathfrak{h}(x_3), x_3] = 0. \tag{22}$$

Substituting x_3^2 in place of x_2 in (22), we get

$$[x_1x_3^2\mathfrak{g}(x_3), x_3] + [\mathcal{H}(x_3)x_3x_1\mathfrak{h}(x_3), x_3] + [x_3\mathfrak{h}(x_3)x_1\mathfrak{h}(x_3), x_3] = 0. \tag{23}$$

Again, replacing x_1 by x_3x_1 and x_2 by x_3 in (22), we obtain

$$x_3[x_1x_3\mathfrak{g}(x_3), x_3] + [\mathcal{H}(x_3)x_3x_1\mathfrak{h}(x_3), x_3] = 0. \tag{24}$$

comparing (24) and (23), we gte

$$[[x_1, x_3]x_3\mathfrak{g}(x_3), x_3] + [x_3\mathfrak{h}(x_3)x_1\mathfrak{h}(x_3), x_3] = 0. \tag{25}$$

Now replace x_1 with x_3x_1 in (25), to get

$$x_3[[x_1,x_3]x_3\mathfrak{g}(x_3),x_3]+[x_3\mathfrak{h}(x_3)x_3x_1\mathfrak{h}(x_3),x_3]=0. \tag{26}$$

Multiplying (25) from the left by x_3 in and then comparing with (26), we find that

$$[x_3[\mathfrak{h}(x_3),x_3]x_1\mathfrak{h}(x_3),x_3]=0. \tag{27}$$

Again putting $x_1=x_1x_3$ in (25), we get

$$[x_3[\mathfrak{h}(x_3),x_3]x_1x_3\mathfrak{h}(x_3),x_3]=0. \tag{28}$$

Now right multiplying (27) by x_3 and comparing with (28), we have $[x_3[\mathfrak{h}(x_3),x_3]x_1[\mathfrak{h}(x_3),x_3],x_3]=0$ and hence

$$[x_3[\mathfrak{h}(x_3),x_3]x_1x_3[\mathfrak{h}(x_3),x_3],x_3]=0 \tag{29}$$

Let $\lambda(x_3)=x_1[\mathfrak{h}(x_3),x_3]$. This implies $[\lambda(x_3)x_1\lambda(x_1),x_3]$, that is,

$$\lambda(x_3)x_1\lambda(x_1)x_3-x_3\lambda(x_3)x_1\lambda(x_3) \tag{30}$$

$\forall\ x_1,x_3\in\mathfrak{I}$. In (30), replacing x_1 with $x_1\lambda(x_3)u_1$, where $u_1\in\mathfrak{I}$, we obtain

$$\lambda(x_3)x_1\lambda(x_3)u_1\lambda(x_3)x_3-x_3\lambda(x_3)x_1\lambda(x_3)u_1\lambda(x_3)=0. \tag{31}$$

Using (30) and (31) gives that

$$\lambda(x_3)x_1x_3\lambda(x_3)u_1\lambda(x_3)-\lambda(x_3)x_1\lambda(x_3)x_3u_1\lambda(x_3)=0$$

that is $\lambda(x_3)x_1[\lambda(x_3),x_3]u_1\lambda(x_3)=0\ \forall\ x_1,u_1,x_3\in\mathfrak{I}$. This implies

$$[\lambda(x_3),x_3]x_1[\lambda(x_3),x_3]u_1[\lambda(x_3),x_3]=0$$

$\forall\ x_1,u_1,x_3\in\mathfrak{I}$ and so $(\mathfrak{I}[\lambda(x_3),x_3])^3=0\ \forall\ x_3\in I$. Since R is semiprime, it contains no nilpotent left ideal, implying $I[\lambda(x_3),x_3]=0\ \forall\ x_3\in\mathfrak{I}$ that is, $\mathfrak{I}[[\mathfrak{h}(x_3),x_3],x_3]=(0)$, as desired.

In the similar manner, we can prove the same conclusion for $\mathcal{G}(x_1x_2)+\mathcal{H}(x_2)\mathcal{H}(x_1)x_2x_1\in Z(\mathfrak{R})\ \forall\ x_1,x_2\in\mathfrak{I}$. □

Theorem 3.10. *Let $\mathfrak{R}$ be a semiprime ring, $\mathfrak{I}$ be a nonzero left ideal of $\mathfrak{R}$. If $\mathfrak{R}$ admits a multiplicative (generalized)-derivations $\mathcal{G}$ and $\mathcal{H}$ associated with the mappings $\mathfrak{g}$ and $\mathfrak{h}$ respectively on $\mathfrak{R}$ such that $\mathcal{G}(x_1x_2)-\mathcal{H}(x_2)\mathcal{H}(x_1)\pm x_2x_1\in Z(\mathfrak{R})\ \forall\ x_1,x_2\in\mathfrak{I}$, then $x_1[\mathfrak{h}(x_1),x_1]_2=(0)\ \forall\ x_1\in\mathfrak{I}$.*

Proof. If we replace $\mathcal{G}$ with $-\mathcal{G}$ and $\mathfrak{h}$ with $-\mathfrak{h}$ in Theorem 3.5, we conclude that $(-\mathcal{G})(x_1x_2)+\mathcal{H}(x_2)\mathcal{H}(x_1)\pm x_2x_1\in Z(\mathfrak{R})\ \forall\ x_1,x_2\in\mathfrak{I}$, implies that $I[(-h)(x_1),x_1]_2=(0)\ \forall\ x_1\in I$, that is $\mathcal{G}(x_1x_2)-\mathcal{H}(x_2)\mathcal{H}(x_2)\mp x_2x_1\in Z(\mathfrak{R})\ \forall\ x_1,x_2\in\mathfrak{I}$, implies that $x_1[\mathfrak{h}(x_1),x_1]_2=(0)\ \forall\ x_1\in\mathfrak{I}$, as desired.

□

Corollary 3.11. *Let $\mathfrak{R}$ be a semiprime ring, $\mathfrak{R}$ admitting two multiplicative (generalized)-derivations $\mathcal{G}$ and $\mathcal{H}$ associated with the mapping $\mathfrak{g}$ and $\mathfrak{h}$ respectively. If $\mathcal{G}(x_1x_2)+\mathcal{H}(x_2)\mathcal{H}(x_1)\pm x_2x_1\in Z(\mathfrak{R})\ \forall\ x_1,x_2\in\mathfrak{R}$, then $\mathfrak{h}$ is a centralizing map on $\mathfrak{R}$.*

In view of Theorem 3.9, Lemma 2.2 and Lemma 2.1 we immediately get the following corollary.

Corollary 3.12. *Let $\mathfrak{R}$ be a semiprime ring, $\mathfrak{R}$ admitting two multiplicative (generalized)-derivations $\mathcal{G}$ and $\mathcal{H}$ associated with a derivation $\mathfrak{h}$ and a mapping $\mathfrak{g}$ respectively. If $\mathcal{G}(x_1x_2)+\mathfrak{H}(x_2)\mathcal{H}(x_1)\pm x_2x_1 \in Z(\mathfrak{R})\ \forall\ x_1,x_2\in\mathfrak{R}$, then $\mathfrak{g}=0$ or $\mathfrak{R}$ contains a nonzero central ideal.*

Theorem 3.13. *Let $\mathfrak{R}$ be a semiprime ring, $\mathfrak{I}$ be a nonzero left ideal of $\mathfrak{R}$ and $\varphi : R\to R$ any mapping. If $\mathfrak{R}$ admits a multiplicative (generalized)-derivations $\mathcal{G}$ and $\mathcal{H}$ associated with the mappings $\mathfrak{g}$ and $\mathfrak{h}$ respectively on $\mathfrak{R}$ such that $\mathcal{G}(x_1x_2)+\mathcal{H}(x_2)\mathcal{H}(x_1)\pm[x_1,\varphi(x_2)]\in Z(\mathfrak{R})\ \forall\ x_1,x_2\in\mathfrak{I}$, then $x_1[\mathfrak{h}(x_1),x_1]_2=0$, $\forall\ x_1\in\mathfrak{I}$.*

Proof. We begin with the hypothesis

$$\mathcal{G}(x_1x_2)+\mathcal{H}(x_2)\mathcal{H}(x_1)+[x_1,\varphi(x_2)]\in Z(\mathfrak{R})\ \forall\ x_1,x_2\in\mathfrak{I}. \tag{32}$$

Now replacing x_1 with x_1x_3,we obtain

$$\begin{aligned}&\mathcal{G}(x_1)x_3x_2+x_1\mathfrak{g}(x_3x_2)+\mathcal{H}(x_2)\mathcal{H}(x_1)x_3+\mathcal{H}(x_2)x_1\mathfrak{h}(x_3)\\&+x_1[x_3,\varphi(x_2)]+[x_1,\varphi(x_2)]x_3\in Z(\mathfrak{R}).\end{aligned}$$

This relation can be re-written as

$$\begin{aligned}&(\mathcal{G}(x_1x_2)+\mathcal{H}(x_2)\mathcal{H}(x_1)+[x_1,\varphi(x_2)])x_3-\mathcal{G}(x_1x_2)x_3+\mathcal{G}(x_1)x_3x_2\\&+x_1\mathfrak{g}(x_3x_2)+\mathcal{H}(x_2)x_1\mathfrak{h}(x_3)+x_1[x_3,\varphi(x_2)]\in Z(\mathfrak{R}).\end{aligned}$$

Now commuting both sides with x_3 and then using equation (32), we obtain

$$[\mathcal{G}(x_1)x_3x_2+x_1\mathfrak{g}(x_3x_2)-\mathcal{G}(x_1x_2)x_3+\mathcal{H}(x_2)x_1\mathcal{H}(x_3)+x_1[x_3,\varphi(x_2)],x_3]=0$$

that is

$$\begin{aligned}&[\mathcal{G}(x_1)[x_3,x_2],x_3]+[x_1\mathfrak{g}(x_3x_2)-x_1\mathfrak{g}(x_2)x_3,x_3]\\&+[\mathcal{H}(x_2)x_1\mathfrak{h}(x_3),x_3]+[x_1[x_3,\varphi(x_2)],x_3]=0.\end{aligned}\tag{33}$$

Now substituting x_3x_1 for x_1 and x_3 for x_2 in above relation, we get

$$\begin{aligned}0&=[x_3x_1\mathfrak{g}(x_3^2)-x_3x_1\mathfrak{g}(x_3)x_3,x_3]+[\mathcal{H}(x_3)x_3x_1\mathfrak{h}(x_3),x_3]\\&\quad+[x_3x_1[x_3,\varphi(x_3)],x_3]\\&=x_3[x_1x_3\mathfrak{g}(x_3),x_3]+[\mathcal{H}(x_3)x_3x_1\mathfrak{h}(x_3),x_3]+x_3[x_1[x_3,\varphi(x_3)],x_3].\end{aligned}\tag{34}$$

Replacing x_2 with x_3^2 in (33), we get

$$[x_1\mathfrak{g}(x_3^3)-x_1\mathfrak{g}(x_3^2)x_3,x_3]+[\mathcal{H}(x_3^2)x_1\mathfrak{h}(x_3),x_3]+[x_1[x_3,\varphi(x_3^2)],x_3]=0,$$

that is

$$\begin{aligned}&[x_1x_3^2\mathfrak{g}(x_3),x_3]+[\mathcal{H}(x_3)x_3x_1\mathfrak{h}(x_3),x_3]+x_3[\mathfrak{h}(x_3)x_1\mathfrak{h}(x_3),x_3]\\&+[x_1[x_3,\varphi(x_3^2)],x_3]=0.\end{aligned}\tag{35}$$

Subtracting (35) from (34), we obtain

$$\begin{aligned}&x_3[x_1x_3\mathfrak{g}(x_3),x_3][x_1x_3^2\mathfrak{g}(x_3),x_3]-x_3[\mathfrak{h}(x_3)x_1\mathfrak{h}(x_3),x_3]\\&+x_3[x_1[x_3,\varphi(x_3)],x_3]-[x_1[x_3,\varphi(x_3^2)],x_3]=0.\end{aligned}\tag{36}$$

Again substituting x_3x_1 in place of x_1 in (36), we get

$$\begin{aligned} &x_3^2[x_1x_3\mathfrak{g}(x_3),x_3]-x_3[x_1x_3^2\mathfrak{g}(x_3),x_3]x_3[\mathfrak{h}(x_3)x_3x_1\mathfrak{h}(x_3),x_3]\\ &+x_3^2[x_1[x_3,\varphi(x_3)],x_3]-x_3[x_1[x_3,\varphi(x_3^2)],x_3]=0. \end{aligned} \tag{37}$$

Left multiplying (36) by x_3 and then subtracting from (37), we obtain

$$[x_3[\mathfrak{h}(x_3),x_3]x_1\mathfrak{h}(x_3),x_3]=0\ \forall\ x_1,x_3\in\mathfrak{I}. \tag{38}$$

Since (38) is same as (27) in Theorem 3.9, hence by same argument of Theorem 3.9 we get the required result.

By the same manner, we can prove that the same conclusion holds for $\mathcal{G}(x_1x_2)+\mathcal{H}(x_2)\mathcal{H}(x_1)-[x_1,\varphi(x_2)]\in Z(\mathfrak{R})\ \forall\ x_1,x_2\in\mathfrak{I}$. □

One can prove the following theorem using the same technique as above.

Theorem 3.14. *Let $\mathfrak{R}$ be a semiprime ring, $\mathfrak{I}$ be a nonzero left ideal of $\mathfrak{R}$ and $\varphi:\mathfrak{R}\to\mathfrak{R}$ any mapping. If $\mathfrak{R}$ admits a multiplicative (generalized)-derivations $\mathcal{G}$ and $\mathcal{H}$ associated with the mappings $\mathfrak{g}$ and $\mathfrak{h}$ respectively on $\mathfrak{R}$ such that $G(x_1x_2)-H(x_2)H(x_1)\pm[x_1,\varphi(x_2)]\in Z(R)\ \forall\ x_1,x_2\in\mathfrak{I}$, then $x_1[\mathfrak{h}(x_1),x_1]_2=0$, $\forall\ x_1\in\mathfrak{I}$.*

Corollary 3.15. *Let $\mathfrak{R}$ be a semiprime ring and $\varphi:\mathfrak{R}\to\mathfrak{R}$ any mapping. Suppose that $\mathcal{G},\mathcal{H}:\mathfrak{R}\to\mathfrak{R}$ be a multiplicative (generalized)-derivation associated with the map $\mathfrak{g},\mathfrak{h}:\mathfrak{R}\to\mathfrak{R}$. If $\mathcal{G}(x_1x_2)+\mathcal{H}(x_2)\mathcal{H}(x_1)\pm[x_1,\varphi(x_2)]\in Z(\mathfrak{R})\ \forall\ x_1,x_2\in\mathfrak{I}$, then $\mathfrak{h}$ is a centralizing map on $\mathfrak{R}$.*

In view of Theorem 3.13, Lemma 2.2 and Lemma 2.1 we immediately get the following corollary.

Corollary 3.16. *Let $\mathfrak{R}$ be a semiprime ring and $\varphi:\mathfrak{R}\to\mathfrak{R}$ any mapping. Suppose that $\mathcal{G}$ and $\mathcal{H}$ are two multiplicative (generalized)-derivations associated with a derivation h and a mapping $\mathfrak{g}$ respectively on $\mathfrak{R}$. If $\mathcal{G}(x_1x_2)+\mathcal{H}(x_2)\mathcal{H}(x_1)\pm[x_1,\varphi(x_2)]\in Z(\mathfrak{R})\ \forall\ x_1,x_2\in\mathfrak{I}$, then $\mathfrak{h}=0$ or $\mathfrak{R}$ contains a nonzero central ideal.*

Theorem 3.17. *Let $\mathfrak{R}$ be a semiprime ring, $\mathfrak{I}$ be a nonzero left ideal of $\mathfrak{R}$ and $\varphi:\mathfrak{R}\to\mathfrak{R}$ any mapping. If $\mathfrak{R}$ admits a multiplicative (generalized)-derivations $\mathcal{G}$ and $\mathcal{H}$ associated with the mappings $\mathfrak{g}$ and $\mathfrak{h}$ respectively on $\mathfrak{R}$ such that $\mathcal{G}(x_1x_2)+\mathcal{H}(x_2)\mathcal{H}(x_1)\pm[\varphi(x_1),x_2]\in Z(\mathfrak{R})\ \forall\ x_1,x_2\in\mathfrak{I}$, then $x_1[\mathfrak{h}(x_1),x_1]_2=0$ $\forall\ x_1\in\mathfrak{I}$.*

Proof. We begin with the assumption

$$\mathcal{G}(x_1x_2)+\mathcal{H}(x_1)\mathcal{H}(x_2)+[\varphi(x_1),x_2]\in Z(\mathfrak{R}). \tag{39}$$

$\forall\ x_1,x_2\in\mathfrak{I}$. Replacing x_2x_3 in place of x_2, we obtain $\mathcal{G}(x_1x_2)x_3+x_1x_2\mathfrak{g}(x_3)+\mathcal{H}(x_1)\mathcal{H}(x_2)x_3+\mathcal{H}(x_1)x_2\mathfrak{h}(x_3)+x_2[\varphi(x_1),x_3]+[\varphi(x_1),x_2]x_3\in Z(\mathfrak{R})$. Commuting with x_3 and using $\mathcal{G}(x_1x_2)+\mathcal{H}(x_1)\mathcal{H}(x_2)+[\varphi(x_1),x_2]\in Z(\mathfrak{R})$, we get

$$[x_1x_2\mathfrak{g}(x_3),x_3]+[\mathcal{H}(x_1)x_2\mathfrak{h}(x_3),x_3]+[x_2[\varphi(x_1),x_3],x_3]=0. \tag{40}$$

Again replace x_1 by x_1^2 in (40), we get

$$\begin{aligned}&[x_1^2x_2\mathfrak{g}(x_3),x_3]+[\mathcal{H}(x_1)x_1x_2\mathfrak{h}(x_3),x_3]\\&+[x_1\mathfrak{h}(x_1)x_2\mathfrak{h}(x_3),x_3]+[x_2[\varphi(x_1^2),x_3],x_3]=0.\end{aligned}\tag{41}$$

In (40), replacing x_1x_2 in place of x_2 and then subtracting from (41), we obtain

$$[x_1\mathfrak{h}(x_1)x_2\mathfrak{h}(x_3),x_3]+[x_2[\varphi(x_1^2),x_3],x_3][x_1x_2[\varphi(x_1),x_3],x_3]=0.\tag{42}$$

In particular for $x_1=x_3$, we have

$$[x_3\mathfrak{h}(x_3)x_2\mathfrak{h}(x_3),x_3]+[x_2[\varphi(x_3^2),x_3],x_3][x_3x_2[\varphi(x_3),x_3],x_3]=0.\tag{43}$$

Again substituting x_3x_2 in place of x_2 in (42), we get

$$[x_3\mathfrak{h}(x_3)x_3x_2\mathfrak{h}(x_3),x_3]+[x_3x_2[\varphi(x_3^2),x_3],x_3][x_3^2x_2[\varphi(x_3),x_3],x_3]=0.\tag{44}$$

Left multiplying (43) by x_3 and then subtracting from (44), we obtain

$$\begin{aligned}0&=[x_3\mathfrak{h}(x_3)x_3x_2\mathfrak{h}(x_3),x_3]-x_3[x_3\mathfrak{h}(x_3)x_2\mathfrak{h}(x_3),x_3]\\&=[x_3\mathfrak{h}(x_3)x_3-x_3^2\mathfrak{h}(x_3)x_2\mathfrak{h}(x_3),x_3]\\&=[x_3[\mathfrak{h}(x_3),x_3]x_2\mathfrak{h}(x_3),x_3].\end{aligned}\tag{45}$$

Since (45) and (27) are identical, by Theorem 3.9 we conclude that $\mathfrak{I}[[\mathfrak{h}(x_3),x_3],x_3]=(0)$.

By the same manner, we can prove that the same conclusion holds for $\mathcal{G}(x_1x_2)+\mathcal{H}(x_1)\mathcal{H}(x_2)[\varphi(x_1),x_2]\in Z(\mathfrak{R})\ \forall\ x_1,x_2\in\mathfrak{I}$. The proof of Theorem is completed. □

Using the same method one can prove the following theorem.

Theorem 3.18. *Let R be a semiprime ring, I be a nonzero left ideal of R and $\varphi: R\to R$ any mapping. If $\mathfrak{R}$ admits a multiplicative (generalized)-derivations $\mathcal{G}$ and $\mathcal{H}$ associated with the mappings $\mathfrak{g}$ and $\mathfrak{h}$ respectively on $\mathfrak{R}$ such that $G(x_1x_2)-H(x_2)H(x_1)\pm[\varphi(x_1),x_2]\in Z(R)\ \forall\ x_1,x_2\in I$, then $x_1[h(x_1),x_1]_2=0$ $\forall\ x_1\in I$.*

Corollary 3.19. *Let $\mathfrak{R}$ be a semiprime ring and $\varphi:\mathfrak{R}\to\mathfrak{R}$ any mapping. Suppose that $\mathcal{G}$ and $\mathcal{H}$ are two multiplicative (generalized)-derivations associated with the mappings $\mathfrak{g}$ and $\mathfrak{h}$ respectively on $\mathfrak{R}$. If $\mathcal{G}(x_1x_2)+\mathcal{H}(x_2)\mathcal{H}(x_1)\pm[\varphi(x_1),x_2]\in Z(\mathfrak{R})\ \forall\ x_1,x_2\in\mathfrak{I}$, then $\mathfrak{h}$ is a centralizing map on $\mathfrak{R}$.*

In view of Theorem 3.17, Lemma 2.2 and Lemma 2.1 we immediately get the following corollary.

Corollary 3.20. *Let $\mathfrak{R}$ be a semiprime ring and $\varphi:\mathfrak{R}\to\mathfrak{R}$ any mapping. Suppose that $\mathcal{G}$ and $\mathcal{H}$ are two multiplicative (generalized)-derivations associated with a derivation $\mathfrak{h}$ and a mapping $\mathfrak{g}$ respectively on $\mathfrak{R}$. If $\mathcal{G}(x_1x_2)+\mathcal{H}(x_2)H(x_1)\pm[\varphi(x_1),x_2]\in Z(\mathfrak{R})\ \forall\ x_1,x_2\in\mathfrak{I}$, then $\mathfrak{h}=0$ or $\mathfrak{R}$ contains a nonzero central ideal.*

The following Theorem is an immediate consequence of Theorem 3.13, and Theorem 3.17.

Theorem 3.21. *Let $\mathfrak{R}$ be a semiprime ring and $\mathfrak{I}$ be a nonzero left ideal of $\mathfrak{R}$. If $\mathfrak{R}$ admits a multiplicative (generalized)-derivations $\mathcal{G}$ and $\mathcal{H}$ associated with the mappings $\mathfrak{g}$ and $\mathfrak{h}$ respectively on $\mathfrak{R}$ such that $\mathcal{G}(x_1x_2)-\mathcal{H}(x_2)\mathcal{H}(x_1)\pm[x_1,x_2]\in Z(\mathfrak{R})$ $\forall$ $x_1,x_2\in\mathfrak{I}$, then $x_1[\mathfrak{h}(x_1),x_1]_2=0$ $\forall$ $x_1\in\mathfrak{I}$.*

Example 3.22. Consider

$$\mathfrak{R}=\left\{\begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix}\mid a,b,c\in\mathbb{Z}\right\}.$$

Let $X=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}$ we note that $X\mathfrak{R}X=0$ but $X\neq 0$ implies that $\mathfrak{R}$ is not semiprime ring. Now, we define $\mathcal{G},\mathcal{H},\mathfrak{g},\mathfrak{h}:\mathfrak{R}\to\mathfrak{R}$ by

$$\mathcal{G}\begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix}=\begin{pmatrix}0&a&0\\0&0&0\\0&0&0\end{pmatrix}\text{ and }\mathfrak{g}\begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix}=\begin{pmatrix}0&a^2&b^2\\0&0&-c\\0&0&0\end{pmatrix}$$

,

$$\mathcal{H}\begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix}=\begin{pmatrix}0&a&0\\0&0&c\\0&0&0\end{pmatrix}\text{ and }\mathfrak{h}\begin{pmatrix}0&a&b\\0&0&c\\0&0&0\end{pmatrix}=\begin{pmatrix}0&ab&b^2\\0&0&c\\0&0&0\end{pmatrix}.$$

Then $\mathcal{G}$ and $\mathcal{H}$ are multiplicative (generalized)-derivations associated with the mappings $\mathfrak{g}$ and $\mathfrak{h}$, respectively and φ is a identity mapping on $\mathfrak{R}$. Let $\mathfrak{I}=\left\{\begin{pmatrix}0&0&b\\0&0&0\\0&0&0\end{pmatrix}\mid b\in\mathbb{Z}\right\}$. It is easy to verify that $\mathfrak{I}$ is an ideal of $\mathfrak{R}$ and satisfying the following conditions: (i) $\mathcal{G}(x_1x_2)\pm\mathcal{H}(x_1)\mathcal{H}(x_2)+x_1x_2\in Z(\mathfrak{R})$, (ii) $\mathcal{G}(x_1x_2)+\mathcal{H}(x_1)\mathcal{H}(x_2)\pm x_2x_1\in Z(\mathfrak{R})$, (iii) $\mathcal{G}(x_1x_2)+\mathcal{H}(x_2)\mathcal{H}(x_1)\pm x_2x_1\in Z(\mathfrak{R})$ and (iv) $\mathcal{G}(x_1x_2)-\mathcal{H}(x_2)\mathcal{H}(x_1)\pm[x_1,x_2]\in Z(\mathfrak{R})$ $\forall$ $x_1,x_2\in\mathfrak{I}$ but $\mathfrak{R}$ is noncommutative. Hence, the hypothesis of semiprimeness in the Theorem 3.1, Theorem 3.5, Theorem 3.9 and Theorem 3.21 cannot be omitted.

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