*J. Appl. Math. & Informatics Vol.* **36***(2018), No. 1 - 2, pp. 69 - 79*
https://doi.org/10.14317/jami.2018.069

# ON GENERALIZED SUBWAY METRIC†

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Abstract. The Euclid metric is well-known and there are many results on the space with that metric. But there are many other metrics which gives more practical and useful results in the plane. In this paper, we introduce new metric function in the plane, which is more useful in city with subway. Finally we generalize to the general metric space and introduce a new metric on $\mathbb{R}^n$.

AMS Mathematics Subject Classification : 51K05.
*Key words and phrases* : Generalized subway metric, new metrics on metric spaces.

## 1. Introduction

It is much helpful to understand geometric properties of spaces to adopt metric function. Euclid metric is easy to understand and useful but it is not practical to guide to the distance of route in city. To improve this fact, taxicab distance was introduced for taxi-moving route [5,7] and, $\alpha$-metric as a generalization of taxicab distance was studied[2,3,4]. Nowadays, there are many papers[1,6,7] that adopt new metrics to reflect more reality in the modern city and study geometric properties of metric given spaces. There are not only taxis but subways, and not only straight street but curved road in modern city. So it is meaningful to introduce new metric with reality. There are some studies about subway metric but there are some problems more such as to force to use subway when it is not efficient. In this point of view, we introduce a new metric in the modern city with subway, and we call it a generalized subway metric in the plane. Finally

Received August 2, 2017. Revised September 16, 2017. Accepted October 9, 2017.
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†This paper was supported by the grant of the Seoul Science High School

we suggest and generalized it to the general metirc space and introduce a new metric on $\mathbb{R}^n$.

## 2. Generalized Subway metric

Subway distance[1,6,7] was defined to think about distance between two point $P(x_1, y_1)$, $Q(x_2, y_2)$ in city. It is given by

$$d(P,Q) = min(d_T(P,Q), d_T(A,L) + d_T(B,L))$$

where $d_T$ is taxicab metric.

In this paper, we define a new metric on the plane with subway like as follow: Let $m = max(|x_1 - x_2|, |y_1 - y_2|), n = min(|x_1 - x_2|, |y_1 - y_2|)$, and let $\alpha$, with $0 \leq \alpha \leq \frac{\pi}{4}$, $d_\alpha = m + n(sec\alpha - tan\alpha)$ be an $\alpha$-distance function.

Let $A, B \in \mathbb{R}^2$ and line subway route $S$ which containe station $P$, $Q$. Consider a set $M(A,B)$ as

$$M(A,B) = \{d | P \in S, Q \in S,\ d(A,B) = d_\alpha(A,P) + kd_E(P,Q) + d_\alpha(Q,B)\}$$

where $0<k<1$ and $d_E$ is an Euclid metric. In this paper, we assume that the route of subway is parallel to $x$-axis. Let us define the function $d_{S'}$ as

$$d_{S'} : \mathbb{R}^2 \times \mathbb{R}^2 \to R, d_{S'}(A,B) = min\{minM(A,B), d_\alpha(A,B)\}.$$

Hereafter, we denote $minM(A,B)$ by $d_S(A,B)$. Then we have

**Theorem 2.1.** *$d_{S'}$ is a distrance function*

*Proof.* Since minM(A, B) $\geq 0$, and $d_\alpha$(A, B) $\geq 0$, we see that $d_{S'}(A,B) \geq 0$. If the starting point and terminal point are determined, then $minM(A,B)$ is fixed. Let us check the conditions of distance function.

(1) $d_{S'}(A,A) = 0$ and $d_{S'}(A,B) = 0 \Leftrightarrow minM(A,B) = d_\alpha(A,B) = 0 \Leftrightarrow$ A= B.

(2) From the definition of $M(A,B)$ and the fact that $d_\alpha$ is a distance function, we see that $d_{S'}(A,B) = d_{S'}(B,A)$.

(3) Divide by 6 cases to prove triangle inequality.

1) AB, AC, BC are more shorter $\alpha$ distance than $d_S$ :
Trivially $d_\alpha(A,B) \leq d_\alpha(A,C) + d_\alpha(B,C)$.

2) If $minM(A,B) \leq d_\alpha(A,B)$ and the other path are more shorter $\alpha$ distance than $d_S$ :
In this case, since $d_S(A,B) \leq d_\alpha(A,B) \leq d_\alpha(A,C) + d_\alpha(B,C)$, the triangle inequality can be satisfied.

3) Only one of the distance $d_{s'}$ of AC or BC is shorter $d_{S'}$ than $\alpha$ distance and another is longer than $\alpha$ distance. Then we get:
$d_\alpha(A,C) + d_\alpha(B,P) + kd_E(P,Q) + d_\alpha(Q,P)$

$\geq d_\alpha(A,Q) + kd_E(P,Q) + d_\alpha(Q,P)$
$\geq d_S(A,B) \geq d_\alpha(A,B)$.
Hence triangle inequality is satisfied.

4) Only one of the $\alpha$ distance of BC or AC is shorter $\alpha$ distance than $d_{S'}$ and another is longer than $d_S$. Then we get:
$d_S(A,C)+d_\alpha(B,C) = d_\alpha(A,P)+kd_E(P,Q)+d_\alpha(Q,C)+d_\alpha(B,C)$
$\geq d_\alpha(A,P) + kd_E(P,Q) + d_\alpha(B,Q) \geq d_S(A,B)$,
so we get desired resilt.

5) If the $d_S$ of AC and BC are shorter than $\alpha$ distance, and another is longer than $\alpha$ distance, then we see that
$d_\alpha(A,C)+kd_E(P,Q)+d_\alpha(Q,C)+d_\alpha(C,P')+kd_E(P',Q')+d_\alpha(Q',B)$
$\geq d_\alpha(A,C) + kd_E(P,Q) + d_\alpha(Q,P') + kd_E(P',Q') + d_\alpha(Q',B)$
$= d_\alpha(A,C) + kd_E(P,Q) + d_E(Q,P') + kd_E(P',Q') + d_\alpha(Q',B)$
$\geq d_\alpha(A,C) + kd_E(P,Q') + d_\alpha(Q',B) \geq d_S(A,B) \geq d_\alpha(A,B)$,
hence the triangle is proved.

6) If all of the $d_S$ of AB, AC and BC are shorter than $\alpha$ distance. Then we can prove by the similar method of 5).

□

**Definition 2.2.** We call the function $d_{S'}$ by generalized subway metric.

Let $A$ be a starting point, $B$ be a terminal point and the line subway route $l$ and station exist continuous on $l$, P and Q are two point on $l$. Then we can assume $A(0,0), B(x_0,y_0)$ $(x_0 > 0, y_0 > 0$ ), $l : y = m$ ($m$ is a constant). Let $P'(x'_1, m)$, $Q'(x'_2, m)$ is points that satisfies $\min M(A,B)$.

**Theorem 2.3.** *The x-coordinate is monotonic when they are moving. That is* $0 < x'_1 < x'_2 < x_0$.

*Proof.* Loss of generality it is sufficient to prove the monotonic increasing case. First We prove for the case $P'$, $Q'$ is the point between 0 and $x_0$.

① When $x'_1 < -x_0$: It is longer than when $0 < x'_1 < x_0$. And distance between $x'_1$ and 0 is larger than $x_0$. It means that it is always true for any point $P'$ and $Q'$. So $P'$ is not minimized point, it is a contradiction.

② When $-x_0 < x'_1 < 0$: Let $P''$ be a symmetric point to y-axis of $P'$. Then, the distance between the starting point and $P'$ is the same with the distance between starting point and $P''$. But distance between $P''$ and $Q$ is not longer than distance between $P'$ and $Q'$. So $P'$ is not minimized point, it is a contradiction.

③ When $x_1' > x_0$, $0 < x_2' < x_0$ : We can prove similar method as above.

Let $x_1' > x_2'$. In this case, if we change the position of $P'$ and $Q'$, then the distance is more shorten than before. So it is a contradiction. Hence we complete a proof. Then do not go back through $x$-axis.

□

**Theorem 2.4.** *The first movement with $\alpha$ distance is parallel to $y$-axis.*

*Proof.* Geomatically prove that it is not need to consider if first-moving is parallel to $x$-axis. Distance difference of $d_\alpha$ and $d_{S''}$ of origin to point on subway is $(1-k)d_S(P,Q)$. So if first moving is parallel to $x$-axis, then the point must be the closer (m, m) to be shorter distance. Hence $d_\alpha((0,0),(m,m))$ is minimum distance.

Let us consider the case of the first moving is parallel to $y$-axis.
If the positions which get into subway are ($t_1$, m),($t_2$, m) (Assume, $t_1 < t_2$), then the distance difference is (m + (sec$\alpha$ - tan$\alpha$)$t_2$) - (m + (sec$\alpha$ - tan$\alpha$)$t_1$ + k$(t_2 - t_1)$) = (sec$\alpha$ - tan$\alpha$ - k)$(t_2 - t_1)$. In this case, we divide into two cases by the magnitude of $\sec\alpha - \tan\alpha$ and $k$.

Then, divide a case which is larger $\sec\alpha - \tan\alpha$ or $k$ .

1) $sec\alpha - tan\alpha > k$: Riding on foot of perpendicular on subway line from starting point (If origin starts, it is $(0,m)$) makes generalized subway metric minimized. So $(1+k)m$ is the minimum distance. But since $d_\alpha((0,0),(m,m)) = (sec\alpha - tan\alpha + 1)m > (1+k)m$, first moving must be parallel to $y$-axis.
2) $sec\alpha - tan\alpha < k$: If the starting point is origin, then riding on $(m,m)$ is maximum. But $d_\alpha((0,0),(m,m))$ is equal unrelated what first moving direction is to $x$-axis or $y$-axis.

Hence we complete the proof. □

**Theorem 2.5.** *If $x_0 > m + |y_0 - m|$, then $d_{S'}$ is given is follows.*

(1) $x_0 > y_0$:
To calculating $minM(A,B)$, consider $Q$ and $B$. And let $X$ be a point with coordinate$(x_0 - y_0 + m, m)$. Q is toward (based on $x$-axis) $X$ because first moving must be parallel to $y$-axis. By the similar method, $T(m,m)$ and $P$ must be more left than $T$ by theorem 2.4. Then $Q$ must exist more right than $X$. Hence we have
$d_\alpha(Q,B) = |y_0 - m| + (sec\alpha - tan\alpha)(x_0 - x_2)$ and $\sqrt{2} - 1 < sec\alpha - tan\alpha < 1$ at $0 \le \alpha \le \frac{\pi}{4}$.

Considering $E(x_3, m), F(x_4, m)$, difference distance between $E$, $B$ and distance between $F$, $B$ is $(sec\alpha - tan\alpha)(x_4 - x_3)$. So, divide case by two cases to get $P, Q$ which make to minimize $minM(A, B)$ by $k$ as

1) $sec\alpha - tan\alpha < k$ : $minQ$ is $X(x_0 - y_0 + m, m)$ and $minP$ is $T(m, m)$.
2) $sec\alpha - tan\alpha > k$ : $minQ$ is $(x_0, m)$ and $minP$ is $(0, m)$.

Then we can get generalized subway metric regarding $sec\alpha - tan\alpha$ and $k$ as follows:

1) $sec\alpha - tan\alpha > k$:
since $P$ is $(0, m)$, $Q$ is $(x_0, m)$, $d_S(A, B) = min(x_0 + (sec\alpha - tan\alpha)y_0, m + |y_0 - m| + kx_0)$. Then we can divide into two cases as follows.

① $m < y_0 < 2m$:
Since $x_0 + (sec\alpha - tan\alpha)y_0 - m - |y_0 - m| - kx_0 = x_0 + (sec\alpha + tan\alpha)y_0 - y_0 - kx_0 = (1 - k)(x_0 - y_0) > 0$, $d_{S'}$ is $m + |y_0 - m| + kx_0$.

② $0 < y_0 < m$:
Since $x_0 + (sec\alpha - tan\alpha)y_0 - m - |y_0 - m| - kx_0 = x_0 + (sec\alpha + tan\alpha + 1)y_0 - 2m - kx_0 = (1 - k)x_0 + (sec\alpha - tan\alpha)y_0 - 2m$, we get $d_{S'}$ as follows.

i) if $y_0 > \frac{2m-(1-k)x_0}{sec\alpha+tan\alpha+1}$, then $d_{S'} = m + |y_0 - m| + kx_0$
ii) if $y_0 < \frac{2m-(1-k)x_0}{sec\alpha+tan\alpha+1}$, then $d_{S'} = x_0 + (sec\alpha + tan\alpha)y_0$.

2) $sec\alpha - tan\alpha < k$:

Since $P$ is (m, m), $Q$ is $(x_0 - y_0 + m, m)$, $d_{S'} = min(x_0 + (sec\alpha - tan\alpha)y_0, (sec\alpha - tan\alpha - k)m + (m + kx_0) + |y_0 - m|(1 - k + sec\alpha - tan\alpha))$. Hence we get $d_{S'}$ divide by two cases.

① If $m < y_0 < 2m$, then $x_0 + (sec\alpha - tan\alpha)y_0 - (sec\alpha + tan\alpha - k)m - (m + kx_0) - |y_0 - m|(1 - k + sec\alpha - tan\alpha) = x_0 + (sec\alpha - tan\alpha)y_0 - (sec\alpha + tan\alpha - k)m - (m + kx_0) - (y_0 - m)(1 - k + sec\alpha - tan\alpha) = -y_0 + (1 - k)x_0 - (sec\alpha + tan\alpha - k)y_0 + (sec\alpha + tan\alpha)y_0 = (1 - k)(x_0 - y_0) > 0$.
So $d_{S'} = (sec\alpha + tan\alpha - k)m + (m + kx_0) + |y_0 - m|((1 - k + sec\alpha + tan\alpha)$.

② If $0 < y_0 < m$, then it occurs two cases.

i) $y_0 > \frac{(1-k)(2m-x_0)+2m(sec\alpha-tan\alpha)}{1-k+2(sec\alpha-tan\alpha)}$ : $d_{S'} = (sec\alpha - tan\alpha - k)m + (m + kx_0) + |y_0 - m|(1 - k + sec\alpha - tan\alpha)$.
ii) $y_0 < \frac{(1-k)(2m-x_0)+2m(sec\alpha-tan\alpha)}{1-k+2(sec\alpha-tan\alpha)}$ : $d_{S'} = x_0 + (sec\alpha - tan\alpha)y_0$.

(2) $x_0 < y_0$ : Since $x_0 > m + |y_0 - m|$, $y_0 > m$. Then the quantity of $minM$ is fixed and only $d_\alpha(A, B)$ is change from $x_0 + (sec\alpha - tan\alpha)y_0$ to $y_0 + (sec\alpha - tan\alpha)x_0$. Hence by the similar method of the case $x_0 > y_0$, we get $d_{S'}$ as follows:

1) $sec\alpha - tan\alpha > k$:
$d_{S'} = min(y_0 + (sec\alpha - tan\alpha)x_0, m + |y_0 - m| + kx_0)$
In this case, $y_0 + (sec\alpha - tan\alpha)x_0 - m - |y_0 - m| - kx_0 > y_0 - m - |y_0 - m| > 0$, so $d_{S'} = m + |y_0 - m| + kx_0 = y_0 + kx_0$.
Hence we $d_{S'}$ is given as
① $y_0 > \frac{2m-(1-k)x_0}{sec\alpha - tan\alpha + 1}$ : $m + |y_0 - m| + kx_0$.

② $y_0 < \frac{2m-(1-k)x_0}{sec\alpha - tan\alpha + 1}$ : $x_0 + (sec\alpha - tan\alpha)y_0$.

2) $sec\alpha - tan\alpha < k$:
$d_{S'} = min(y_0 + (sec\alpha - tan\alpha)x_0, (sec\alpha - tan\alpha - k)m + (m + kx_0) + |y_0 - m|(1 - k + sec\alpha - tan\alpha)) = (sec\alpha - tan\alpha - k)m + (m + kx_0) + |y_0 - m|(1 - k + sec\alpha - tan\alpha)$.
Hence we get $d_{S'}$ as follows:

① $y_0 > \frac{(1-k)(2m-x_0)+2m(\sec\alpha - \tan\alpha)}{1-k+2(\sec\alpha-\tan\alpha)}$:
$(\sec\alpha - \tan\alpha - k)m + (m + kx_0) + |y_0 - m|(1 - k + \sec\alpha - \tan\alpha)$.
② $y < \frac{(1-k)(2m-x_0)+2m(\sec\alpha - \tan\alpha)}{1-k+2(\sec\alpha-\tan\alpha)}$: $y_0 + (\sec\alpha - \tan\alpha)x_0$.

**Theorem 2.6.** *If $m + |y_0 - m| > x_0$, then $d_{S'}$ is given as follows.*

Since $P$ is boarding gate and $Q$ is outing gate, $P$ must exist more right than $Q$. But it is contradicts to Theorem 2.3. So $\min M(A, B)$ is determined when $Q$ is boarding subway point and $P$ is boarding subway point. Then, it needs to a difference formula.

Let starting at $A(0, 0)$ and riding at $T(x_t, m)$, getting off at $W(x_w, m)$ and end point be $B(x_0, y_0)$. Let fixed terminal point and change starting point to get $\min M(A, B)$. Let us consider $d_{S'}$ by dividing three cases:

(1) $x_0 > y_0$:
Decide two boarding points. Let $T_1(x_{t_1}, m) and T_2(x_{t_2}, m)$ are station, and let $x_{t_1} < x_{t_2}$. Then, distance difference between dropping by $T_1$ and $T_2$, fixed getting-off-point $(x_w, m)$, is

$$\begin{aligned} m + (\sec\alpha - \tan\alpha)x_{t_2} + k(x_w - x_{t_2}) \quad &+ \quad |y_0 - m| + (x_0 - x_w)(\sec\alpha - \tan\alpha) - \\ (m + (\sec\alpha - \tan\alpha)x_{t_1} + k(x_w - x_{t_1}) \quad &+ \quad |y_0 - m| + (x_0 - x_w)(\sec\alpha - \tan\alpha)) \\ &= (\sec\alpha - \tan\alpha - k)(x_{t_2} - x_{t_1}). \end{aligned}$$

Hence we see that if the boarding point is more closer to $(0, m)$ then $d_{S'}$ is minimized, when $sec\alpha - tan\alpha > k$. On the other hand if the boarding point is more closer to $(m, m)$, then $d_{S'}$ is minimized when

$sec\alpha - tan\alpha < k$. Boaring point cannot exist more right than $(m, m)$ by Theorem 2.3. Hence we see that if the boarding point is closer to $(x_0, m)$, then $d_{S'}$ is minimized when $sec\alpha - tan\alpha > k$. On the other hand, if the boarding point is more closer to $(x_0 - |y_0 - m|, 0)$, then $d_{S'}$ is minimized when $sec\alpha - tan\alpha < k$. Hence we get $d_{S'} = m + |y_0 - m| + kx_0$ if $y_0 > \frac{2m-(1-k)x_0}{\sec\alpha - \tan\alpha + 1}$ and $d_{S'} = x_0 + (sec\alpha - tan\alpha)y_0$ if $0 < y_0 < \frac{2m-(1-k)x_0}{\sec\alpha - \tan\alpha + 1}$. But if $sec\alpha - tan\alpha < k$, then it occurs a unusual case. In this case, the distance difference is given by $(k - sec\alpha - tan\alpha)(dw + dt) > 0$ if the boarding and getting off points are $(x_t, m)$, $(x_w, m)$ and $(x_t + dt, m)$, $(x_w - dw, m)$ respectively. After all $minM(A, B)$ is occured if $x_t = x_w$, that is $d_{S'} = d_\alpha$, it is a case of nonusing subway.

(2) $|y_0 - m| < x_0 < y_0$ : In this case we see that $d_{S'}$ is same of the case (1) by the similar way of (1).

(3) If $x_0 < |y_0 - m|$, we can get $d_{S'}$ dividing by three cases

1) $x_0 > y_0$: Then $y_0 < m$.
So we consider two cases

① $sec\alpha - tan\alpha > k$: The boarding and getting off points must be $(0, m)$ and $(x_0, m)$ respectively to minimize $M(A, B)$. Then the distance difference of $minM(A, B)$ and $d_\alpha(A, B)$ is $2m - y_0 + kx_0 - (x_0 + (sec\alpha - tan\alpha)y_0)$ and which is smaller than $(1 - (sec\alpha - tan\alpha))y_0 - (1 - k)^{x_0} < (1 - k)(y_0 - x_0) < 0$. Hence $d_{S'} = minM(A, B)$

② $sec\alpha - tan\alpha < k$: In this case, the boarding point and getting off point must be $(0, m)$ and $(x_0, m)$ respectively to minimize $M(A, B)$. Hence we get $d_{S'} = d_\alpha$.

2) $x_0 < y_0$: By the smilar way of the above case, we get $d_{S'}$ from two cases.

① $sec\alpha - tan\alpha < k$: $d_{S'} = d_\alpha$

② $sec\alpha - tan\alpha > k$: the distance difference of $minM(A, B)$ and $d_\alpha(A, B)$ is $m + |y_0 - m| + kx_0 - (y_0 + (sec\alpha - tan\alpha)x_0) = m - y_0 + |y_0 - m| - (sec\alpha - tan\alpha)x_0$. Hence we get $d_{S'} = minM(A, B)$.

## 3. New metrics on metirc spaces

**3.1. New metrics on a metric space using generalized subway metirc.** In this section, it will be shown to be able to construct new metric function from given metric function on $X$.

Suppose that there is a metric function $d$ defined on $X$ and the other metric function $d_S$ defined on a finite set $S \subset X$. If $d_S(P, Q) \le d(P, Q)$ for all $P, Q \in S$, define $m_s : X \times X \to R_0^+$ and $d \diamond d_s$ as

$$m_s(x, y) = \min\{d(x, P) + d_s(P, Q) + d(Q, y) | P, Q \in S\}$$

$$d \diamond d_s(x, y) = \min\{d(x, y), m_s(x, y)\}$$

Then we see that

$$d \diamond d_s(x, y) = d(x, y) \text{ or } d \diamond d_s(x, y) = m_s(x, y)$$
$$d \diamond d_s(x, y) \leq d(x, y),\ d \diamond d_s(x, y) \leq m_s(x, y)$$

Since $S$ is finite, there exists $P, Q \in S$ satisfying $m_s(x, y) = d(x, P) + d_s(P, Q) + d(Q, y)$.

**Theorem 3.1.** *$d \diamond d_s$ is a metric function on $X$.*

*Proof.*
*(1) Claim : $d \diamond d_s(x, y) = 0$ iff $x = y$.*
*If $d \diamond d_s(x, y) = 0$, $d(x, y) = 0$ or $m_s(x, y) = 0$.*
*If $d(x, y) = 0$, then $x = y$ because $d$ is an metric function.*
*If $m_s(x, y) = 0$, there exists $P, Q \in S$ which satisfies $d(x, P) + d_s(P, Q) + d(Q, y) = 0$. But every term in $m_s(x, y)$ is non-negative thus they are 0. Now $d(x, P) = d_s(P, Q) = d(Q, y) = 0$ so $x = P = Q = y$.*

*Conversely if $x = y$, then $d(x, y) = 0$. So $d \diamond d_s(x, y) = min\{0, m_s(x, y)\} = 0$.*

*(2) symmetry : $d \diamond d_s(x, y) = d \diamond d_s(y, x)$*
*Suppose that $m_s(x, y) \neq m_x(y, x)$.*
*Then we can assume that $m_s(x, y) < m_s(y, x)$ without lost of generality.*
*So there are $P, Q \in S$ which satisfy $m_s(x, y) = d(x, P) + d_s(P, Q) + d(Q, y) < m_s(y, x)$, and $m_s(x, y) = d(y, Q) + d_s(Q, P) + d(P, x) < m_s(y, x)$ by the definition of metric function. But it violates definition of $m_s(y, x)$. It is contradiction and now it is true that $m_s(x, y) = m_s(y, x)$.*

*(3) triangle inequality*
*It is divided into four cases to show $d \diamond d_s(x, y) + d \diamond d_s(y, x) \geq d \diamond d_s(x, z)$.*

1) $d \diamond d_s(x, y) = d(x, y), d \diamond d_s(y, z) = d(y, z)$:

$$d \diamond d_s(x, y) + d \diamond d_s(y, x) = d(x, y) + d(y, x) \geq d(x, z) \geq d \diamond d_s(x, z)$$

2) $d \diamond d_s(x, y) = d(x, y)$, $d \diamond d_s(y, z) = m_s(y, z) = d(y, P) + d_s(P, Q) + d(Q, z)(P, Q \in S)$:

$$\begin{aligned} & d \diamond d_s(x, y) + d \diamond d_s(y, x) \\ & = d(x, y) + d(y, P) + d_s(P, Q) + d(Q, z) \\ & \geq d(x, P) + d_s(P, Q) + d(Q, z) \\ & \geq m_s(x, z) \\ & \geq d \diamond d_s(x, z) \end{aligned}$$

3) $d \diamond d_s(x, y) = m_s(x, y) = d(x, P) + d_s(P, Q) + d(Q, y)$, $d \diamond d_s(y, z) = d(x, y)$. *So, we can prove by the similar way of the case 2).*

4) *If* $d \diamond d_s(x,y) = m_s(x,y) = d(x,P) + d_s(P,Q) + d(Q,y)(P,Q \in S)$ *and* $d \diamond d_s(y,z) = m_s(y,z) = d(y,R) + d_s(R,T) + d(T,z)(R,T \in S)$, *then*

$$\begin{aligned}
&d \diamond d_s(x,y) + d \diamond d_s(y,z)\\
&= m_s(x,y) + m_s(y,z)\\
&= d(x,P) + d_s(P,Q) + d(Q,y) + d(y,R) + d_s(R,S) + d(S,z)\\
&\geq d(x,P) + d_s(P,Q) + d_s(Q,y) + d(y,R) + d_s(R,S) + d(S,z)\\
&\geq d(x,P) + d_s(P,R) + d_s(R,S) + d(S,z)\\
&\geq m_s(x,z)\\
&\geq d \diamond d_s(x,z)
\end{aligned}$$

**3.2. New metric on $R^n$.** *For angles* $0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n < \frac{\pi}{2}$ *and points* $X = (x_1, x_2, \ldots, x_n)$, $Y = (y_1, y_2, \ldots, y_n) \in R^n$, *assume that*

$$|x_{i_1} - y_{i_1}| \geq |x_{i_2} - y_{i_2}| \geq \ldots \geq |x_{i_n} - y_{i_n}|$$

*are satisfied for* $\{i_1, i_2, \ldots, i_n\} = \{1, 2, \ldots, n\}$. *Define the function* $d_{\{\alpha_h\}} : (R^n)^2 \to R$ *by*

$$d_{\{\alpha_h\}}(X,Y) := d_{\alpha_1,\alpha_2,\ldots,\alpha_n}(X,Y) = \sum_{k=1}^{n} (sec\alpha_k - tan\alpha_k)|x_{i_k} - y_{i_k}|$$

*Then we have*

**Theorem 3.2.** *The function $d_{\{\alpha_h\}}$ becomes a distance function*

*Proof Since* $f(x) = secx - tanx$ *satisfies* $f(0) = 1$ *and*

$$f'(x) = secxtanx - sec^2x = -secxf(x) < 0$$

*on* $x \in \left[0, \frac{\pi}{2}\right]$, $f(x)$ *is a monotone decreasing function. Moreover we see that*

*(1)* $d_{\{\alpha_k\}}(X,Y) \geq 0$ *is trivial because* $f(\alpha_k) > 0$.
*(2) Since* $f(\alpha_k) > 0$ *for all* $1 \leq k \leq n$,

$$d_{\{\alpha_k\}}(X,Y) = 0 \text{ iff } \forall_{1 \leq i \leq n} : x_i = y_i, \text{ that is } X = Y.$$

*(3)* $d_{\{\alpha_k\}}(X,Y) = d_{\{\alpha_k\}}(Y,X)$ *is trivial.*
*(4) For arbitrary points* $X = (x_1, x_2, \ldots, x_n)$, $Y = (y_1, y_2, \ldots, y_n)$, $Z = (z_1, z_2, \ldots, z_n)$, *if we assume that*

$$\begin{aligned}
|x_{i_1} - y_{i_1}| \geq |x_{i_2} - y_{i_2}| \geq \ldots \geq |x_{i_n} - y_{i_n}|, \{i_1, i_2, \ldots, i_n\} = \{1, 2, \ldots, n\},\\
|x_{j_1} - z_{j_1}| \geq |x_{j_2} - z_{j_2}| \geq \ldots \geq |x_{j_n} - z_{j_n}|, \{j_1, j_2, \ldots, j_n\} = \{1, 2, \ldots, n\},\\
|y_{k_1} - z_{k_1}| \geq |y_{k_2} - z_{k_2}| \geq \ldots \geq |y_{k_n} - z_{k_n}|, \{k_1, k_2, \ldots, k_n\} = \{1, 2, \ldots, n\},
\end{aligned}$$

*then by the rearrangement inequality we see that*

$$d_{\{\alpha_h\}}(X,Z)$$

$$\begin{aligned}
&= \sum_{h=1}^{n}(sec\alpha_h - tan\alpha_h)|x_{j_h} - z_{j_h}| \\
&\le \sum_{h=1}^{n}(sec\alpha_h - tan\alpha_h)|x_{j_h} - y_{j_h}| + \sum_{h=1}^{n}(sec\alpha_h - tan\alpha_h)|y_{j_h} - z_{j_h}| \\
&\le \sum_{h=1}^{n}(sec\alpha_h - tan\alpha_h)|x_{i_h} - y_{i_h}| + \sum_{h=1}^{n}(sec\alpha_h - tan\alpha_h)|y_{k_h} - z_{k_h}|(\because) \\
&= d_{\{\alpha_h\}}(X,Y) + d_{\{\alpha_k\}}(Y,z).
\end{aligned}$$

*Hence the function* $d_{\{\alpha_h\}} : (R^n)^2 \to R$ *becomes a distance function.*

**Remark 3.3.** In Theorem 3.2, if we consider the function $f(x)$ as a positive monotone decreasing function instead of $f(x) = \sec x - \tan x$ ,then the theorem is also true.

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