J. Appl. Math. & Informatics Vol. **36***(2018), No. 1 - 2, pp. 51 - 57*
https://doi.org/10.14317/jami.2018.051

PREGROUPS AND PRE-B-ALGEBRAS†

GANG WU AND YOUNG HEE KIM*

Abstract. In this paper, we introduce the notions of pregroups, postgroups and pre-B-algebras, and we investigate their relations. Using this notions we give another proof that the notion of B-algebras coincides with the notion of pregroups.

AMS Mathematics Subject Classification : 06F35.
Key words and phrases : pregroup, postgroup, pre-B-algebra, B-algebra.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([6, 7]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [3, 4], Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. J. Neggers and H. S. Kim ([14]) introduced a new notion which appears to be of some interest, i.e., that of a B-algebra, and studied some of its properties. M. Kondo and Y. B. Jun ([12]) proved that the class of B-algebras is equivalent in one sense to the class of groups by using the property: $x = 0 * (0 * x)$, for all $x \in X$. J. Neggers and H. S. Kim ([14]) argued slightly differently in taking their position. J. R. Cho and H. S. Kim ([2]) discussed further relations between B-algebras and other classes of algebras, such as quasigroups. It is well-known that every group determines a B-algebra, called a *group-derived* B-algebra. It is natural to consider the problem whether or not all B-algebras are so group-derived. J. Neggers and H. S. Kim ([15]) introduced the notion of normality in B-algebras, and obtained a fundamental theorem of B-homomorphism for B-algebras. C. B. Kim and H. S. Kim ([9]) introduced the notion of a BM-algebra which is a specialization of B-algebras, and they proved the following: the class of

Received September 28, 2017. Revised November 3, 2017. Accepted November 8, 2017.
*Corresponding author.
†This work was financially supported by the Research Year of Chungbuk National University in 2017.

BM-algebras is a proper subclass of B-algebras, and showed that a BM-algebra is equivalent to a 0-commutative B-algebra, and the class of Coxeter algebras is a proper subclass of BM-algebras. H. K. Park and H. S. Kim ([16]) introduced the notion of a quadratic B-algebra which is a medial quasigroup, and obtained that every quadratic B-algebra on a field X with $|X| \geq x$, is a BCI-algebra. Y. B. Jun et al. ([8]) considered the fuzzification of (normal) B-subalgebras, and investigated some related properties. They characterized fuzzy B-algebras. P. J. Allen et al. ([1]) gave another proof of the close relationship of B-algebras with groups using the zero adjoint mapping. H. S. Kim and H. G. Park ([11]) showed that if X is a 0-commutative B-algebra, then $(x * a) * (y * b) = (b * a) * (y * x)$. Using this property they showed that the class of p-semisimple BCI-algebras is equivalent to the class of 0-commutative B-algebras. A. Walendziak ([17]) obtained some systems of axioms defining a B-algebra, and he also obtained a simplified axiomatization of 0-commutative B-algebras. C. B. Kim and H. S. Kim ([10]) introduced the notion of a BA-algebra, and showed that the class of BA-algebras is equivalent to the class of B-algebras. For general reference on BCK/BCI-algebra we refer to ([5, 13, 18]).

In this paper, we introduce the notions of pregroups, postgroups and pre-B-algebras, and we investigate their relations. Using this notions we give another proof that the notion of B-algebras coincides with the notion of pregroups.

2. Preliminaries

A *B-algebra* ([14]) is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(I) $x * x = 0$,
(II) $x * 0 = x$,
(III) $(x * y) * z = x * (z * (0 * y))$ for all x, y, z in X.

Example 2.1 ([14])**.** Let X be the set of all real numbers except for a negative integer $-n$. Define a binary operation $*$ on X by

$$x * y := \frac{n(x-y)}{n+y}$$

Then $(X, *, 0)$ is a B-algebra.

Example 2.2. Let $X := \{0, 1, 2, 3, 4, 5\}$ be a set with the following table:

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Then $(X, *, 0)$ is a B-algebra (see [14]).

Theorem 2.3 ([14, 17])**.** *If* $(X, *, 0)$ *is a* B*-algebra, then*

(1) $x * (y * z) = (x * (0 * z)) * y$,
(2) $0 * (0 * x) = x$,
(3) $0 * (x * y) = y * x$ *for all* $x, y, z \in X$.

Proposition 2.4 ([2, 14])**.** *In any* B*-algebra, the left and the right cancelation laws hold.*

Theorem 2.5 ([1])**.** *Let* $(X, \bullet)$ *be a group with identity* e_X*. If we define* $x * y := x \bullet y^{-1}$*, then* $(X, *, e_X)$ *is a* B*-algebra.*

3. Pregroups and postgroups

Let $(X, \bullet)$ be a group and let $\varphi : X \to X$ be a function. A groupoid $(X, *)$ is said to be a *pregroup* of $(X, \bullet)$ with respect to φ if $x \bullet y := x * \varphi(y)$ for all $x, y \in X$. Moreover, a groupoid $(X, *)$ is said to be a *postgroup* of $(X, \bullet)$ with respect to φ if $x * y := x \bullet \varphi(y)$ for all $x, y \in X$.

Example 3.1. Consider $X := \{0, 1, 2, 3\}$ with the following table:

$\bullet$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Then $(X, \bullet)$ is a group which is isomorphic with $\mathbf{Z}_4$. If we define a map $\varphi : X \to X$ by $\varphi(0) = 0, \varphi(1) = 3, \varphi(2) = 2$ and $\varphi(3) = 1$, then the groupoid $(X, *)$ with the following table:

$*$	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

is a pregroup of $(X, \bullet)$ with respect to φ. Note that $(X, *)$ is not a group, since it has no identity.

Example 3.2. Let $(X, \bullet)$ be a group. Define a binary operation $*$ on X by $x * y := x \bullet y^{-1}$ for all $x, y \in X$ and define a map $\varphi : X \to X$ by $\varphi(x) := x^{-1}$ for all $x \in X$. Then $x * \varphi(y) = x * y^{-1} = x \bullet (y^{-1})^{-1} = x \bullet y$. This shows that $(X, *)$ is a pregroup of $(X, \bullet)$ with respect to φ.

Proposition 3.3. *Let* $(X, *)$ *be a pregroup of a group* $(X, \bullet)$ *with respect to a function* $\varphi : X \to X$*. Then*

(1) $(Im\varphi, *)$ *has only one idempotent,*
(2) φ *is injective,*
(3) *every finite pregroup is also a postgroup.*

Proof. (1) Let $u := \varphi(e_X)$ where e_X is an identity for a group $(X, \bullet)$. Then $x = x \bullet e_X = x * \varphi(e_X) = x * u$ for all $x \in X$. It follows that $u = u * u$, i.e., u is an idempotent element of $Im\varphi$. Assume $v = \varphi(w)$ is an idempotent in $Im\varphi$ for some $w \in X$. Then $v \bullet w = v * \varphi(w) = v * v = v$. Since $(X, \bullet)$ is a group, we obtain $w = e_X$ and hence $v = \varphi(w) = \varphi(e_X) = u$.

(2) If $\varphi(y) = \varphi(z)$ for any $y, z \in X$, then $y = e_x \bullet y = e_X * \varphi(y) = e_X * \varphi(z) = e_X \bullet z = z$, proving that φ is injective.

(3) Assume X is finite. Since φ is injective, it is also an onto function, i.e., φ is a bijective function. Let $\varphi^{-1} : X \to X$ be an inverse function of φ. Then $x \bullet \varphi^{-1}(y) = x * \varphi(\varphi^{-1}(y)) = x * y$ for all $x, y \in X$, which proves that $(X, *)$ is a postgroup of $(X, \bullet)$. □

Proposition 3.4. *If $(X, *)$ is a pregroup of a group $(X, \bullet)$ with respect to a function $\varphi : X \to X$, then φ is onto.*

Proof. Since $(X, \bullet)$ is a group, we have $x \bullet X = X$ for any $x \in X$. It follows from $(X, *)$ is a pregroup of a group $(X, \bullet)$ with respect to a function $\varphi : X \to X$ that $x \bullet y = x * \varphi(y)$, and hence $x * Im\varphi = x \bullet X = X$ for all $x \in X$. This shows that $|X| = |x * Im\varphi| \le |Im\varphi| \le |X|$, proving that $Im\varphi = X$, i.e., φ is onto. □

Theorem 3.5. *Every left-zero semigroup $(X, *)$, $|X| \ge 2$, is a postgroup of any group, and it can not be a pregroup of any group.*

Proof. Let $(X, \bullet)$ be a group with identity e_X. Define a map $\varphi : X \to X$ by $\varphi(x) := e_X$ for all $x \in X$. Then $x * y = x = x \bullet e_X = x \bullet \varphi(y)$ for all $x, y \in X$, proving that $(X, *)$ is a postgroup of $(X, \bullet)$.

Assume that $(X, *)$ is a pregroup of a group $(X, \bullet)$. Then there is a function $\varphi : (X, *) \to (X, \bullet)$ such that $x \bullet y = x * \varphi(y)$ for all $x, y \in X$. It follows that $x \bullet y = x * \varphi(y) = x = x \bullet e_X$. Since $(X, \bullet)$ is a group, we obtain $y = e_X$ for all $y \in X$, i.e., $|X| = 1$, a contradiction. □

Remark. Not every groupoid is a postgroup. See the following example.

Example 3.6. Let $a \in X$. Define a binary operation $x * y := a$ for all $x, y \in X$. Then $(X, *)$ is a groupoid. Assume $(X, *)$ is a postgroup of a group $(X, \bullet)$ with respect to $\varphi : X \to X$. Then $x * y = x \bullet \varphi(y)$ for all $x, y \in X$. Hence $x \bullet \varphi(y) = x * y = a$ for all $x, y \in X$. Since $(X, \bullet)$ is a group, we have $\varphi(y) = x^{-1} \bullet a$ for all $x \in X$. Let $x \ne z$ in X. Then $x^{-1} \bullet a = \varphi(y) = z^{-1} \bullet a$. Since $(X, \bullet)$ is a group, we have $x^{-1} = z^{-1}$ and hence $x = z$, a contradiction. Hence $(X, *)$ is not a postgroup.

Theorem 3.7. *Let $(X, *)$ be a pregroup of a group $(X, \odot, \widehat{e})$ with respect to ψ and let $(X, *)$ be a postgroup of a group $(X, \bullet, e)$ with respect to φ. Then $(\varphi \circ \psi)(x) = (\widehat{e})^{-1} \bullet x$ for all $x \in X$*

Proof. Let $(X, *)$ be a pregroup of a group $(X, \odot, \widehat{e})$ with respect to ψ. Then $x \odot y = x * \psi(y)$ for all $x, y \in X$. Let $(X, *)$ be a postgroup of a group $(X, \bullet, e)$ with respect to φ. Then $x * y = x \bullet \varphi(y)$ for all $x, y \in X$. It follows that

$x \odot y = x * \psi(y) = x \bullet \varphi(\psi(y))$ for all $x, y \in X$. Hence $x = \widehat{e} \odot x = \widehat{e} \bullet \varphi(\psi(x))$, which shows that $(\varphi \circ \psi)(x) = (\widehat{e})^{-1} \bullet x$ for all $x \in X$. □

The following proposition can be easily proved:

Proposition 3.8. *The direct product of pregroups is a pregroup and the direct product of postgroups is a postgroup.*

Remark. Given a non-empty set X, not every groupoid $(X, *)$ can be a pregroup of a group $(X, \bullet)$ defined on X.

Example 3.9. Let $\mathbf{N} := \{0, 1, 2, \cdots\}$. Assume $(\mathbf{N}, +)$ is a pregroup of a group $(\mathbf{N}, \bullet)$ with respect to a mapping $\varphi : X \to X$. Let e be an identity for $(\mathbf{N}, \bullet)$. Then $x = x \bullet e = x + \varphi(e)$ for all $x \in \mathbf{N}$, which shows that $\varphi(e) = 0$. Moreover, $x = e \bullet x = e + \varphi(x)$ and hence $\varphi(x) = x - e$ for all $x \in \mathbf{N}$. Thus we obtain $x \bullet y = x + \varphi(y) = x + y - e$ for all $x, y \in \mathbf{N}$. It follows that $e = x \bullet x^{-1} = x + x^{-1} - e$ and hence $x^{-1} = 2e - x \geq 0$ for all $x \in \mathbf{N}$. This shows that $x \leq 2e$ for all $x \in \mathbf{N}$, a contradiction.

Proposition 3.10. *Let $(X, *)$ be a pregroup of a group $(X, \bullet)$ with respect to a function $\varphi : X \to X$. If φ is onto, then the left and right cancelation laws hold in $(X, *)$.*

Proof. Assume $x * y = z * y$ where $x, y, z \in X$. Since φ is onto, there exists $a \in X$ such that $\varphi(a) = y$. It follows that $x \bullet a = x * \varphi(a) = x * y = z * y = z * \varphi(a) = z \bullet a$. Since $(X, \bullet)$ is a group, we obtain $x = z$. Similarly, the left cancelation law holds. □

4. Pre-B-algebras and postgroups

Definition 4.1. Let $(X, *, 0)$ be a B-algebra and let $\varphi : X \to X$ be a mapping. An algebra $(X, \bullet, 0)$ is said to be a *pre-B-algebra* with respect to φ if $x * y := x \bullet \varphi(y)$ for all $x, y \in X$.

Proposition 4.2. *If $(X, \bullet, 0)$ is a pre-B-algebra, then*

(1) $x \bullet \varphi(x) = 0$ *and* $x = x \bullet \varphi(0)$,
(2) $x = 0 \bullet \varphi(0 \bullet \varphi(x))$,
(3) $(x \bullet \varphi(y)) \bullet \varphi(y) = x \bullet \varphi(z \bullet \varphi(0 \bullet \varphi(y)))$,
(4) $x \bullet \varphi(y \bullet \varphi(z)) = (x \bullet \varphi(0 \bullet \varphi(z))) \bullet \varphi(y)$,

for all $x, y, z \in X$ and for some mapping $\varphi : X \to X$.

Proof. If $(X, \bullet, 0)$ is a pre-B-algebra, then there exists a B-algebra $(X, *, 0)$, where $x * y := x \bullet \varphi(y)$, for all $x, y \in X$, for some mapping $\varphi : X \to X$. (1) Given $x \in X$, we have $0 = x * x = x \bullet \varphi(x)$ and $x = x * 0 = x \bullet \varphi(0)$. (2) Given $x \in X$, we have $x = 0 * (0 * x) = 0 \bullet \varphi(0 * x) = 0 \bullet \varphi(0 \bullet \varphi(x))$. (3) For any $x, y, z \in X$, we have $(x * y) * z = (x \bullet \varphi(y)) \bullet \varphi(y)$ and $x * (z * (0 * y)) = x \bullet \varphi(z * (0 * y)) = x \bullet \varphi(z \bullet \varphi(0 * y)) = x \bullet \varphi(z \bullet \varphi(0 \bullet \varphi(y)))$, which proves (3), since $(X, *, 0)$ is a B-algebra. (4) For any $x, y, z \in X$, we have $x * (y * z) = x \bullet \varphi(y \bullet \varphi(z))$

and $(x*(0*z))*y = (x*(0*z))\bullet\varphi(y) = (x\bullet\varphi(0*z))\bullet\varphi(y) = (x\bullet\varphi(0\bullet\varphi(z)))\bullet\varphi(y)$, which proves (4), since $(X, *, 0)$ is a B-algebra. □

Theorem 4.3. *Every group is a pre-B-algebra.*

Proof. Let $(X, \bullet)$ be a group with identity e_X. Define a map $\varphi : X \to X$ by $\varphi(x) := x^{-1}$ for all $x \in X$. If we define a binary operation "$*$" on X by $x * y := x \bullet \varphi(y)$, then $x * y = x \bullet y^{-1}$ for all $x, y \in X$. Given $x \in X$, we have $x * x = x \bullet x^{-1} = e_X$ and $x * e_X = x \bullet {e_X}^{-1} = x$. Given $x, y, z \in X$, we have $(x * y) * z = (x \bullet y^{-1}) \bullet z^{-1} = x \bullet (y^{-1} \bullet z^{-1})$ and $x * (z * (e_X * y)) = x * (z \bullet (e_X \bullet y^{-1})^{-1}) = x * (z \bullet y) = x \bullet (z \bullet y)^{-1} = x \bullet (y^{-1} \bullet z^{-1})$, proving that $(X, *, e_X)$ is a B-algebra, i.e., $(X, \bullet, e_X)$ is a pre-B-algebra. □

Proposition 4.4. *Every B-algebra is a postgroup of a group.*

Proof. It follows immediately from Theorem 2.5. □

Question. Can non-isomorphic groupoids $(X, \cdot_1)$ and $(X, \cdot_2)$ produce isomorphic B-algebras through the proper choices of identities e_1, e_2 and mappings φ_1, φ_2 ?

Theorem 4.5. *Let $(X, \bullet)$ be a group with identity e_X and let $(X, *, e_X)$ be a B-algebra, where $x \bullet y := x * \psi(y)$ and $x * y := x \bullet \varphi(y)$ for all $x, y \in X$. Then ψ, φ are bijective and $\psi^{-1} = \varphi$.*

Proof. Given $x, y \in X$, we have $x * y = x \bullet \varphi(y) = x * \psi(\varphi(y)) = x * (\psi \circ \varphi)(y)$. By Proposition 2.4, we obtain $y = (\psi \circ \varphi)(y)$.

Given $x, y \in X$, we have $x \bullet y = x * \psi(y) = x \bullet \varphi(\psi(y)) = x \bullet (\varphi \circ \psi)(y)$. Since every group has cancelation laws, we obtain $y = (\varphi \circ \psi)(y)$, proving the theorem. □

References

1. P.J. Allen, J. Neggers and H.S. Kim, *B-algebras and groups*, Sci. Math. Japo. Online **9** (2003), 9-17.
2. Jung R. Cho and H.S. Kim, *On B-algebras and quasigroups*, Quasigroups and related systems **7** (2001), 1-6.
3. Q.P. Hu and Xin Li, *On BCH-algebras*, Math. Seminar Notes **11** (1983), 313-320.
4. Q.P. Hu and Xin Li, *On proper BCH-algebras*, Math. Japonica **30** (1985), 659-661.
5. A. Iorgulescu, Algebras of logic as BCK-algebras, Editura ASE, Bucharest, 2008.
6. K. Iséki and S. Tanaka, *An introduction to theory of BCK-algebras*, Math. Japonica **23** (1978), 1-26.
7. K. Iséki, *On BCI-algebras*, Math. Seminar Notes **8** (1980), 125-130.
8. Y.B. Jun, E.H. Roh and H.S. Kim, *On fuzzy B-algebras*, Czech. Math. J. **52**(2002), 375-384.
9. C.B. Kim and H.S. Kim, *On BM-algebras*, Sci. Math. Japo. 63(2006), 421-427.
10. C.B. Kim and H.S. Kim, *Another axiomatization of B-algebrs*, Demonstratio Math. **41** (2008), 259-262.
11. H.S. Kim and H.G. Park, *On 0-commutative B-algebras*, Sci. Math. Japo. **62** (2005), 31-36.

12. M. Kondo and Y.B. Jun, *The class of B-algebras coincides with the class of groups*, Sci. Math. Japo. Online **7** (2002), 175-177.
13. J. Meng and Y.B. Jun, BCK-algebras, Kyungmoon Sa, Seoul, 1994.
14. J. Neggers and H.S. Kim, *On B-algebras*, Mate. Vesnik **54** (2002), 21-29.
15. J. Neggers and H.S. Kim, *A fundamental theorem of B-homomorphism for B-algebras*, Intern. Math. J. **2**(2002), 207-214.
16. H.K. Park and H.S. Kim, *On quadratic B-algebras*, Quasigroups and related systems **8** (2001), 67-72.
17. A. Walendziak, *Some axiomatizations of B-algebras*, Math. Slovaca **56** (2006), 301-306.
18. H. Yisheng, BCI-algebras, Science Press, Beijing, 2006.

Gang Wu Department of Mathematics and Applied Mathematics, Harbin University of Commerce, Harbin, 150076, China.
e-mail: `w3393@163.com`

Young Hee Kim Department of Mathematics, Chungbuk National University, Cheongju, 28644, Korea.
e-mail: `yhkim@chungbuk.ac.kr`