J. Appl. Math. & Informatics Vol. **36**(2018), No. 1 - 2, pp. 29 - 38
https://doi.org/10.14317/jami.2018.029

SYMMETRIC IDENTITIES FOR DEGENERATE q-POLY-BERNOULLI NUMBERS AND POLYNOMIALS[†]

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Abstract. In this paper, we introduce a degenerate q-poly-Bernoulli numbers and polynomials include q-logarithm function. We derive some relations with this polynomials and the Stirling numbers of second kind and investigate some symmetric identities using special functions that are involving this polynomials.

AMS Mathematics Subject Classification : 11B68, 11B73, 11B75.
Key words and phrases : degenerate poly-Bernoulli polynomials, degenerate q-poly-Bernoulli polynomials, Stirling numbers of the second kind, q-polylogarithm function.

1. Introduction

Throughout this paper, we use the following notations. $\mathbb{N} = \{1, 2, 3, \dots\}$ denotes the set of natural numbers, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ denotes the set of non-negative integer, $\mathbb{Z}$ denotes the set of integers, and $\mathbb{C}$ denotes the set of complex numbers, respectively.

Also in this paper, we use the notation ;

$$[x]_q = \frac{1-q^x}{1-q}.$$

Hence, $\lim_{q\to 1}[x]_q = x$.

The classical Bernoulli numbers B_n and polynomials $B_n(x)$ are given by the generating functions(see[1-14]);

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!},$$

Received May 23, 2017. Revised August 21, 2017. Accepted September 15, 2017.
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[†]This work was supported by 2017 Hannam University Research Fund

and

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}.$$

Many researchers have studied about the generalizations of these numbers and polynomials. And various attempts have been made for the study of the classical Bernoulli numbers and polynomials. In [1-3], there are definitions and properties of the poly Bernoulli numbers and their zeta function. In [4], L. Carlitz introduced the degenerate Bernoulli polynomials $B_n(x;\lambda)$ that the generating function is given as below:

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}B_n(x;\lambda)\frac{t^n}{n},\quad (\lambda\in\mathbb{C}). \tag{1.1}$$

When $x\neq 0, B_n(0|\lambda)=B_n(\lambda)$ is called the degenerate Bernoulli numbers.

The first few are

$$\begin{aligned}
&B_0(x;\lambda)=1,\\
&B_1(x;\lambda)=x-\frac{1}{2}+\frac{1}{2}\lambda,\\
&B_2(x;\lambda)=x^2-x+\frac{1}{6}-\frac{1}{6}\lambda^2,\\
&B_3(x;\lambda)=x^3-\frac{3}{2}x^2+\frac{1}{2}x-\frac{3}{2}\lambda x+\frac{1}{4}\lambda^3-\frac{1}{4}\lambda,\quad \cdots.
\end{aligned}$$

Note that $(1+\lambda t)^{\frac{1}{\lambda}}$ tend to e^t as $\lambda\to 0$. It is certain that the Equation(1.1) reduces to the generating function of the classical Bernoulli polynomials :

$$\lim_{\lambda\to 0}\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}.$$

The generalized falling factorial $(x|\lambda)_n$ with increment λ is defined by

$$(x|\lambda)_n=\Pi_{k=1}^{n}(x-\lambda(k-1))\quad (see[6,11,13]).$$

For $n\geq 0$, we have

$$B_n(x;\lambda)=\sum_{l=0}^{n}\binom{n}{l}B_l(\lambda)(x|\lambda)_{n-l}, \tag{1.2}$$

where $(x|\lambda)_n=x(x-1)\cdots(x-(n-1)\lambda)$ is generalized falling factorial(see [4,11,13]).

The polylogarithm function Li_k is defined by

$$Li_k(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^k}$$

for $k \in \mathbb{Z}$ (see[1,2,3,6,7,8,9]). By using polylogarithm function, Kaneko, in [7], defined a sequence of rational numbers, which is refered to as poly-Bernoulli numbers,

$$\frac{Li_k(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}.$$

The k-th q-polylogarithm function $Li_{k,q}$ is introduced by

$$Li_{k,q}(x) = \sum_{n=1}^{\infty} \frac{x^n}{[n]_q^k}, \quad (k \in \mathbb{Z}). \tag{1.3}$$

For nonnegative integer k, the q-polylogarithm function is represented by a rational function,

$$Li_{-k,q}(x) = \frac{1}{(1-q)^k} \sum_{l=0}^{k} (-1)^l \binom{k}{l} \frac{q^l x}{1-q^l x}.$$

Recently, in [6], we introduced q-poly-Bernoulli polynomials defined by

$$\frac{Li_{k,q}(1-e^{-t})}{e^t-1} e^{xt} = \sum_{n=0}^{\infty} B_{n,q}^{(k)}(x) \frac{t^n}{n!}. \tag{1.4}$$

The Stirling number of the first kind is given by

$$(x)_n = \sum_{m=0}^{n} S_1(n,m) x^m (n \geq 0)$$

and

$$\sum_{n=m}^{\infty} S_1(n,m) \frac{t^n}{n!} = \frac{(log(1+t))^m}{m!},$$

where $(x)_n$ is falling factorial(see[4,5,6,12]).

The Stirling numbers of the second kind is defined by

$$\sum_{n=m}^{\infty} S_2(n,m) \frac{t^n}{n!} = \frac{(e^t-1)^m}{m!}. \tag{1.5}$$

In this paper, we consider the degenerate q-poly-Bernoulli polynomials. We investigate several properties of the polynomials and derive some relation with other polynomials. We also find some symmetric identities of degenerate q-poly-Bernoulli polynomials by using special functions.

2. Degenerate q-poly-Bernoulli polynomials

In this section, we define the generating function of degenerate q-poly-Bernoulli numbers $B_{n,q}^{(k)}(\lambda)$ and polynomials $B_{n,q}^{(k)}(x;\lambda)$. From the definition, we get some identities that is similar to the classical Bernoulli polynomials.

Definition 2.1. For $k \in \mathbb{Z}, \quad n \geq 0, \quad 0 \leq q < 1$, we define the degenerate q-poly-Bernoulli polynomials by:

$$\frac{Li_{k,q}(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} B_{n,q}^{(k)}(x;\lambda)\frac{t^n}{n!} \tag{2.1}$$

where

$$Li_{k,q}(t) = \sum_{n=1}^{\infty} \frac{t^n}{[n]_q^k}$$

is the k-th q-polylogarithm function.

When $x = 0$, $B_{n,q}^{(k)}(\lambda) = B_{n,q}^{(k)}(0;\lambda)$ are called the degenerate q-poly-Bernoulli numbers. Note that $lim_{q\to 1}[n]_q = n$, and $lim_{q\to 1}B_{n,q}^{(k)}(x;\lambda) = B_n^{(k)}(x;\lambda)$.

It is trivial that the Equation(2.1) is reduced to the q-poly-Bernoulli polynomials which is introduced in Equation(1.4),

$$\lim_{\lambda\to 0} B_{n,q}^{(k)}(x;\lambda) = B_{n,q}^{(k)}(x).$$

From the Equation(2.1), we get the relation between the degenerate q-poly-Bernoulli numbers and the degenerate q-poly-Bernoulli polynomials.

Theorem 2.2. *Let $n \geq 0, k \in \mathbb{Z}, 0 \leq q < 1$. We have*

$$B_{n,q}^{(k)}(x;\lambda) = \sum_{l=0}^{n} \binom{n}{l} B_{l,q}^{(k)}(\lambda)(x|\lambda)_{n-l}. \tag{2.2}$$

where $(x|\lambda)_{n-l}$ is the generalized falling factorial.

Proof. For $n \geq 0, k \in \mathbb{Z}, 0 \leq q < 1$, we can derive the following result:

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n,q}^{(k)}(x;\lambda)\frac{t^n}{n!} &= \frac{Li_{k,q}(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}B_{l,q}^{(k)}(\lambda)(x|\lambda)_{n-l}\right)\frac{t^n}{n!}.\end{aligned}$$

Hence, we get

$$B_{n,q}^{(k)}(x;\lambda) = \sum_{l=0}^{n} \binom{n}{l} B_{l,q}^{(k)}(\lambda)(x|\lambda)_{n-l}.$$

□

Replacing x by $x + y$ in the Equation(2.2), we have an addition theorem.

Theorem 2.3. *For $n \geq 0, k \in \mathbb{Z}, 0 \leq q < 1$, we obtain*

$$B_{n,q}^{(k)}(x+y;\lambda) = \sum_{l=0}^{n} \binom{n}{l} B_{l,q}^{(k)}(x;\lambda)(y|\lambda)_{n-l}.$$

Proof. Let $n \geq 0, k \in \mathbb{Z}, 0 \leq q < 1$. Then we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n,q}^{(k)}(x+y;\lambda)\frac{t^n}{n!} &= \frac{Li_{k,q}(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x+y}{\lambda}} \\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}B_{l,q}^{(k)}(x;\lambda)(y|\lambda)_{n-l}\frac{t^n}{n!}.\end{aligned}$$

Thus, we get the addition theorem as below;

$$B_{n,q}^{(k)}(x+y;\lambda) = \sum_{l=0}^{n}\binom{n}{l}B_{l,q}^{(k)}(x;\lambda)(y|\lambda)_{n-l}.$$

□

In the Equation(1.4), the definition of q-polylogarithm function $Li_{k,q}$, is represented by

$$\begin{aligned}Li_{k,q}(1-e^{-t}) &= \sum_{l=1}^{\infty}\frac{(1-e^{-t})^l}{[l]_q^k} \\ &= \sum_{n=1}^{\infty}(-1)^l\frac{(e^{-t}-1)^l}{[l]_q^k} \\ &= \sum_{n=1}^{\infty}\sum_{l=1}^{n}\frac{(-1)^{l+n}}{[l]_q^k}l!S_2(n,l)\frac{t^n}{n!}.\end{aligned}$$

From above result, we get

$$\frac{1}{t}Li_{k,q}(1-e^{-t}) = \sum_{n=0}^{\infty}\sum_{l=1}^{n+1}\frac{(-1)^{l+n+1}}{[l]_q^k}l!\frac{S_2(n+1,l)}{n+1}\frac{t^n}{n!}. \tag{2.3}$$

Using the Equation(2.3), we obtain next theorem.

Theorem 2.4. *For $n \geq 0, k \in \mathbb{Z}$, we have*

$$B_{n,q}^{(k)}(x;\lambda) = \sum_{i=0}^{n}\binom{n}{i}\sum_{l=1}^{i+1}\frac{(-1)^{l+i+1}l!S_2(i+1,l)}{[l]_q^k(i+1)}B_{n-i,q}(x;\lambda).$$

Proof. Let $n \geq 0, k \in \mathbb{Z}, 0 \leq q < 1$. By the relation between the q-polylogarithm function and stirling numbers, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} B_{n,q}^{(k)}(x;\lambda)\frac{t^n}{n!} &= \frac{Li_{k,q}(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} \\
&= \frac{Li_{k,q}(1-e^{-t})}{t}\left(\frac{t(1+\lambda t)^{\frac{x}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right) \\
&= \sum_{n=0}^{\infty}\sum_{l=1}^{n+1}\frac{(-1)^{l+n+1}}{[l]_q^k}l!\frac{S_2(n+1,l)}{n+1}\frac{t^n}{n!}\sum_{n=0}^{\infty}B_{n,q}(x;\lambda)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}\sum_{l=1}^{i+1}\frac{(-1)^{l+i+1}l!S_2(i+1,l)}{[l]_q^k(i+1)}B_{n-i,q}(x;\lambda)\frac{t^n}{n!}.
\end{aligned}$$

Therefore, we get

$$B_{n,q}^{(k)}(x;\lambda) = \sum_{i=0}^{n}\binom{n}{i}\sum_{l=1}^{i+1}\frac{(-1)^{l+i+1}l!S_2(i+1,l)}{[l]_q^k(i+1)}B_{n-i,q}(x;\lambda).$$

□

From the definition of degenerate q-poly-Bernoulli polynomials, a recurrence formula is derived as the following theorem.

Theorem 2.5. *For $n \geq 1, k \in \mathbb{Z}, 0 \leq q < 1$, we get*

$$\begin{aligned}
&B_{n,q}^{(k)}(x+1;\lambda) - B_{n,q}^{(k)}(x;\lambda) \\
&\quad = \sum_{r=1}^{n}\binom{n}{r}\left(\sum_{l=0}^{r-1}\frac{(-1)^{l+1+r}}{[l+1]_q^k}(l+1)!S_2(r,l+1)\right)(x|\lambda)_{n-r}.
\end{aligned}$$

Proof. Let $n \geq 1, k \in \mathbb{Z}, 0 \leq q < 1$. From the Definition 2.1, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} B_{n,q}^{(k)}(x+1;\lambda)\frac{t^n}{n!} - \sum_{n=0}^{\infty} B_{n,q}^{(k)}(x;\lambda)\frac{t^n}{n!} \\
&\quad = \frac{Li_{k,q}(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x+1}{\lambda}} - \frac{Li_{k,q}(1-e^{-t})}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} \\
&\quad = \sum_{l=0}^{\infty}\frac{(1-e^{-t})^{l+1}}{[l+1]_q^k}(1+\lambda t)^{\frac{x}{\lambda}} \\
&\quad = \sum_{n=0}^{\infty}\sum_{r=1}^{n}\binom{n}{r}\left(\sum_{l=0}^{r-1}\frac{(-1)^{l+1+r}}{[l+1]_q^k}(l+1)!S_2(r,l+1)\right)(x|\lambda)_{n-r}\frac{t^n}{n!}.
\end{aligned}$$

Therefore, the formula is appeared as follows;

$$B_{n,q}^{(k)}(x+1;\lambda) - B_{n,q}^{(k)}(x;\lambda) = \sum_{r=1}^{n} \binom{n}{r} \left(\sum_{l=0}^{r-1} \frac{(-1)^{l+1+r}}{[l+1]_q^k} (l+1)! S_2(r, l+1) \right) (x|\lambda)_{n-r}.$$

□

3. Symmetric identities for the degenerate q-poly-Bernoulli polynomials

In this section, we consider some generating functions and investigate general symmetric identities for the degenerate q-poly-Bernoulli polynomials by given special functions.

Theorem 3.1. *For $x \in \mathbb{R}$ and $n \geq 0$, $a, b > 0 (a \neq b)$, we have the following identity;*

$$\sum_{m=0}^{n} \binom{n}{m} a^{n-m} b^m B_{n-m,q}^{(k)}\left(bx; \frac{\lambda}{a}\right) B_{m,q}^{(k)}\left(ax; \frac{\lambda}{b}\right) = \sum_{m=0}^{n} \binom{n}{m} a^m b^{n-m} B_{m,q}^{(k)}\left(bx; \frac{\lambda}{a}\right) B_{n-m,q}^{(k)}\left(ax; \frac{\lambda}{b}\right).$$

Proof. For $x \in \mathbb{R}$ and $n \geq 0$, $a, b > 0 (a \neq b)$, We consider the generating function,

$$F(t) = \left(\frac{Li_{k,q}(1-e^{-at}) Li_{k,q}(1-e^{-bt})}{((1+\lambda t)^{\frac{a}{\lambda}} - 1)((1+\lambda t)^{\frac{a}{\lambda}} - 1)} \right) (1+\lambda t)^{\frac{2abx}{\lambda}}.$$

The generating function, $F(t)$, is written by

$$\begin{aligned} F(t) &= \left(\frac{Li_{k,q}(1-e^{-at}) Li_{k,q}(1-e^{-bt})}{((1+\lambda t)^{\frac{a}{\lambda}} - 1)((1+\lambda t)^{\frac{a}{\lambda}} - 1)} \right) (1+\lambda t)^{\frac{2abx}{\lambda}} \\ &= \sum_{n=0}^{\infty} B_{n,q}^{(k)}\left(bx; \frac{\lambda}{a}\right) \frac{(at)^n}{n!} \sum_{m=0}^{\infty} B_{n,q}^{(k)}\left(ax; \frac{\lambda}{b}\right) \frac{(bt)^m}{m!} \qquad (3.1) \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} a^{n-m} b^m B_{n-m,q}^{(k)}\left(bx; \frac{\lambda}{a}\right) B_{m,q}^{(k)}\left(ax; \frac{\lambda}{b}\right) \frac{t^n}{n!}. \end{aligned}$$

Similarly, we can get

$$F(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} a^m b^{n-m} B_{m,q}^{(k)}\left(bx; \frac{\lambda}{a}\right) B_{n-m,q}^{(k)}\left(ax; \frac{\lambda}{b}\right) \frac{t^n}{n!}. \qquad (3.2)$$

From Equation(3.1) and (3.2), we can easily get the above result. □

By substituting $b = 1$, we obtain next corollary.

Corollary 3.2. *For $a>0, x\in\mathbb{R}$ and $n\geq 0$, we have*

$$\sum_{m=0}^{n}\binom{n}{m}a^{n-m}B_{n-m,q}^{(k)}\left(x;\frac{\lambda}{a}\right)B_{m,q}^{(k)}\left(ax;\lambda\right)$$
$$=\sum_{m=0}^{n}\binom{n}{m}a^{m}B_{n-m,q}^{(k)}\left(ax;\lambda\right)B_{m,q}^{(k)}\left(x;\frac{\lambda}{a}\right).$$

In [14], a generalized factorial sum $\sigma_k(n;\lambda)$ is introduced by

$$\frac{(1+\lambda t)^{\frac{(n+1)}{\lambda}}-1}{(1+\lambda t)^{\frac{1}{\lambda}}-1}=\sum_{k=0}^{\infty}\sigma_k(n;\lambda)\frac{t^k}{k!} \qquad (3.3).$$

Using the generalized factorial sum, we get a symmetric relation of degenerate q-poly-Bernoulli polynomials.

Theorem 3.3. *For $x,y\in\mathbb{R}$, $n\geq 0$, $a,b>0$ and $a\neq b$, we have*

$$\sum_{m=0}^{n}\binom{n}{m}a^{n-m}b^{m-1}B_m\left(ax;\frac{\lambda}{b}\right)\sigma_{n-m}\left(b-1;\frac{\lambda}{a}\right)$$
$$=\sum_{m=0}^{n}\binom{n}{m}a^{m-1}b^{n-m}B_m\left(bx;\frac{\lambda}{a}\right)\sigma_{n-m}\left(a-1;\frac{\lambda}{b}\right)$$

Proof. Let $x,y\in\mathbb{R}$, $n\geq 0$, $a,b>0$ and $a\neq b$.

We consider the generating function:

$$F(t)=\frac{tLi_{k,q}(1-e^{-at})Li_{k,q}(1-e^{-bt})((1+\lambda t)^{\frac{ab}{\lambda}}-1)(1+\lambda t)^{\frac{abx}{\lambda}}}{((1+\lambda t)^{\frac{a}{\lambda}}-1)^2((1+\lambda t)^{\frac{b}{\lambda}}-1)^2}.$$

The Equation follows as below

$$\begin{aligned}F(t)&=\sum_{n=0}^{\infty}B_{n,q}^{(k)}\left(\frac{\lambda}{a}\right)\frac{(at)^n}{n!}\sum_{n=0}^{\infty}B_{n,q}^{(k)}\left(\frac{\lambda}{b}\right)\frac{(bt)^n}{n!}\\&\quad\times\sum_{n=0}^{\infty}\sigma_n\left(b-1;\frac{\lambda}{a}\right)\frac{(at)^n}{n!}b^{-1}\sum_{n=0}^{\infty}B_n\left(ax,\frac{\lambda}{b}\right)\frac{(bt)^n}{n!}\\&=\sum_{n=0}^{\infty}B_{n,q}^{(k)}\left(\frac{\lambda}{a}\right)\frac{(at)^n}{n!}\sum_{n=0}^{\infty}B_{n,q}^{(k)}\left(\frac{\lambda}{b}\right)\frac{(bt)^n}{n!}\\&\quad\times\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}a^{n-m}b^{m-1}B_n\left(ax,\frac{\lambda}{b}\right)\sigma_n\left(b-1;\frac{\lambda}{a}\right)\frac{t^n}{n!}.\end{aligned} \qquad (3.4)$$

In similar method, we have

$$\begin{aligned}F(t) &= \sum_{n=0}^{\infty} B_{n,q}^{(k)}\left(\frac{\lambda}{a}\right)\frac{(at)^n}{n!}\sum_{n=0}^{\infty} B_{n,q}^{(k)}\left(\frac{\lambda}{b}\right)\frac{(bt)^n}{n!} \\ &\qquad \times \sum_{n=0}^{\infty}\sigma_n\left(a-1;\frac{\lambda}{b}\right)\frac{(bt)^n}{n!}a^{-1}\sum_{n=0}^{\infty}B_n\left(bx,\frac{\lambda}{a}\right)\frac{(at)^n}{n!} \\ &= \sum_{n=0}^{\infty} B_{n,q}^{(k)}\left(\frac{\lambda}{a}\right)\frac{(at)^n}{n!}\sum_{n=0}^{\infty} B_{n,q}^{(k)}\left(\frac{\lambda}{b}\right)\frac{(bt)^n}{n!} \\ &\qquad \times \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}a^{m-1}b^{n-m}B_m\left(bx,\frac{\lambda}{a}\right)\sigma_{n-m}\left(a-1;\frac{\lambda}{b}\right)\frac{t^n}{n!}.\end{aligned} \tag{3.5}$$

Comparing the coefficient of the Equation(3.4) and (3.5), then it gives the symmetric identity;

$$\begin{aligned}&\sum_{m=0}^{n}\binom{n}{m}a^{n-m}b^{m-1}B_n\left(ax,\frac{\lambda}{b}\right)\sigma_n\left(b-1;\frac{\lambda}{a}\right) \\ &\qquad = \sum_{m=0}^{n}\binom{n}{m}a^{m-1}b^{n-m}B_m\left(bx,\frac{\lambda}{a}\right)\sigma_{n-m}\left(a-1;\frac{\lambda}{b}\right).\end{aligned}$$

□

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